Trees in tournaments.

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Abstract

A digraph is said to be *n-unavoidable* if every tournament of order n contains it as a subgraph. Let $f(n)$ be the smallest integer such that every oriented tree is $f(n)$-unavoidable. Sumner (see [7]) noted that $f(n) \geq 2n-2$ and conjetured that equality holds. Häggkvist and Thomason established the upper bounds $f(n) \leq 12n$ and $f(n) \leq (4+o(1))n$. Let $g(k)$ be the smallest integer such that every oriented tree of order n with k leaves is $(n+g(k))$-unavoidable. Häggkvist and Thomason [1] proved that $g(k) \leq 2^{512k^3}$. Havet and Thomassé conjectured that $g(k) \leq k-1$. We study here the special case where the tree is a merging of paths (the union of disjoint paths emerging from a common origin). We prove that a merging of order n of k paths is $(n+\frac{3}{2}(k^2-3k)+5)$-unavoidable. In particular, a tree with three leaves is $(n+5)$-unavoidable, i.e. $g(3) \leq 5$. By studying trees with few leaves, we then prove that $f(n) \leq \frac{38}{5}n - 6$.

keywords: tournament, tree, unavoidable.

1 Introduction.

A *tournament* is an orientation of a complete graph. An *oriented tree* is an orientation of a tree; in particular, an *oriented path* is an orientation of a path. Throughout this paper, since we only consider oriented trees and oriented paths, we abbreviate 'oriented tree' to 'tree' and 'oriented path' to 'path'. A digraph is said to be *n-unavoidable* if every tournament of order n contains it as a subgraph. Let $f(n)$ be the smallest integer such that every (oriented) tree of order n is $f(n)$-unavoidable. Sumner (see [7]) noted that

$f(n) \geq 2n-2$. Indeed, the regular tournaments of order $2n-3$ have no vertex with outdegree greater than $n-2$ and thus do not contain the oriented tree consisting of a root dominating $n-1$ leaves. He also conjectured that equality holds.

Häggkvist and Thomason [1] proved that $f(n) \leq (4+o(1))n$, thereby improving the previous upper bound $(1+o(1))n\log_2 n$ obtained by Wormald [10] . They also established the absolute bound $f(n) \leq 12n$. Their proof is based on trees with few leaves and the existence of a minimal function $g(k) \leq 2^{512k^3}$ such that every tree of order n with k leaves is $(n+g(k))$-unavoidable.

With Stéphan Thomassé, we conjecture the following:

Conjecture 1 (Havet et Thomassé) Every tree of order n with k leaves is $(n+k-1)$-unavoidable, i.e. $g(k) \leq k-1$.

This conjecture implies Sumner's because a tree of order n has at most $n-1$ leaves. Since trees with two leaves are paths, a result of Thomason [9] (see corollary 2.1.1) confirms this conjecture for $k=2$.

Definition 1 Let $P=(x_1,\cdots,x_n)$ be a path. We say that x_1 is the *origin* of P and x_n is the *terminus* of P.

Let $P_1, P_2, \dots P_k$ be paths. The *merging* of $P_1, P_2, \dots, P_k$, denoted $\bigvee_{i=1}^k P_i$, is the digraph $\bigcup_{i=1}^k P'_i$ where P'_i is isomorphic to P_i with origin x and $V(P'_i) \cap V(P'_j) = \{x\}$, $1 \leq i < j \leq k$. We say that x is the *origin* of $\bigvee_{i=1}^k P_i$.

A tree with three leaves is the merging of three paths. In this paper, we prove (theorem 3.2) that a merging of order n of k paths is $(n+\frac{3}{2}(k^2-3k)+5)$-unavoidable. In particular, the trees with three leaves are $(n+5)$-unavoidable. By studying the unavoidabilty of trees with few leaves, we then improve the proof of Häggkvist and Thomason and obtain the bound $f(n) \leq \frac{38}{5}n-6$ (theorem 5.1).

2 Preliminaries: paths in tournaments.

Definition 2 Let T be a tournament. Let x and y be two vertices of T, we write $x \rightarrow y$ if (x,y) is an edge of T. In the same way, let X and Y be two subdigraphs of T. We write $X \rightarrow Y$ if $x \rightarrow y$ for all pairs $(x,y) \in V(X) \times V(Y)$. Let $A_1, A_2, \dots, A_k$ be a family of sets of vertices of T. We denote by $T[A_1, A_2, \dots, A_k]$ the subtournament induced by T on the set of vertices $\bigcup_{1\leq i\leq k} V(A_i)$ and by $T-[A_1, A_2, \dots, A_k]$ the subtournament induced by T on the set of vertices $V(T)\backslash \bigcup_{1\leq i\leq k} V(A_i)$. We often abbreviate $T-[A_1]$ in $T-A_1$ and $T-[\{x\}]$ in $T-x$.

Let $P=(x_1,\cdots,x_n)$ be a path.

If $x_1 \rightarrow x_2$, P is an *outpath*, otherwise P is an *inpath*. The *directed outpath* of order n is the path $P=(x_1,\cdots,x_n)$ in which $x_i \rightarrow x_{i+1}$ for all i, $1 \leq i < n$; the dual notion is that of a *directed inpath*. The *length* of a path is its number of edges. We denote the path $(x_2, \cdots, x_n)$ by $^*P$.

The *blocks* of P are the maximal subdipaths of P. We enumerate the blocks of P from the origin to the terminus. The first block of P is denoted $B_1(P)$ and its length $b_1(P)$. Likewise, the i$^{\text{th}}$ block of P is denoted $B_i(P)$ and its length $b_i(P)$. The path P is totally described by the signed sequence $sgn(P)(b_1(P), b_2(P), \cdots, b_k(P))$ where k is the number of blocks of P and $sgn(P) = +$ if P is an outpath and $sgn(P) = -$ if P is an inpath.

A tournament is *strong* (or *strongly connected*) if for any two vertices x and y there exists a directed outpath with origin x and terminus y. A nonstrong tournament is said to be *reducible*. The *strong component* of a tournament are its maximal subgraphs. The strong component C of a tournament T such that $T - C \to C$ is the *maximal strong component* of T.

Let X be a set of vertices of T. The *outsection* generated by X in T is the set of vertices y to which there exists a directed outpath from $x \in X$; we denote this set by $S_T^+(X)$ (or by $S_T^+(x)$ if $X = \{x\}$ and $S_T^+(x, y)$ if $X = \{x, y\}$). The dual notion, the *insection*, is denoted by $S_T^-(X)$. We also write $s_T^+(X)$ (resp. $s_T^-(X)$) for the number of vertices of $S_T^+(X)$ (resp. $S_T^-(X)$). An *outgenerator* of T is a vertex x of T such that $S_T^+(x) = V(T)$; the dual notion is an *ingenerator*.

Theorem 2.1 (Havet and Thomassé, [2]) *Let T be a tournament of order $n+1$, P an outpath of order n and x, y two distinct vertices of T. If $s^+(x, y) \geq b_1(P) + 1$ then x or y is an origin of P in T.*

Corollary 2.1.1 (Thomason, [9]) *Let T be a tournament of order $n+1$ and P a path of order n. At least two vertices of T are origins of P . In particular, T contains P.*

The following corrollary is a special case of lemma 1 of [2].

Corollary 2.1.2 *Let T be a tournament of order $n+1$, P an outpath of order n with $b_1(P) \geq 2$ and x a vertex of T. If $d^+(x) \geq 2$ and $s^+(x) \geq b_1(P) + 1$ then x is an origin of P in T.*

Proof. Let y be an outgenerator of $T[N^+(x)]$ and z an outneighbour of x distinct from y. In $T - x$, $s^+(y, z) = s^+(x) - 1 \geq b_1(P) = b_1(^*P) + 1$, so by theorem 2.1, y or z is an origin of $^*P$ in $T - x$. Thus, x is an origin of P in T. $\square$

Theorem 2.2 (Havet and Thomassé, [2]) *Let T be a tournament of order $n \geq 8$, and P a path of order n. Then T contains P.*

3 Mergings of paths.

In this section, we study the unavoidability of mergings of paths. A special class of mergings of paths, the class of *claws* (mergings of directed outpaths), has already been investigated (see [8], [3], [4], [5] and [6]). The *degree* of a claw its number of paths. Lu, Wang et Wong [6] proved that claws of order n and degree at most $\frac{19}{50}n$ are n-unavoidable and that there exist claws of order n and degree $\frac{11}{23}n$ that are not n-unavoidable.

The main result of this section is theorem 3.2, which states that a merging of order n of k paths is $(n + \frac{3}{2}(k^2 - 3k) + 5)$-unavoidable. In order to prove this, we first prove lemma 3.3, which allows us restrict our consideration to mergings of outpaths.

3.1 Reduction to mergings of outpaths.

Lemma 3.1 (Havet and Thomassé, [2]) *Let P be an outpath of order n_1, Q an inpath of order n_2, T a tournament of order at least $n_1 + n_2$ and x a vertex of T. If x is an origin of P and an origin of Q in T, then x is an origin of the merging $P \vee Q$ in T.*

We prove here a generalization of this lemma.

Definition 3 Let R be a merging of paths with origin x. $^*R$ denote the digraph $R - x$.

A path $P = (x_1, x_2, \dots, x_n)$ is *quasi-ascending* if $x_1 \leftarrow x_2$ and $x_i \rightarrow x_{i+1}$, $2 \leq i \leq n-1$.

Lemma 3.2 *Let R_1 be a merging of order n_1 of outpaths and R_2 a merging of order n_2 of inpaths and T a tournament of order at least $n_1 + n_2$. If x is an origin of R_1 and an origin of R_2 in T, then x is an origin of $R_1 \vee R_2$ in T.*

Proof. Set $R_1 := \bigvee_{i=1}^{l} P_i$ and $R_2 := \bigvee_{j=1}^{k} Q_j$. By duality, we may suppose that $d^-(x) \geq |R_2|$.

We will prove by induction on k, the number of paths of R_2, that each origin x of R_1 in T such that $d^-(x) \geq |R_2|$ is an origin of $R_1 \vee R_2$.

- If $k = 1$ then set $R_2 = Q$. Let us prove the result by induction on l, the number of paths of R_1.

  If $l = 1$, corollary 3.1 yields the result.

  If $l > 1$, set $q_i := |N^-(x) \cap P_i|$, $q := |N^-(x) \cap T - R_1|$ and $R_1' := \bigvee_{i=1}^{l-1} P_i$. If $q + q_1 + q_2 + \cdots + q_{l-1} \geq |R_2|$, x is an origin of $R_1' \vee Q$ in $T -^* P_l$ by the induction hypothesis. So x is an origin of $R_1 \vee Q$. We may suppose, therefore, that $q_l \neq 0$.

  If $b_1(Q) \geq 2$, let Q' be a directed inpath of length $q + q_1 + q_2 + \cdots + q_{l-1}$. Clearly, x is an origin of Q' in $T -^* P_l$, because its indegree is $q + q_1 + q_2 + \cdots + q_{l-1}$. By the induction hypothesis, in $T -^* P_l$, x is an origin of $R_1' \vee Q'$. Therefore, in $T -^* R_1'$, we have $d^-(x) \geq 2$ (one inneighbour in $^*Q'$ and one in $N^-(x) \cap P_l$) and

$s^-(x) \geq q + q_1 + q_2 + \cdots + q_{l-1} + q_l \geq n_2$. Thus, by corollary 2.1.2, x is an origin of Q in $T -^* R'_1$. And since x is an origin of P_l, by corollary 3.1 x is an origin of $P_l \vee Q$. Hence, in T, x is an origin $R_1 \vee Q$.

If $b_1(Q) = 1$, let Q' be a quasi-ascending path of length $q + q_1 + q_2 + \cdots + q_{l-1}$. By the induction hypothesis, in $T -^* P_l$, x is an origin of $R' \vee Q'$. Therefore, in $T -^* R'_1$, we have $d^-(x) \geq 2$ and $s^+(N^-(x)) \geq q + q_1 + q_2 + \cdots + q_{l-1} + q_l - 1 \geq n_2 - 1$. Thus, by lemma 2.1.2, x is an origin of Q in $T -^* R'_1$. And since x is an origin of P_l, by corollary 3.1 x is an origin of $P_l \vee Q$. Hence, in T, x is an origin $R_1 \vee Q$.

- If $k \geq 2$, set $R'_2 := \bigvee_{j=1}^{k-1} Q_j$. By the previous case, in T, x is an origin of $R_1 \vee Q_k$. In $T -^* Q_k$, x is an origin of R_1 and $d^-(x) \geq |R_2| - |^*Q_k| = |R'_2|$. Thus, by the induction hypothesis, x is an origin of $R_1 \vee R'_2$ in $T -^* Q_k$. Therefore, x is an origin $R_1 \vee R_2$ in T. $\square$.

Using this lemma, we prove the following useful lemma:

Lemma 3.3 *Let R_1 be a merging of order n_1 of outpaths such that every tournament of order $f_1(n_1)$ contains a_1 origins of R_1, and R_2 a merging of order n_2 of inpaths such that every tournament of order $f_2(n_2)$ contains a_2 origins of R_2. Then $R = R_1 \vee R_2$ is m-unavoidable, where $m = \max\{f_1(n_1) + f_2(n_2) - a_1 - a_2 + 1; n_1 + n_2\}$.*

In particular, if R_1 is (n_1+l_1)-unavoidable and R_2 is (n_2+l_2)-unavoidable with $l_1+l_2 \geq 1$, then R is $(n_1 + n_2 + l_1 + l_2 - 1)$-unavoidable.

Proof.

Let T be a tournament of order m. In T, there are $m - f_1(n_1) + a_1$ origins of R_1 and $m - f_2(n_2) + a_2$ origins of R_2. Since $m \geq f_1(n_1) + f_2(n_2) - a_1 - a_2 + 1$, there is a vertex x which is both an origin of R_1 and an origin of R_2. By lemma 3.2, since $m \geq n_1 + n_2$, x is an origin of $R_1 \vee R_2$ in T. $\square$

This lemma allows us to limit our study to mergings of outpaths. Indeed, if we prove that each merging of order n_1 of k_1 outpaths is $(n_1 + l(k_1))$-unavoidable, for some function l, then, by duality, each merging of order n_2 of k_2 inpaths is $(n_1 + l(k_2))$-unavoidable. Thus by lemma 3.3, the merging of order n of k_1 outpaths and k_2 inpaths is $(n + l(k_1) + l(k_2))$-unavoidable.

Lemma 3.3 yields readily the following corollaries:

Corollary 3.3.1 *The merging of order n of two outpaths and one inpath is $(n+1)$-unavoidable.*

Proof. Corollary 2.1.1 states that in every tournament of order $n_1 + 1$ there are two origins of a prescribed path of order n_1. And the merging of order n_2 of two (out)paths is a path and therefore is $(n_2 + 1)$-unavoidable. $\square$

The bound $n+1$ of this lemma is best possible: a reducible tournament of order n in which the maximal strong component is a 3-cycle does not contain the merging $P_1 \vee P_2 \vee P_3$, where P_1 and P_2 are outpaths of length one and P_3 a directed inpath of length $n-3$.

Corollary 3.3.2 *A merging of order $n \geq 14$ of two outpaths and two inpaths is $(n+1)$-unavoidable.*

Proof. Let $R = P_1 \vee P_2 \vee Q_1 \vee Q_2$ with P_1 and P_2 two outpaths and Q_1 and Q_2 two inpaths. Set $R_1 = P_1 \vee P_2$ and $R_2 = Q_1 \vee Q_2$ and let n_1 be the order of R_1 and n_2 the order of R_2. We have $n_1 + n_2 = n + 1$. Since $n \geq 14$ and by duality, we may suppose that $n_1 \geq 8$. Thus by theorem 2.2, R_1 is n_1-unavoidable. And, by corollary 2.1.1, R_2 is (n_2+1)-unavoidable. □

The bound $n+1$ of this lemma is the best possible: a reducible tournament of order n in the which maximal strong component is a 3-cycle does not contain the merging $P_1 \vee P_2 \vee P_3 \vee P_4$, where P_1 and P_2 are outpaths of length one and P_3 and P_4 are directed inpaths of orders n_3 and n_4, respectively, with $n_3 + n_4 = n - 1$.

Definition 4 An *anticlaw of degree d* is the dual of a claw of degree d, that is, the merging of d directed inpaths.

Corollary 3.3.3 *A merging of order n of a claw of order n_1 and degree at most $\frac{19}{50}n_1$ and an anticlaw of order n_2 and degree at most $\frac{19}{50}n_2$ is $(n+1)$-unavoidable.*

Proof. By the theorem of Lu, Wang and Wong [6] noted above, a claw (or anticlaw) of order n_i and degree at most $\frac{19}{50}n_i$ is n_i-unavoidable. □

3.2 Mergings of outpaths.

3.2.1 Mergings of outpaths with first blocks of length at least two.

Definition 5 The *fork* $F = [x, y, \{u, v\}]$ is the digraph defined by $V(F) = \{x, y, u, v\}$ and $E(F) = \{(x, y), (y, u), (y, v)\}$. We call x the *origin* of F and u and v the *points* of F. The *merging* of forks is defined analogously to the merging of paths.

A tree is *quasi-suitable* (resp. *suitable*) if it is a merging of k_1 forks, k_2 directed outpaths of length two and k_3 directed outpaths of length one (resp. with $k_1 \leq k_3$).

In [5], Lu proved the following lemma. An elementary proof was given by Lu, Wang and Wong [6]. We give here an even shorter proof due to J. A. Bondy (personnal communication). We need the following definition:

Definition 6 A *k-matching* is a digraph $D = (A \to B)$ that satisfies the three following properties:

(A, B) is a partition of $V(D)$;

$(x, y) \in E(D)$ only if $x \in A$ and $y \in B$;

for all $a \in A$, $d^+(a) \leq k$ and for all $b \in B$, $d^-(b) = 1$.

Lemma 3.4 (Lu, [5]) *Let T be a tournament. Every vertex with maximum outdegree is origin of a spanning suitable tree.*

Proof. (Bondy) Let x be a vertex with maximal outdegree. It is easy to see that the existence of a spanning suitable tree with origin x is equivalent to the existence of a 2-matching $(N^+(x) \to N^-(x))$ in $T - x$. Now, an immediate consequence of Hall's theorem is the following: Let A and B be two disjoint sets of vertices of a digraph D. Then D contains a k-matching $(A \to B)$ if and only if, for all $Y \subseteq B$, $N^-(Y) \cap A \geq \frac{|Y|}{k}$. We will show by contradiction that this condition is satisfied with $A = N^+(x)$, $B = N^-(x)$ and $k = 2$, yielding the result.

Let $Y \subset N^-(x)$, and set $X := N^-(Y) \cap N^+(x)$ and $Z := N^+(x) \setminus X$. Suppose, by way of contradiction, that $2|X| < |Y|$. Let y be a vertex with maximal outdegree in $T[Y]$. Y dominates $Z \cup \{x\}$ and at least $(|Y| - 1)/2 \geq |X|$ vertices of $T[Y]$. Thus $d^+(y) > d^+(x)$, contradicting the assumption that x is a vertex with maximal outdegree in T. $\square$

Lemma 3.5 *Let $R = \bigvee_{i=1}^k P_i$ be a merging of order n of $k \geq 2$ outpaths with first block of length at least two and T a tournament of order $n + k - 1$. Every vertex which is an origin of a spanning suitable tree in T is an origin of R in T.*

In particular, every vertex with maximal outdegree in T is an origin of R in T.

Proof.

We will prove the result by induction on k. The idea of the proof is to find a subtournament T_1 of T such that $|T_1| = |P_1| + 1$, x is an origin of P_1 in T and, denoting V_1 the set of vertices of $T_1 - x$, the tree $A_2 = A - V_1$ is suitable and spans $T - V_1$. Then, by the induction hypothesis, x is an origin of $\bigvee_{i=2}^k P_i$ in $T - V_1$ so x is an origin of R in T.

Let A be a spanning suitable tree with origin x, that is, the merging of forks $F_1, F_2, \ldots, F_{k_1}$, directed outpaths of length two $R_1, R_2, \ldots, R_{k_2}$ and directed outpath of length one $Q_1, Q_2, \ldots, Q_{k_3}$.

Let T_1 be the subtournament of T defined as follows:

If $3k_1 + 1 \geq |P_1|$, let i be the smallest integer such that $3i \geq |P_1|$.

If $3i + 1 = |P_1| + 1$, $T_1 := T(F_1, F_2, \ldots, F_i)$.

If $3i + 1 = |P_1|$, $T_1 := T(F_1, F_2, \ldots, F_i, Q_{k_3})$.

If $3i + 1 = |P_1| + 2$, $T_1 := T(F_1, F_2, \ldots, F_{i-1}, Q_{k_3}, y)$, with y a point of F_i.

If not, $3k_1 + 1 < |P_1|$.

If $3k_1 + 2k_2 + 1 \geq |P_1|$, let i be the smallest integer such that $3k_1 + 2i + 1 \geq |P_1|$.

If $3k_1 + 2i + 1 = |P_1| + 1$, $T_1 := T(F_1, \ldots, F_{k_3}, R_1, \ldots R_i)$.

If $3k_1 + 2i + 1 = |P_1|$, $T_1 := T(F_1, F_2, \ldots, F_{k_3}, R_1, \ldots R_i, Q_{k_3})$.
If not, $3k_1 + 2k_2 + 1 < |P_1|$.
Let i be the smallest integer such that $3k_1 + 2k_2 + i = |P_1|$.
$T_1 := T(F_1, F_2, \ldots, F_{k_3}, R_1, \ldots R_{k_2}, Q_1, \ldots, Q_i)$.

Now let A' be the tree induced on A by the vertices of $V(T - T_1) \cup \{x\}$. It is easy to check that A' is a spanning tree of order $|\bigvee_{j=2}^{k} P_j| + k - 1$. Moreover, we have $|T_1| \geq |P_1| + 1$ and $s^+_{T_1}(x) \geq |P_1|$. If $|P_1| \geq 4$, $d^+(x) \geq 2$ and by lemma 2.1.2, x is an origin of P_1 in T_1. If $|P_1| = 3$, P_1 is the directed outpath of length two and x (which is an outgenerator of T_1) is an origin of P_1 in T_1.

If $k = 2$, set $T_2 = T -^* P_1$. We have $|T_2| = |P_2| + 1$ and $s^+_{T_2}(x) \geq |P_2|$. If $|P_2| \geq 4$, $d^+_{T_2}(x) \geq 2$ and, by lemma 2.1.2, x is an origin of P_2 in T_2. If $|P_2| = 3$, P_1 is the directed outpath of length two and x is an origin of P_2 in T_2. Thus x is an origin of $P_1 \vee P_2$ in T.

If $k \geq 3$, by the induction hypothesis, x is an origin of $\bigvee_{j=1}^{k-1} P_j$ in $T(A')$. Consequently, x is an origin of R. $\square$

Remark 3.1 Note that if P_i is not directed, it suffices that $s^+_{T_2}(x) \geq |P_2| - 1$. Applying the same method, one can prove that a merging of order n of k outpaths whose first blocks are of length at least two and k_1 of which are directed, is $\max(n+1, n+k_1-1)$-unavoidable.

The above lemma and lemma 3.3 yield the following corollary:

Corollary 3.5.1 *A merging of order n of k paths with first blocks of length at least two is $(n + k - 1)$-unavoidable.*

One can prove the following lemma by the method of proof of lemma 3.5:

Lemma 3.6 *Let $R = \bigvee_{i=1}^{k} P_i$ be a merging of order n of $k \geq 2$ outpaths with first block of length at least two and T a tournament of order $n + 2k - 2$. Every origin of a spanning quasi-suitable tree is an origin of R in T.*

3.2.2 Mergings of outpaths with first blocks of length one.

Definition 7 Let $h(k) := \frac{1}{2}(k^2 + k + 2)$. The function $h(k)$ satisfies the following recursive relation: $h(k) = 2h(k-1) - h(k-2) + 1$ with $h(0) = 1$ and $h(1) = 2$.

Lemma 3.7 *Let T be a tournament of order $n + h(k) - 1$ and $P_1, P_2, \ldots, P_k$, k paths of orders $n_1, n_2, \ldots, n_k$ respectively, such that $\sum_{i=1}^{k} n_i = n + k - 1$ and $b_1(P_i) = 1$, $1 \leq i \leq k$. Let K be a set of $h(k)$ vertices. In K, there is an origin of $R = \bigvee_{i=1}^{k} P_i$ such that $|K \setminus V(R)| \geq h(k-1) - 1$.*

Proof. By induction on k. If $k=1$, theorem 2.1 yields the result.

If $k \geq 2$, set $R' = \bigvee_{i=1}^{k-1} P_i$. We have $R = R' \vee P_k$. Let (A_1, A_2) be a partition of $V(T-K)$ such that $|A_1| = \sum_{i=1}^{k-1}(n_i - 1)$ and $|A_2| = n_k - 1$. Let K_1 be a subset of $h(k-1)$ vertices of K. By the induction hypothesis, there is a vertex x of K_1 that is an origin of R' in $T(K_1, A_1)$, with $K_3 := K_1 \setminus V(R')$ of size at least $h(k-2)-1$. Set $K_2 := K_3 \cup (K \setminus K_1)$. Since $h(k) = 2h(k-1) - h(k-2) + 1$, we have $|K_2| \geq h(k-1)$. Thenrefore, by the induction hypothesis, there is a vertex y of K_2 that is an origin of R' in $T(K_2, A_1)$ with $K_4 := K_2 \setminus V(R')$ of size at least $h(k-2)-1$. Now, by theorem 2.1, in $T(x, y, A_2)$ x or y is an origin of P_k. Without loss of generality, we may suppose that it is x. Hence, in $T(K_1 \setminus K_3, y, A_1, A_2)$, x is an origin of $R' \vee P_k = R$. Moreover $K_2 \setminus \{y\} \subseteq K \setminus V(R)$, so $|K \setminus V(R)| \geq h(k-1) - 1$. □

3.2.3 General case.

Lemma 3.8 *Let R be a merging of order n of $k \geq 2$ outpaths P_i with first block of length one and one outpath Q with first block of length at least two. R is $(n + 2h(k) - k - 1)$-unavoidable.*

Proof. Let T be a tournament of order $n + 2h(k) - k - 1$. Let R_1 be the merging of $h(k)$ outpaths of length one and set $R_2 := R_1 \vee^* Q$. R_1 is $2h(k)$-unavoidable. Since $b_1(Q) \geq 2$, ${}^*Q$ is an outpath and, by corollary 2.1.1, in a tournament of order $|Q|$ there are two origins of ${}^*Q$. Hence, by lemma 3.3, R_2 is $(|{}^*Q| + 2h(k) - 1)$-unavoidable. Since $n + 2h(k) - k - 1 \geq |Q| + 2h(k)$, there is an origin x of R_2 in T. Set $T' := T -^* Q$ and $R' := \bigvee_{i=1}^{k} P_i$, and let $x_1, \ldots, x_{h(k)}$ be the vertices of $R_2 -^* Q$. We have $|T'| \geq |R'| + h(k) - 1$, so by lemma 3.7, one of the x_i, say x_1, is an origin of R' in T'. Thus x_1 is an origin of $R' \vee (x, ^*Q) = R$ in T. □

Proposition 3.1 *If x is an origin of a quasi-suitable tree of order n in T then, for all $y \in V(T) \setminus \{x\}$, x is an origin of a quasi-suitable tree of order $n-3$ in $T-y$.*

Proof. Let A be a suitable tree of order n in T. Let S be $N_A^+(y) \cup \{y\}$. S is of size at most three. Thus $T - S$ is a suitable tree or order $n-3$ in $T-y$ with origin x. □

Lemma 3.9 *Let R be a merging of order n of k_1 outpaths $P_i, 1 \leq i \leq k_1$, with first blocks of length one and $k_2 \geq 2$ outpaths with first blocks of length at least two. R is $(n + 3h(k_1) + 2k_2 - 5)$-unavoidable.*

Proof. Let $R_1 := \bigvee_{i=1}^{k_1} P_i$ and $R_2 := \bigvee_{i=1}^{k_2} Q_i$, and n_1 and n_2 their respective orders. Let T be a tournament of order $n + 3h(k_1) + 2k_2 - 5$, T' a subtournament of T of order $n_2 + 3h(k_1) + 2k_2 - 5$ and T'' the tournament $T - T'$. Let us consider the vertices $x_i, 1 \leq i \leq h(k_1)$, such that x_i is an origin of a spanning quasi-suitable tree A_i in $T - (x_1, \ldots, x_{i-1})$. According to proposition 3.1, x_i is an origin of a quasi-suitable tree of order at least

(n_2+2k_2-2) in $T'-(\bigcup_{j\neq i} x_j)$. Thus, by lemma 3.6, x_i is an origin of R_2 in $T'-(\bigcup_{j\neq i} x_j)$. Moreover, $T_1 = T[V(T''), \bigcup_{i=1}^{h(k_1)} x_i]$ has order $n_1+h(k)-1$. Therefore, by lemma 3.7, one of the x_i, say x_1, is an origin of R_1 in T_1. Hence x_1 is an origin of R in T. □

With the aid of the above four lemmas, one can prove the following theorem for any merging of outpaths:

Theorem 3.1 *A merging of order n of k outpaths is $(n+3h(k-2)-1)$-unavoidable.*

Since $h(k-2) = \frac{1}{2}(k^2-3k)+2$, this theorem and lemma 3.3 yield the following:

Theorem 3.2 *A merging of order n of $k \geq 3$ paths is $(n+\frac{3}{2}(k^2-3k)+5)$-unavoidable. In particular, a tree with three leaves is $(n+5)$-unavoidable.*

4 The method of Häggkvist and Thomason.

Definition 8 Let A_1 be a subtree of a tree A. We denote by $A^+(A_1)$, (resp. $A^-(A_1)$) the forest that is the union of the components of the forest $A-A_1$ which are joined positively (resp. negatively) to A_1, i.e. the set of maximal subtrees A_2 of $A-A_1$ for which there exist $x \in A_1$ and $y \in A_2$ such that x dominates (resp. is dominated by) y in A.

Let p be an integer greater than two. The *p-heart* $h_p(A)$ of a tree A of order n, is the subtree spanned by those edges $e \in E(A)$ for which each of the two components of $A-e$ has order at least n/p. If there is no such edge, $h_p(A)$ will denote the unique vertex of A whose deletion leaves components each of which is of order less than n/p. Note that $h_p(A)$ is connected, so it is indeed a tree. Moreover, $h_p(A)$ has at most $p-1$ leaves. Indeed, the removal from A of an edge of $h_p(A)$ adjacent to a leaf l of $h_p(A)$ leaves a component of order at least n/p whose intersection with $h_p(A)$ is l. Hence, if $h_p(A)$ has k leaves, $h_p(A)$ is of order at most $n-k\lceil n/p\rceil+k$. Note, furthermore, that all components of $A-h_p(A)$ are of order less than n/p, that is, at most $\lceil n/p\rceil-1$.

The approach of Häggkvist and Thomason [1] is to establish an upper bound on $f(n)$ by induction. Their method for finding a tree A in a tournament T is first to find the p-heart $h_p(A)$ of A in the subtournament induced by the vertices of sufficiently large outdegree and indegree. Then, by the induction hypothesis, one finds the components of $A-h_p(A)$ in the remaining subtournament these components being joined in T to the leaves of $h_p(A)$ so as to form A. Since the p-heart has few leaves, our goal, in following this approach, is to find (small) functions $\alpha_k(n)$ such that every tree of order n with k leaves is $\alpha_k(n)$-unavoidable. In order to guarantee that the vertices of the vertices of the tournament which constitute the p-heart have large indegrees and outdegrees, we use the following proposition, whose easy proof is omitted.

Proposition 4.1 *Let T be a tournament. At most $2k-1$ vertices of T have outdegree less than k.*

Suppose that, for $2 \leq k \leq p$, there exists a function $\alpha_k(n)$ such that every tree with k leaves is $\alpha_k(n)$-unavoidable. Suppose also that every tree of order $m < n$ is $\alpha(m)$-unavoidable. We will find conditions on $\alpha(n)$ which guarantee that trees of order n are $\alpha(n)$-unavoidable.

Let A be a tree of order n, $h_p(A)$ its p-heart and k the number of leaves of $h_p(A)$. The order t of $h_p(A)$ is at most $n - k\lceil n/p \rceil + k$.

Let a^+ and a^- be the orders of $A^+(h_p(A))$ and $A^-(h_p(A))$, respectively. By duality, we may suppose that $a^+ \leq a^-$. Since $a^+ + a^- = n - t$, we have $a^+ \leq (n-t)/2$.

Let us show that one can find A in a tournament if one can find $\alpha_k(t)$ vertices v such that

$$d^+(v) \geq t - 1 + a^+ + \alpha(n/p) - n/p \ (1)$$

and

$$d^-(v) \geq t - 1 + a^+ + a^- + \alpha(n/p) - n/p \ (2)$$

In the subtournament induced by the $\alpha_k(t)$ vertices satisfying the above two inequalities, one can find $h_p(A)$. Let $A_1^+, A_2^+, \ldots, A_r^+$ be the components joined positively to $h_p(A)$ and $a_1, a_2, \ldots, a_r$ the vertices of $h_p(A)$ to which they are respectively joined. We will construct the components A_j^+ in succession. We wish to find A_j^+ in the subtournament induced by the outneighbours of a_j which are not in the tree already built: $h_p(A) \cup \bigcup_{1 \leq l < j} A_l^+$. Since $|A_j^+| < n/p < n$, by the induction hypothesis one can find A_j^+ in every tournament of order $\alpha(|A_j^+|)$. To guarantee the existence of a subtree A_j^+ correctly joined to $h_p(A)$, it suffices that a_j have outdegree at least $t - 1 + \sum_{1 \leq l < j} |A_l^+| + \alpha(|A_j^+|)$. Since $\sum_{1 \leq l \leq r} |A_l^+| = a^+$ and $|A_j^+| < n/p$, this follows from (1).

Let $A_1^-, A_2^-, \ldots, A_s^-$ be the components joined negatively to $h_p(A)$ and $b_1, b_2, \ldots, b_s$ the vertices of $h_p(A)$ to which they are respectively joined. We will construct the components A_m^- in succession. We wish to find A_m^- in the subtournament induced by the outneighbours of b_j which are not in the tree already built: $h_p(A) \cup \bigcup_{1 \leq l \leq r} A_l^+ \cup \bigcup_{1 \leq l < m} A_l^-$. Since $|A_m^-| < n/p < n$, by the induction hypothesis, one can find A_m^- in every tournament of order $\alpha(|A_m^-|)$. To guarantee the existence of a subtree A_m^-, correctly joined to $h_p(A)$, it suffices that b_j have indegree at least $t - 1 + a^+ + \sum_{1 \leq l < m} |A_l^-| + \alpha(|A_m^-|)$. Since $\sum_{1 \leq l \leq s} A_l^-| = a^-$ and $|A_m^-| < n/p$, this follows from (2).

According to proposition 4.1, there are $\alpha_k(t)$ vertices with outdegrees satisfying (1) and indegrees satisfying (2) in every tournament of order $\alpha(n)$ such that:

$$\alpha(n) \geq \alpha_k(t) + 4t - 5 + 4a^+ + 2a^- + 4\alpha(n/p) - 4n/p.$$

Since $a^+ + a^- = n - t$ and $a^+ \leq \frac{1}{2}(n-t)$, this inequality holds if

$$\alpha(n) \geq \alpha_k(t) + 4t - 5 + 3(n-t) + 4\alpha(n/p) - 4n/p.$$

And since $t \leq n - k\lceil n/p \rceil + k$, this is satisfied if

$$\alpha(n) - 4\alpha(n/p) \geq \alpha_k(n - k\lceil n/p \rceil + k) + (n - k\lceil n/p \rceil + k) + 3n - 4n/p - 5 \quad (\star_k).$$

Because $h_p(A)$ has at least two and at most $p-1$ leaves, if α satisfies $(\star_k)$ for $2 \leq k \leq p-1$, every tree of order n is $\alpha(n)$-unavoidable. So, in order to find the smallest possible function α, we should aim at finding the smallest possible function α_k.

By proving $\alpha_k(n) \leq 2^k n$, Häggkvist and Thomason [1] obtained the following result:

Theorem 4.1 (Häggkvist and Thomason, [1]) *A tree of order n is $12n$-unavoidable.*

Proof. For the 10-heart of a tree, with $\alpha_k(n) = 2^k n$ and $\alpha(n) = 12n$, all the inequalities $(\star_k)$, $2 \leq k \leq 9$, are satisfied. □

5 Trees with few leaves.

In this section, for small values of k, we find upper bounds $\alpha_k(n)$ on the order of a tournament that contains all trees of order n with k leaves. Using the method of Häggkvist et Thomason, these yield the upper bound of $f(n) \leq \frac{38n}{5} - 6$.

Corollary 2.1.1 states that a tree of order n with two leaves is $(n+1)$-unavoidable (i.e. $\alpha_2(n) = n+1$) and corollary 3.2 that a tree of order n with three leaves is $(n+5)$-unavoidable (i.e. $\alpha_3(n) \leq n+5$).

Lemma 5.1 *A tree of order n with four leaves is $(2n+9)$-unavoidable.*

Proof. Let A be a tree of order n with four leaves. There exists a vertex v of A such that $A - v$ is a forest whose components are paths, together with one tree with at most three leaves. We may suppose that this tree is contained in $A^+(v)$; the proof is dual is the other case. If there is a vertex x such that $d^+(x) \geq |A^+(v)| + 5$ and $d^-(x) \geq |A^-(v)| + 1$, we may find $A^+(v)$ in $N^+(x)$ and $A^-(v)$ in $N^-(x)$, and thus we have A in T. And we can find such a vertex in every tournament of order $2(|A^+(v)|+5) + 2(|A^-(v)|+1) - 1 = 2n+9$. □

Lemma 5.2 *A tree of order n with five leaves is $(2n+15)$-unavoidable.*

Proof. Let A be a tree of order n with five leaves. There exists a vertex v of A such that $A - v$ is a forest whose components consist of one path and trees with at most three leaves. We may suppose that the path is contained in $A^-(v)$; the proof is dual is the other case. If there is a vertex x such that $d^+(x) \geq |A^+(v)| + 5$ and $d^-(x) \geq |A^-(v)| + 4$, we may find $A^+(v)$ in $N^+(x)$ and $A^-(v)$ in $N^-(x)$ and thus we have A in T. And we can find such a vertex in every tournament of order $2(|A^+(v)|+5) + 2(|A^-(v)|+4) - 1 = 2n+15$. □

Lemma 5.3 *A tree of order n with six or seven leaves is* $(4n+8)$*-unavoidable.*

Proof. Let A be a tree of order n with six or seven leaves. There exists a subtree of A, A_1, with at most four leaves such that $A - A_1$ is a forest whose components are all joined in the same way, say positively, to A_1, satisfying in addition one of the following two conditions:

a) $A - A_1$ is of order at least four and its components are trees with at most three leaves.

b) $A - A_1$ is of order at least three and its components are paths.

a) By lemma 5.1, if there exist $2|A_1|+9$ vertices x such that $d^+(x) \geq |A^+(A_1)|+|A_1|+4$, we may find A in T. And we can find that many such vertices in every tournament of order $2|A_1|+9+2(|A^+(A_1)|+|A_1|+4)-1 \leq 4n-2|A^+(A_1)|+16 \leq 4n+8$.

b) By lemma 5.1, if there exist $2|A_1|+9$ vertices x such that $d^+(x) \geq |A^+(A_1)|+|A_1|$, we may find A in T. And we can find that many such vertices in every tournament of order $2|A_1|+9+2(|A^+(A_1)|+|A_1|)-1 \leq 4n-2|A^+(A_1)|+8 \leq 4n+8$. □

Lemma 5.4 *A tree of order n with 8, 9 or 10 leaves is* $5n$*-unavoidable.*

Proof. Let A be a tree of order n with 8, 9 or 10 leaves. Either a) there exists a subtree A_1 of A with at most three leaves such that $A - A_1$ is a forest of order at least 9 whose components are trees with at most three leaves, or b) there is a path A_1 such that $A - A_1$ is a forest whose components are paths.

a) If there exist $|A_1|+5$ vertices of outdegree at least $|A^+(A_1)|+|A_1|+5$ and indegree at least $|A^-(A_1)|+|A^+(A_1)|+|A_1|+4$, we may find A in T. And we can find that many such vertices in every tournament of order $|A_1|+5+2(|A^+(A_1)|+|A_1|+4)+2(|A^-(A_1)|+|A^+(A_1)|+|A_1|+4) \leq 5n+21-|A^+(A_1)|-2A^-(A_1)| \leq 5n$.

b) If there exist $|A_1|+1$ vertices of outdegree at least $|A^+(A_1)|+|A_1|$ and indegree at least $|A^-(A_1)|+|A^+(A_1)|+|A_1|$, we may find A in T. And we can find that many such vertices in every tournament of order $|A_1|+1+2(|A^+(A_1)|+|A_1|)+2(|A^-(A_1)|+|A^+(A_1)|+|A_1|) \leq 5n$.

□

Theorem 5.1 *A tree of order n is* $(\frac{38}{5}n-6)$*-unavoidable.*

Proof. Applying the method of Häggkvist and Thomason to the 9-heart, $(\star_k)$ is verified for all $2 \leq k \leq 8$, with $\alpha(n) = \frac{38}{5}n-6$. □

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