Oriented Hamiltonian Paths in Tournaments.

A proof of Rosenfeld's conjecture.

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We prove that with three exceptions, every tournament of order n contains each oriented path of order n. The exceptions are the antidirected paths in the 3-cycle, in the regular tournament on 5 vertices and in the Paley tournament on 7 vertices.

Tournaments are very rich structures and many questions deal with their subgraphs. In particular, much work has been done concerning oriented paths in tournaments. The first result on this topic, and maybe the very first on tournaments, is Rédei's theorem which asserts that every tournament contains an odd number of directed hamiltonian paths (and thus at least one). Instead of just looking for directed paths, one can seek arbitrary orientations of paths. Such a path may be specified by the signed sequence of the lengths of its blocks, that is, its maximal directed subpaths; the sign of this sequence is + if the first arc of the path is oriented forward and − otherwise. In this vein, Grünbaum proved in [6] that, with three exceptions, every tournament contains an antidirected hamiltonian path (one of type $\pm(1, 1, \cdots, 1)$); the exceptions are the cycle on 3 vertices, the regular tournament on 5 vertices and the Paley tournament on 7 vertices. A year later, in 1972, Rosenfeld [7] gave an easier proof of a stronger result: in a tournament on at least 9 vertices, each vertex is the origin of an antidirected hamiltonian path. He also made the following conjecture: there is an integer $N > 7$ such that every tournament on n vertices, $n \geq N$, contains any orientation of the hamiltonian path. The condition $N > 7$ results from Grünbaum's counterexamples. Several papers gave partial answers to this conjecture: for paths with two blocks (Alspach and Rosenfeld [1], Straight [8]), and for paths having the i^{th} block of length at least $i + 1$ (Alspach and Rosenfeld [1]); also Forcade proved in [5] that there are always an odd number of hamiltonian paths of any type in tournaments with 2^n vertices. Rosenfeld's conjecture was verified by Thomason, who proved in [9] that N exists and is less than 2^{128}. While he did not make any attempt to sharpen this bound, he wrote that $N = 8$ should be the right value.

In this article, we completely settle Rosenfeld's conjecture: any tournament contains each orientation of a hamiltonian path, the sole exceptions being Grünbaum's counterexamples. Our proof is essentially based on brute force applied to a long case analysis. The new idea - the induction hypothesis - is easy to present: just as Thomason did, we introduce our induction hypothesis in the case of paths of order n contained in tournaments of order $n+1$. Thomason [9] proved that every tournament of order $n+1$ contains any orientation of the path of order n. He proved, more precisely, that if P is a path of order n whose first block has k vertices and T is a tournament of order $n+1$, then any k-set of vertices of T contains an origin of P. His proof necessitated checking a number of small cases, and he suggested that an analogous proof for tournaments of order n would require the use of a computer. Our induction is based on 2-subsets instead of k-subsets and, without checking any small tournament, we prove that of any two vertices x and y in an tournament of order $n+1$, there always exists one which is an origin of P provided that the outsection generated by $\{x, y\}$ is greater than the first block of P (here P is an outpath - one with positive sign - and the outsection generated by $\{x, y\}$ is the set of vertices one can reach by directed outpaths with origin x or y). This approach has many interesting corollaries; for example, we give a necessary and sufficient condition for a vertex x to be the origin of a path P of order n in a tournament T of order $n+1$. The problem becomes considerably more messy when we apply the same induction to oriented hamiltonian paths. The main result of this paper is a characterization of all the pairs (T, P) where T is a tournament and P an outpath, both of order n, such that some two vertices in T generate an outsection greater than the first block of P, but neither is an origin of P. Our list comprises 52 small exceptions (of order less than 9) and 14 infinite families. Among these exceptions, we retrieve of course Grünbaum's ones; but in all other cases we show that one can find a vertex which is an origin of P. Conversely, if (T, P) is not an exception, of two vertices whose outsection is the whole tournament, at least one is an origin of P; moreover, there always exist two such vertices. Thus, the theorem of the abstract holds (this is our corollary 4.1). Since we obtain very strong criteria for finding an origin of P among two vertices, we are able to deduce other corollaries. Here are three of them. Let T be a tournament of order $n \geq 9$. If T is 3-strong, every vertex is the origin of any hamiltonian oriented path (corollary 4.2). If T is 2-strong and P is an oriented hamiltonian path, every vertex is the origin of P or $-P$ (corollary 4.3). If the first block of P has length 1 and the second block is short (of length at most $\lceil\frac{n-5}{2}\rceil$), then every vertex of T is an origin of P or $-P$ (corollary 4.5).

1. ORIENTED PATHS OF ORDER N IN TOURNAMENTS OF ORDER $N+1$.

DEFINITION 1.1. - A *tournament* is an orientation of the arcs of a complete graph. Throughout this paper, we always use the letter T for a tournament, and since we consider only one tournament at a time, we will systematically omit the letter T in our notation (for example, in the notation for degrees or neighbourhoods). Similarly, V is always the set of vertices of T. If X is a subset of V, the subtournament of T induced by X is denoted by $T(X)$; for convenience we denote $T(V \setminus X)$ by $T\backslash X$. Let D be a digraph. We say that T *contains* D if D is a subgraph of T. The *order* of a digraph D, denoted by $|D|$, is its number of vertices.

- An *oriented path of order* n is an orientation of the path $P=(x_1,\cdots,x_n)$. Occasionally, we abbreviate 'oriented path' to 'path'. Let $P=(x_1,\cdots,x_n)$ be a path. We say that x_1 is the *origin* of P and x_n is the *terminus* of P. If $x_1 \to x_2$, P is an *outpath*, otherwise P is an *inpath*. The *directed outpath* of order n is the path $P=(x_1,\cdots,x_n)$ in which $x_i \to x_{i+1}$ for all i, $1 \le i < n$; the dual notion is *directed inpath*; both directed out- and inpaths are *dipaths*. The *length* of a path is its number of arcs. We denote the path $(x_1,\cdots,x_{n-1})$ by P^* and the path $(x_2,\cdots,x_n)$ by ${}^*P$. Let P be a path. The *blocks* of P are the maximal subdipaths of P. We enumerate the blocks of P from the origin to the terminus. The first block of P is denoted by $B_1(P)$ and its length by $b_1(P)$. Likewise, the i^{th} block of P is denoted by $B_i(P)$ and its length by $b_i(P)$. The path P is totally described by the signed sequence $sgn(P)(b_1(P),b_2(P),\cdots,b_k(P))$ where k is the number of blocks of P and $sgn(P)=+$ if P is an outpath and $sgn(P)=-$ if P is an inpath.

- Let X be a subset of vertices of T. We denote by I_X (resp. O_X) an arbitrary hamiltonian directed inpath (resp. outpath) of $T(X)$.

- The *dual* (sometimes called the converse) of a tournament T is the tournament $-T$ on the same set of vertices such that $x \to y$ is an arc of $-T$ if and only if $y \to x$ is an arc of T. In the same way, the *dual* of a path P is the path, denoted $-P$, on the same set of vertices such that $x \to y$ is an arc of $-P$ if and only if $y \to x$ is an arc of P.

- Let X and Y be two disjoint subsets of $V(T)$. We simply write $X \to Y$ when $x \to y$ for every pair $(x,y) \in X \times Y$. If $X \to V(T) \setminus X$, the subset X is *minimal in* T and the subset $V(T) \setminus X$ is *maximal in* T. A tournament is *strong* if for any two vertices x and y there exists a directed outpath with origin x and terminus y. A (strong) *component* of T is a strong subtournament of T which is maximal by inclusion. A tournament is $k-strong$, if $|V(T)| \geq k+1$ and $T-X$ is strong for any set X of $k-1$ vertices.

- Let X be a set of vertices of T. The *outsection* generated by X in T is the set of vertices y to which there exists a directed outpath from $x \in X$; we denote this set by $S^+(X)$ (note that $X \subseteq S^+(X)$ since we allow paths of length zero). We abbreviate $S^+(\{x\})$ to $S^+(x)$ and $S^+(\{x,y\})$ to $S^+(x,y)$. The dual notion, the *insection*, is denoted by $S^-(X)$. We also write $s^+(X)$ (resp. $s^-(X)$) for the number of vertices of $S^+(X)$ (resp. $S^-(X)$). If $X \subseteq Y \subseteq V$, we write $S^+_Y(X)$ instead of $S^+_{T(Y)}(X)$. An *outgenerator* of T is a vertex $x \in T$ such that $S^+(x) = T$, the dual notion is an *ingenerator*.

Lemma 1.1. *Let T be a tournament. Then the strong components of a tournament can be ordered in a unique way $T_1, \ldots, T_\alpha$ such that $T_i \rightarrow T_j$ for all $1 \leq i < j \leq \alpha$. Moreover the set of outgenerators of T is $V(T_1)$.*

Throughout the paper, we make wide (and often implicit) use of the above lemma. It follows easily from this result that x is an outgenerator of T if and only if x is an origin of a hamiltonian directed outpath of T. Another easy consequence is that if x is an outgenerator and $\{x\}$ is not minimal then there is a hamiltonian directed outpath P of T such that x is the origin of ${}^*P$.

The following theorem is a strengthening of Thomason's first theorem of [9]. While our proof is not simpler than his, we do not have to check small cases to establish the result. Consequently, we are able to check by hand the small cases arising in the proof concerning tournaments of order n (Theorem 4.1).

Theorem 1.1.
Let T be a tournament of order $n+1$, P an outpath of order n and x, y two distinct vertices of T. If $s^+(x,y) \geq b_1(P)+1$ then x or y is an origin of P in T.

Proof. We proceed by induction on n, the result holding trivially for $n = 1$. Suppose that both theorem 1.1 and the dual form of theorem 1.1 (that is the same statement where P is an inpath and $s^-(x,y) \geq b_1(P)+1$) hold for $|T| = n$, where $n \geq 1$. Let x and y be two vertices of T such that $x \rightarrow y$ and $s^+(x,y) \geq b_1(P)+1$. We distinguish two cases:

Case 1 : $b_1(P) \geq 2$. If $d^+(x) \geq 2$, let $z \in N^+(x)$ be an outgenerator of $T(S^+(x)\backslash\{x\})$ and let $t \in N^+(x)$, $t \neq z$. By definition of z, $s^+_{V\backslash\{x\}}(t,z) = s^+(x) - 1 > b_1(^*P)$. Since $^*P$ is an outpath, by the induction hypothesis, either t or z is an origin of $^*P$ in $T\backslash\{x\}$. Thus x is an origin of P in T.

So we may assume that y is the unique outneighbour of x. Let z be an outgenerator of $T(N^+(y))$. (z exists since $s^+(x,y) \geq 3$). Then $z \rightarrow x$ and z is an outgenerator of $T(S^+(x,y)\backslash\{y\})$. It follows that $s^+_{V\backslash\{y\}}(x,z) = s^+(x,y) - 1$, so by the induction hypothesis, either x or z is an origin of $^*P$ in $T\backslash\{y\}$. Since $d^+_{V\backslash\{y\}}(x) = 0$, this origin is certainly z. We conclude that y is an origin of P in T.

Case 2 : $b_1(P) = 1$. Assume first that $d^+(x) \geq 2$. We denote by X the set $S^-_{V\backslash\{x\}}(N^+(x))$. Consider the partition $(X, Y, \{x\})$ of V where $Y = V\backslash(X \cup \{x\})$. We have $Y \rightarrow x$, $X \rightarrow Y$ and $y \in X$. If $|X| \geq b_2(P)+1$ then x is an origin of P in T; indeed, let $z \in N^+(x)$ be an ingenerator of $T(X)$ and let $u \in N^+(x)$ $u \neq z$. By the induction hypothesis, z or u is an origin of $^*P$ in $T\backslash\{x\}$. Hence x is an origin of P in T. If $|X| \leq b_2(P)$, we have $|Y| > 1$ since $b_2(P) \leq n-2$ and $|X| + |Y| = n$. Let $z \in Y$ be an ingenerator of $T(Y)$. Notice that since $d^+(x) > 1$, $S^-_{V\backslash\{y\}}(z) = V\backslash\{y\}$. Let $u \in Y$ $u \neq z$. By the induction hypothesis, u or z is an origin of $^*P$ in $T\backslash\{y\}$, consequently y is an origin of P in T.

Now assume that $d^+(x) = 1$, thus $N^+(x) = \{y\}$. If $d^+(y) < 2$ then $N^-_{V\backslash\{x\}}(y)$ has at least $n-2$ vertices. By the induction hypothesis one can find $^{**}P$ in $T(N^-_{V\backslash\{x\}}(y))$, thus x is an origin of P in T. If $d^+(y) \geq 2$, denote $S^-_{V\backslash\{y\}}(N^+(y))$ by Y and consider the partition $(X, Y, \{x\}, \{y\})$ of V with $X = V\backslash(Y \cup \{x,y\})$. By definition, $X \rightarrow \{x,y\}$, $Y \rightarrow X \cup \{x\}$. If $|Y| \geq b_2(P)+1$ then y is an origin of P by the previous argument. If $|Y| \leq b_2(P)$, then $b_2(P) \geq d^+(y) \geq 2$. If $|X| \geq 2$, let $z \in X$ be an ingenerator of $T\backslash\{x,y\}$ and let $u \in X$ $u \neq z$. Since $b_2(P) \geq 2$, $^{**}P$ is an inpath and by the induction hypothesis, z or u is an origin of $^{**}P$ in $T\backslash\{x,y\}$. Thus x, (via y) is an origin of P in T. Finally, if $|X| = 1$ then $|Y| = n-2$ and since $n-2 \geq b_2(P) \geq |Y|$ we have $b_2(P) = n-2$. This means that $^*P$ is a directed inpath. Since y is an ingenerator of $T\backslash\{x\}$, x is an origin of P in T. ∎

Corollary 1.1. (Thomason [9]) *Any tournament T of order $n+1$ contains each oriented path P of order n. Moreover, any subset of $b_1(P)+1$ vertices contains an origin of P. In particular, at least two vertices of T are origins of P .*

Corollary 1.2. *Let T be a strong tournament of order $n+1$ and P an outpath of order n. Then:*

(i) every vertex of T except possibly one is an origin of P, and
(ii) if $b_1(P) \geq 2$, every vertex of outdegree at least 2 is an origin of P.

Corollary 1.3. *Let T be a 2-strong tournament of order $n+1$ and P a path of order n. Then every vertex of T is an origin of P.*

Our goal now is to give a necessary and sufficient condition for a vertex to be an origin of a given path P of order n in a tournament of order $n+1$. In essence, the condition says that a vertex x is not an origin if and only if there is not enough room for P to start at x.

Definition 1.2. - A *separation* of a path $P=(x_1,\cdots,x_n)$ is a pair of paths (P_1,P_2) where $P_1=(x_1,\cdots,x_k)$ and $P_2=(x_k,\cdots,x_n)$ for some k, $1\leq k\leq n$; for example $((x_1,x_2),^*P)$ is a separation of P. We note that $|P_1|+|P_2|=|P|+1$ for any separation (P_1,P_2) of P.
- Let x be a vertex of a tournament T of order n and $P=(x_1,\dots,x_k)$ an outpath with $k\leq n$. We say that (x,P) is *blocked in* T if $s^+(x)\leq b_1(P)$ or if $b_1(P)=1$ and $s^-_{V\setminus\{x\}}(N^+(x))\leq b_2(P)$. Let x be a vertex of a tournament T of order n and $P=(x_1,\dots,x_k)$ an inpath with $k\leq n$. We say that (x,P) is *blocked in* T if $s^-(x)\leq b_1(P)$ or if $b_1(P)=1$ and $s^+_{V\setminus\{x\}}(N^-(x))\leq b_2(P)$.
- We say that (x,P) is *forced in* T if for any $k'\leq k$, there is an unique path isomorphic to $(x_1,\cdots,x_{k'})$ in T with origin x. We say that (x,P) is *forced to* y if y is the terminus of the path $(x_1,\cdots,x_k)$ with origin x. Note that if P is an outpath, this means that x has a unique outneighbour z, and then, depending on whether $^*P$ is an in- or an outpath, that z has a unique in- or outneighbour in $T\setminus\{x\}$. This is a very strong restriction on T.

Lemma 1.2. *Let T be a tournament of order $n+1$, P a path of order n and x a vertex of T. The following are equivalent :*
i) The vertex x is not an origin of P in T.
ii) There exists a separation (P_1,P_2) of P such that (x,P_1) is forced to y in T and (y,P_2) is blocked in $T\setminus P_1^*$.

Proof. We proceed by induction on n. We suppose that P is an outpath. If $d^+(x)=1$ we simply apply the induction hypothesis to the unique outneighbour y of x replacing T by $T\setminus\{x\}$, P by $^*P$ and x by y. Suppose that $d^+(x)\geq 2$ and x is not blocked. If $b_1(P)=1$, consider an ingenerator y of $N^+(x)$ and an outgenerator z of $N^+(x)$ distinct of y. Since x is not blocked, $S^-_{T\backslash\{x\}}(z,y) = S^-_{T\backslash\{x\}}(N^+(x))$, so $s^-_{T\backslash\{x\}}(z,y) \geq b_1({}^*P)+1$. Thus by theorem 1.1, y or z is an origin of ${}^*P$ in $T\backslash\{x\}$ and x is an origin of P. A similar argument holds if $b_1(P) \geq 2$. ▌

This easy lemma gives an algorithm in $O(n^3)$ to find paths of order n in tournaments of order $n+1$. Let T be a tournament of order $n+1$ and P a path of order n.

Step 1: $i := 1$, $P := P$ and $T := T$. Pick an outgenerator x of T and an ingenerator y of T distinct of x. (It costs $O(n)$.) By theorem 1.1, x or y is an origin of P in T.

Step 2: Find which of x and y is not blocked. Call it u(i). (It costs $O(n^2)$.)

Step 3: If P is an outpath then pick an outgenerator x' of $N^+_T(u(i))$ and an ingenerator y' of $N^+_T(u(i))$. (It costs $O(n)$.) By the proof of lemma 1.2, x' or y' is an origin of ${}^*P$ in $T\backslash\{u(i)\}$.

If P is an inpath then pick an outgenerator x' of $N^-_T(u(i))$ and an ingenerator y' of $N^-_T(u(i))$. (It costs $O(n)$.) By the proof of lemma 1.2, x' or y' is an origin of ${}^*P$ in $T\backslash\{u(i)\}$.

Step 4: $i := i+1$; $P :={}^* P$; $T := T\backslash\{u(i)\}$; $x := x'$ and $y := y'$. Goto Step 2.

Lemma 1.3.

Let P be an outpath of order n_1 and Q be an inpath of order n_2. Let x be a vertex of T, a tournament of order n_1+n_2. If x is an origin of P and Q in T then x is the origin of paths P and Q in T with $V(P)\cap V(Q) = \{x\}$.

Proof. Let P be an outpath and Q be an inpath of T, both with origin x. Since $d^+(x)+d^-(x) = |P|+|Q|-1$, $d^+(x) \geq |P|$ or $d^-(x) \geq |Q|$. By duality, we may suppose that $d^+(x) \geq |P|$. Clearly $d^-(x) \neq 0$, since x is the origin of the inpath Q. If $d^-(x) = 1$, we consider $T_1 = T \setminus^* Q$. Note that x is a minimal vertex in T_1. Since $|T_1\backslash\{x\}| = |P|$, we can apply Theorem 1.1 in order to find a copy of ${}^*P$ in $T_1\backslash\{x\}$. Thus x is an origin of a copy of P in T_1, disjoint from Q apart in x. So we may assume that $d^-(x) \geq 2$. If $b_1(Q) \geq 2$, since x is origin of Q, we know that $s^-(x) \geq b_1(Q)+1$. Consider a subset X such that $X \supseteq N^-(x)\cup\{x\}$, $s^-_X(x) \geq b_1(Q)+1$ and $|X| = |Q|+1$. By Lemma 1.2, x is an origin of a copy Q' of Q in $T(X)$. In $T\backslash Q'$ there are $|P|$ vertices, so by Corollary 1.1 we may find two distinct origins of ${}^*P$. Since x dominates every vertex of $T\backslash Q'$ except possibly one, we can find a copy of P with origin x which is disjoint from ${}^*Q'$. If $b_1(Q) = 1$, consider a subset X such that $X \supseteq N^-(x)\cup\{x\}$, $s^-_X(N^-(x)) \geq b_2(Q)+1$ and $|X| = |Q|+1$. By Lemma 1.2, x is an origin of a copy Q' of Q in X. In $T\backslash Q'$ there are $|P|$ vertices, so we may find two distinct origins of a copy of $^*P$. Since x dominates every vertex of $T\backslash Q'$ except possibly one, we can find a copy of P with origin x which is disjoint from $^*Q'$. ∎

2. THE FINITE EXCEPTIONS.

Now we would like to prove an analogue of Theorem 1.1 for paths of order n in tournaments of order n instead of paths of order n in tournaments of order $n+1$. Unfortunately, such a statement admits several exceptions. The next two sections are devoted to the enumeration of these.

Definition 2.1.

- An *exception* is a pair $(T;P)$, where T is a tournament, P an outpath with $|T|=|P|$, such that there exist two vertices x,y of T, neither of which is an origin of P but for which $s^+(x,y)\geq b_1(P)+1$.
- *Grünbaum's exceptions* are the three pairs $(3A;(1,1))$, $(5A;(1,1,1,1))$ and $(7A;(1,1,1,1))$ where $3A$, $5A$ and $7A$ are the tournaments depicted in the following pictures.

We give now a list of exceptions. We call them (together with their duals) the *finite exceptions*. First we give some pictures of labelled tournaments involved in the list. We then enumerate these 52 finite exceptions. Our notation of an exception is the following: we write $[T;P;S;P_1;\dots;P_k]$ where T is a tournament, P is an outpath, S is the set of vertices of T which are not origin of P and the paths $P_1,\dots,P_k$ are the paths of T whose origins are precisely the vertices of $V\backslash S$.

Exc 0 : $[3A;(1,1);\{1,2,3\}]$.
Exc 1 : $[4A;(1,1,1);\{1,2,3\};4213]$.
Exc 2 : $[4A;(1,2);\{3,4\};1324;2314]$.
Exc 3 : $[4A;(2,1);\{1,2\};3421;4132]$.
Exc 4 : $[5A;(1,1,1,1);\{1,2,3,4,5\}]$.
Exc 5 : $[5B;(2,1,1);\{1,2,3\};45213;51423]$.
Exc 6 : $[5C;(1,1,2);\{4,5\};12534;23514;31524]$.
Exc 7 : $[5C;(2,1,1);\{1,2,3,4\};51423]$.
Exc 8 : $[5D;(1,1,1,1);\{2,5\};12543;35124;42153]$.
Exc 9 : $[5E;(1,1,1,1);\{2,4,5\};12453;35421]$.
Exc 10 : $[5E;(1,2,1);\{3,5\};12435;23145;45312]$.
Exc 11 : $[5E;(2,2);\{1,2\};34215;42315;52314]$.
Exc 12 : $[5E;(1,1,2);\{1,2\};35412;41523;51423]$.
Exc 13 : $[6A;(3,1,1);\{3,4\};156324;256143;562341;612345]$.
Exc 14 : $[6B;(2,1,1,1);\{3,4\};154326;254316;562143;612345]$.
Exc 15 : $[6C;(1,1,2,1);\{1,2,3,6\};435261;534261]$.
Exc 16 : $[6C;(1,2,1,1);\{4,5,6\};163425;263415;362415]$.

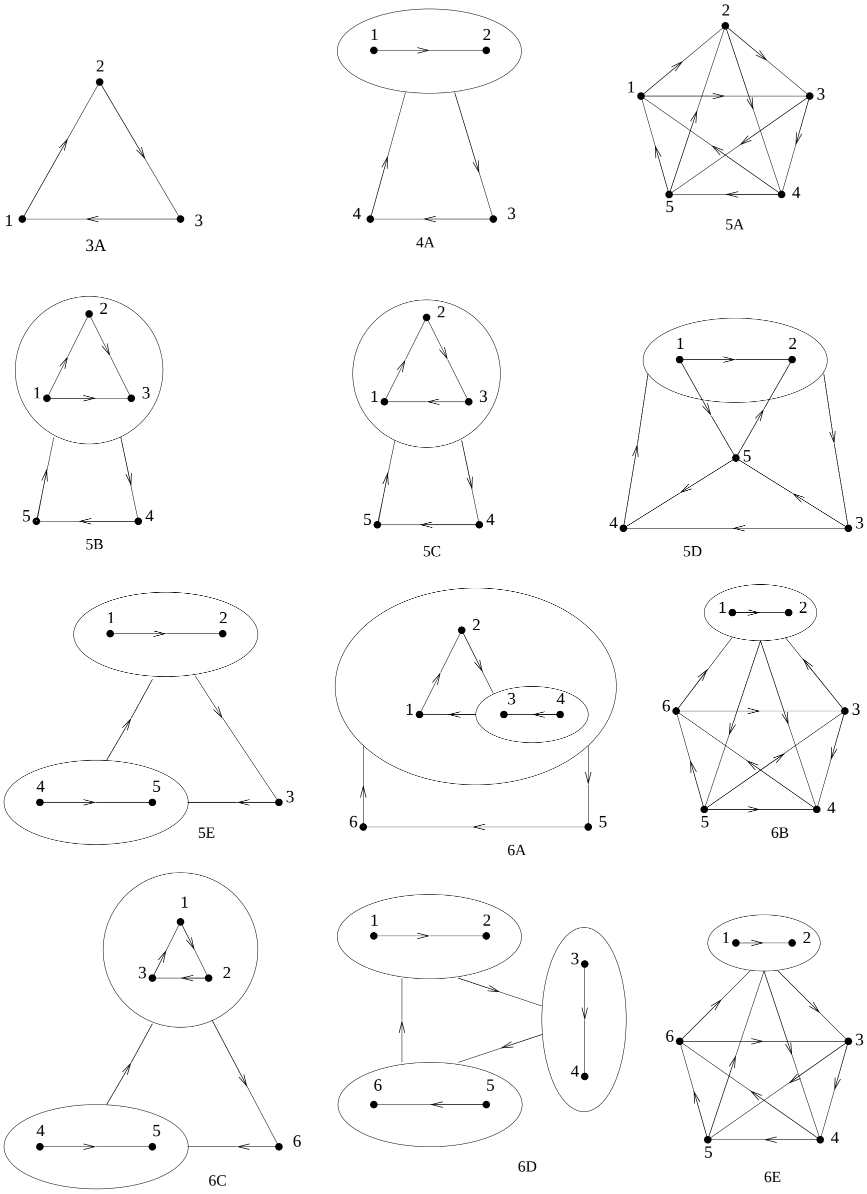

FIG. 1. Finite exceptions

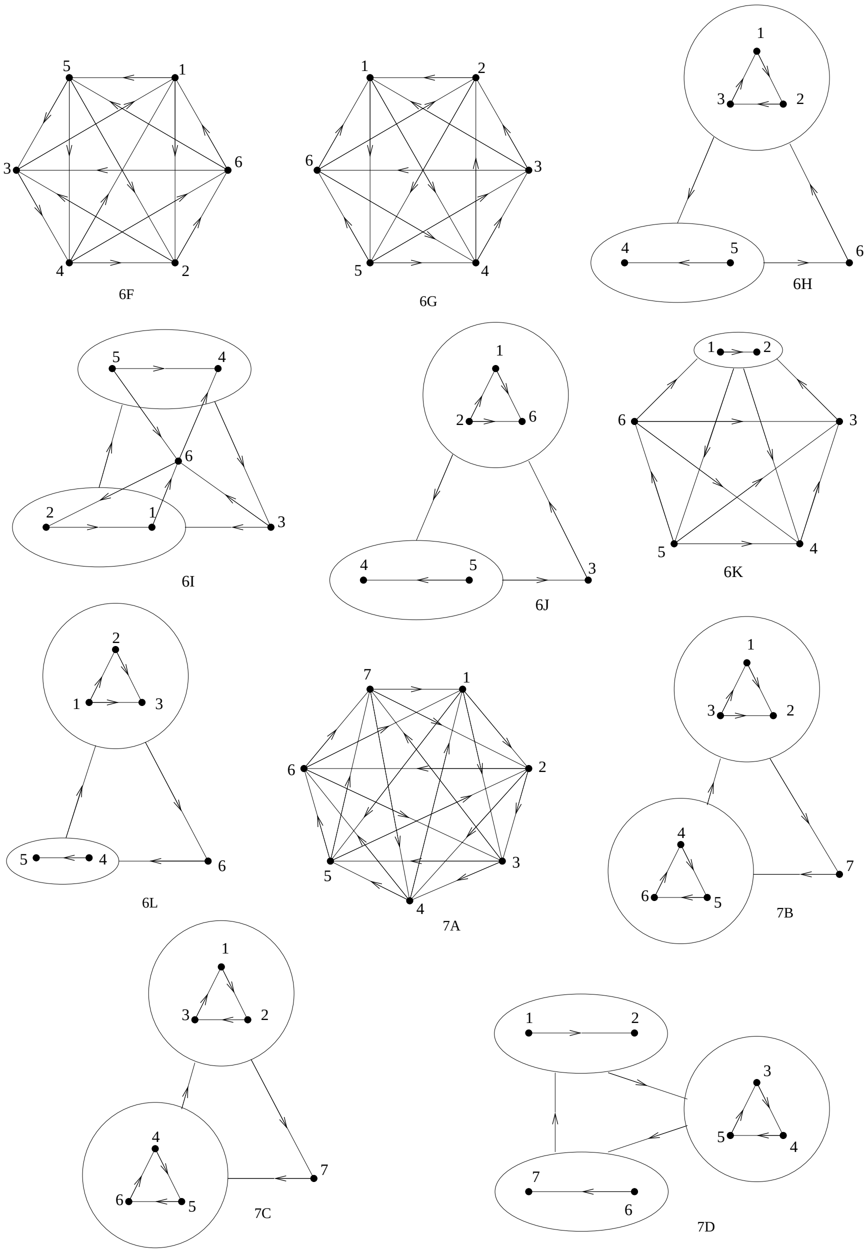

FIG. 2. Finite exceptions

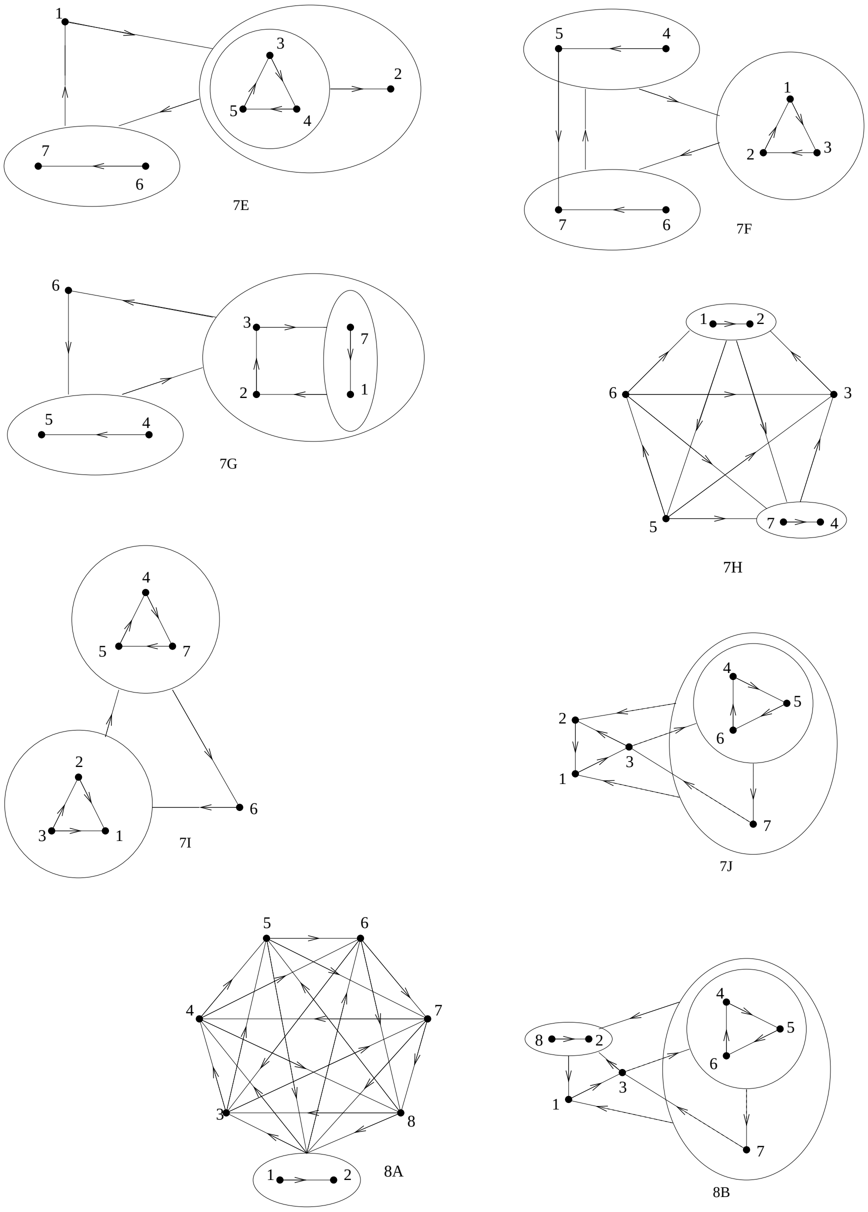

FIG. 3. Finite exceptions

Exc 17 : $[6D; (2,1,1,1); \{2,4,6\}; 124365; 346521; 562143]$.
Exc 18 : $[6D; (1,2,2); \{2,4,6\}; 126345; 341562; 564123]$.
Exc 19 : $[6D; (1,1,1,2); \{2,4,6\}; 125643; 341265; 563421]$.
Exc 20 : $[6E; (1,1,1,1,1); \{1,2\}; 341256; 465213; 516324; 621435]$.
Exc 21 : $[6E; (2,1,1,1); \{1,2\}; 346521; 452136; 562143; 634125]$.
Exc 22 : $[6F; (1,1,1,1,1); \{1,2,3\}; 421563; 532641; 613452]$.
Exc 23 : $[6G; (1,1,1,1,1); \{4,6\}; 145632; 216453; 326415; 546132]$.
Exc 24 : $[6H; (1,1,1,1,1); \{1,2,3,4\}; 543162; 613425]$.
Exc 25 : $[6H; (1,1,1,2); \{4,5\}; 142536; 243516; 341526; 613452]$.
Exc 26 : $[6H; (1,1,3); \{4,5,6\}; 145623; 245631; 345612]$.
Exc 27 : $[6H; (1,3,1); \{4,6\}; 126534; 236514; 316524; 543261]$.
Exc 28 : $[6H; (2,1,2); \{4,5\}; 124563; 234561; 314562; 614235]$.
Exc 29 : $[6I; (1,1,1,1,1); \{4,6\}; 145632; 213654; 365421; 546231]$.
Exc 30 : $[6J; (1,1,1,1,1); \{4,6\}; 162453; 261453; 312465; 542631]$.
Exc 31 : $[6K; (1,2,2); \{3,4\}; 146532; 246531; 541632; 634125]$.
Exc 32 : $[6L; (1,2,1,1); \{5,6\}; 163425; 263415; 361425; 456132]$.
Exc 33 : $[7A; (1,1,1,1,1,1); \{1,2,3,4,5,6,7\}]$.
Exc 34 : $[7B; (1,1,2,1,1); \{1,2,3\}; 4576132; 5674132; 6475132; 7541263]$.
Exc 35 : $[7B; (2,1,3); \{1,2,3\}; 4315627; 5316427; 6314527; 7435612]$.
Exc 36 : $[7B; (2,3,1); \{1,2\}; 3125476; 4567132; 5647132; 6457132; 7421356]$.
Exc 37 : $[7C; (1,1,1,1,1,1); \{4,5,6\}; 1243567; 2341567; 3142567; 7541632]$.
Exc 38 : $[7C; (1,1,2,1,1); \{1,2,3\}; 4156327; 5164327; 6145327; 7541263]$.
Exc 39 : $[7C; (2,1,3); \{1,2,3\}; 4315627; 5316427; 6314527; 7435612]$.
Exc 40 : $[7D; (1,1,1,2,1); \{1,2\}; 3412756; 4512736; 5312746; 6215437; 7215436]$.
Exc 41 : $[7D; (1,1,1,3); \{6,7\}; 1546327; 2546317; 3745216; 4753216; 5734216]$.
Exc 42 : $[7D; (2,2,1,1); \{6,7\}; 1342675; 2341675; 3465127; 4563127; 5364127]$.
Exc 43 : $[7E; (1,1,2,1,1); \{2,7\}; 1236745; 3214756; 4213756; 5213746; 6734215]$.
Exc 44 : $[7F; (1,1,1,3); \{6,7\}; 1732546; 2713546; 3721546; 4127635; 5127634]$.
Exc 45 : $[7G; (2,1,2,1); \{1,7\}; 2654317; 3654721; 4367125; 5367124; 6517234]$.
Exc 46 : $[7H; (2,2,2); \{4,7\}; 1746532; 2746531; 3126574; 5321674; 6247531]$.
Exc 47 : $[7I; (1,1,2,1,1); \{4,5,7\}; 1456237; 2456137; 3456127; 6135427]$.
Exc 48 : $[7J; (1,1,2,1,1); \{1,2\}; 3245167; 4235167; 5234167; 6234157; 7234156]$.
Exc 49 : $[8A; (1,1,1,1,1,1,1); \{1,2\}; 35461278; 46527183; 56487213; 67341285; 74358216; 85347216]$.
Exc 50 : $[8A; (2,1,1,1,1,1); \{1,2\}; 34652718; 46752138; 56734128; 68214375; 78216453; 83412576]$.

Exc 51 : $[8B; (2,1,2,1,1); \{2,8\}; 13245867; 32145867; 42156873; 52164873; 62145873; 73245861]$.

LEMMA 2.1. *Let T be a tournament of order n, x a vertex of T and P an outpath of order n such that $(T\backslash\{x\},^* P)$ is a finite exception and $^*P$ is an outpath. If $d^+(x) \geq 2$ and x is not an origin of P in T then exactly one of the following holds:*
i) The pair (T, P) is a finite exception.
ii) Every outneighbour of x is an origin of P.

Proof. We enumerate the finite exceptions and extend them in all possible ways by a vertex x of outdegree at least 2. To shorten the proof, we use the following notations: assume that $(T\backslash\{x\},^* P)$ is the exception j. Two cases may arise. i) $[S; \text{Exc k}; (x_1 \dots x_{n+1})]$ means that the outneighbourhood of x in T is S and (T, P) is the exception k. The isomorphism between (T, P) and Exc k is given by $(x_1 \dots x_{n+1})$: $x_1 \dots x_n$ are the images of $1 \dots n$ and x_{n+1} is the image of x. ii) $[S; P_1; \dots; P_k]$ means that the outneighbourhood of x in T is S, and that any vertex of S is an origin of P in T. We list those $|S| = k$ different paths.
Exc 0: $(2,1)$ $[\{1,2,3\}; 123x; 231x; 312x]$ $[\{1,2\}; \text{Exc } 3; (2341)]$ The other possible neighbourhoods of x are equivalent.
Exc 1: $(2,1,1)$ $[\{1,2\}; 13x42; 23x41]$ $[\{1,3\}; \text{Exc } 7; (12453)]$ $[\{2,3\}; \text{Exc } 5; (13452)]$ $[\{1,2,3\}; \text{Exc } 5; (23451)]$.
Exc 2: $(2,2)$ $[\{3,4\}; \text{Exc } 11; (45231)]$.
Exc 3: $(3,1)$ $[\{1,2\}; 1342x; 2341x]$.
Exc 4: $(2,1,1,1)$ $[\{1,2\}; 1243x5; 23x514]$ $[\{1,3\}; 1342x5; 3514x2]$ $[\{1,2,3\}; \text{Exc } 21; (234561)]$ $[\{1,2,4\}; 134x25; 23541x; 4125x3]$ $[\{1,2,3,4\}; 134x25; 23541x; 352x14; 451x32]$ $[\{1,2,3,4,5\}; 134x25; 245x31; 351x42; 412x53; 523x14]$. The other possible neighbourhoods of x are equivalent.
Exc 5: $(3,1,1)$ $[\{1,2,3\}; 1452x3; 2451x3; 3451x2]$ $[\{1,2\}; 14x253; 24x153]$ $[\{1,3\}; 14x352; 34x152]$ $[\{2,3\}; 24x351; 34x251]$.
Exc 6: $(2,1,2)$ $[\{4,5\}; \text{Exc } 28; (123465)]$.
Exc 7: $(3,1,1)$ $[\{1,2,3,4\}; 1453x2; 2453x1; 3452x1; 4512x3]$ $[\{1,2,4\}; \text{Exc } 13; (312564)]$ $[\{1,4\}; 14532x; 452x31]$ $[\{1,2\}; 14x253; 24x153]$ $[\{1,2,3\}; 14x352; 24x351; 34x251]$. The other possible neighbourhoods of x are equivalent.
Exc 8: $(2,1,1,1)$ $[\{2,5\}; 2351x4; 5421x3]$.
Exc 9: $(2,1,1,1)$ $[\{2,4,5\}; \text{Exc } 14; (546123)]$ $[\{4,5\}; \text{Exc } 17; (123564)]$ $[\{2,4\}; 23x514; 4125x3]$ $[\{2,5\}; 23x415; 5124x3]$
Exc 10: $(2,2,1)$ $[\{3,5\}; 345x12; 523x41]$.
Exc 11: $(3,2)$ $[\{1,2\}; 13x254; 23x154]$.
Exc 12: $(2,1,2)$ $[\{1,2\}; 13x452; 23x451]$.
Exc 13: $(4,1,1)$ $[\{3,4\}; 3125x64; 4125x63]$.

Exc 14: $(3,1,1,1)$ $[\{3,4\}; 346x152; 46321x5]$.
Exc 15: $(2,1,2,1)$ $[\{1,2,3,6\}; 1654x32; 2654x13; 3654x21; 6524x13]$ $[\{1,2,6\}; \text{Exc } 45; (1234567)]$ $[\{1,2,3\}; 12345x6; 23145x6; 31245x6]$ $[\{1,2\}; 12345x6; 23145x6]$ $[\{1,6\}; 16542x3; 6524x13]$. The other possible neighbourhoods of x are equivalent.
Exc 16: $(2,2,1,1)$ $[\{4,5\}; \text{Exc } 42; (3451267)]$ $[\{4,6\}; 45x2163; 65x2413]$ $[\{5,6\}; 516x234; 64x1532]$ $[\{4,5,6\}; \text{Exc } 42; (3451276)]$.
Exc 17: $(3,1,1,1)$ $[\{2,4,6\}; 24563x1; 46125x3; 62341x5]$ $[\{2,4\}; 24563x1; 46125x3]$. The other possible neighbourhoods of x are equivalent.
Exc 18: $(2,2,2)$ $[\{2,4,6\}; 234x561; 456x123; 612x345]$ $[\{2,4\}; 234x561; 45631x2]$. The other possible neighbourhoods of x are equivalent.
Exc 19: $(2,1,1,2)$ $[\{2,4,6\}; 23x5641; 45x1263; 61x3425]$ $[\{2,4\}; 23x5641; 45x1263]$. The other possible neighbourhoods of x are equivalent.
Exc 20: $(2,1,1,1,1)$ $[\{1,2\}; 13x4652; 23x4651]$.
Exc 21: $(3,1,1,1)$ $[\{1,2\}; 13526x4; 23516x4]$.
Exc 22: $(2,1,1,1,1)$ $[\{1,2\}; 1236x54; 2314x65]$ $[\{1,2,3\}; 1236x54; 2314x65; 3125x46]$. The other possible neighbourhoods of x are equivalent.
Exc 23: $(2,1,1,1,1)$ $[\{4,6\}; 4213x56; 6145x32]$.
Exc 24: $(2,1,1,1,1)$ $[\{1,2,3,4\}; 1245x63; 2345x61; 3145x62; 461x325]$ $[\{1,2,4\}; 1245x63; 2345x61; 46x3512]$ $[\{1,4\}; 1245x63; 46x3512]$ $[\{1,2\}; 1245x63; 2345x61]$ $[\{1,2,3\}; 1245x63; 2345x61; 3145x62]$.
Exc 25: $(2,1,1,2)$ $[\{4,5\}; 46135x2; 56134x2]$.
Exc 26: $(2,1,3)$ $[\{4,5,6\}; \text{Exc } 35; (4562173)]$ $[\{4,5\}; 46x1235; 56x1234]$ $[\{4,6\}; \text{Exc } 35; (4562371)]$ $[\{5,6\}; \text{Exc } 39; (4561372)]$.
Exc 27: $(2,3,1)$ $[\{4,6\}; \text{Exc } 36; (4562371)]$.
Exc 28: $(3,1,2)$ $[\{4,5\}; 46x5123; 56x4123]$.
Exc 29: $(2,1,1,1,1)$ $[\{4,6\}; 4312x56; 6213x54]$.
Exc 30: $(2,1,1,1,1)$ $[\{4,6\}; 43x1625; 6435x21]$.
Exc 31: $(2,2,2)$ $[\{3,4\}; \text{Exc } 46; (1234567)]$.
Exc 32: $(2,2,1,1)$ $[\{5,6\}; 536x421; 645x132]$.
Exc 33: $(2,1,1,1,1,1)$. The Paley tournament is arc transitive, so without loss of generality we may suppose that $\{1,2\} \subseteq N^+(x)$. Let us prove first that 2 is an origin of $(2,1,1,1,1,1)$. If $x \to 5$ then $P = 245x1673$, so we assume that $5 \to x$. If $6 \to x$ then $P = 234716x5$, and conversely if $x \to 6$ then $P = 241x6573$. Thus 2 is an origin of P in any case. Now we determine when 1 is not an origin of P: if $x \to 4$ we have $134x2576$, so $4 \to x$. If $5 \to x$ we have $123675x4$, so $x \to 5$. If $x \to 6$ we have $126x5347$, so $6 \to x$. If $3 \to x$ we have $1243x675$, so $x \to 3$. If $x \to 7$ we have $123x7564$, so $7 \to x$. Finally this extension is isomorphic to Exc 50, the isomorphism being (23456781).
Exc 34: $(2,1,2,1,1)$ $[\{1,2,3\}; 17x45362; 27x45163; 37x45162]$ $[\{1,2\}; 17x45362; 27x45163]$ $[\{1,3\}; 17x45362; 37x45162]$ $[\{2,3\}; 27x45163; 37x45162]$.

Exc 35: $(3,1,3)$ $[\{1,2,3\}; 175x6432; 275x6431; 375x6412]$ $[\{1,2\}; 175x6432; 275x6431]$ $[\{1,3\}; 175x6432; 375x6412]$ $[\{2,3\}; 275x6431; 375x6412]$.
Exc 36: $(3,3,1)$ $[\{1,2\}; 1742365x; 2741365x]$.
Exc 37: $(2,1,1,1,1,1)$ $[\{4,5,6\}; 451632x7; 561432x7; 641532x7]$ $[\{4,5\}; 451632x7; 561432x7]$. The other possible neighbourhoods of x are equivalent.
Exc 38: $(2,1,2,1,1)$ $[\{1,2,3\}; 17x45362; 27x45163; 37x45162]$ $[\{1,2\}; 17x45362; 27x45163]$ $[\{1,3\}; 17x45362; 37x45162]$ $[\{2,3\}; 27x45163; 37x45162]$.
Exc 39: $(3,1,3)$ $[\{1,2,3\}; 175x6423; 275x6431; 375x6412]$ $[\{1,2\}; 175x6423; 275x6431]$ $[\{1,3\}; 175x6423; 375x6412]$ $[\{2,3\}; 275x6431; 375x6412]$.
Exc 40: $(2,1,1,2,1)$ $[\{1,2\}; 13x45267; 23x45167]$.
Exc 41: $(2,1,1,3)$ $[\{6,7\}; 61x34275; 71x34265]$.
Exc 42: $(3,2,1,1)$ $[\{6,7\}; 613754x2; 713654x2]$.
Exc 43: $(2,1,2,1,1)$ $[\{2,7\}; 26x13745; 71x34652]$.
Exc 44: $(2,1,1,3)$ $[\{6,7\}; 64127x35; 74126x35]$.
Exc 45: $(3,1,2,1)$ $[\{1,7\}; 165x2374; 765x2314]$.
Exc 46: $(3,2,2)$ $[\{4,7\}; 431265x7; 731265x4]$.
Exc 47: $(2,1,2,1,1)$ $[\{4,5\}; 46x32517; 56x32417]$ $[\{4,5,7\}; 46x32517; 56x32417; 76x32415]$. The other possible neighbourhoods of x are equivalent.
Exc 48: $(2,1,2,1,1)$ $[\{1,2\}; \text{Exc } 51; (12345678)]$.
Exc 49: $(2,1,1,1,1,1,1)$ $[\{1,2\}; 134265x78; 234165x78]$.
Exc 50: $(3,1,1,1,1,1)$ $[\{1,2\}; 1345827x6; 2345817x6]$.
Exc 51: $(3,1,2,1,1)$ $[\{2,8\}; 21x834756; 81x234756]$. ∎

LEMMA 2.2. *Let T be a tournament of order n, x a vertex of T and P an inpath of order n such that $(T\setminus\{x\}, {}^*P)$ is a finite exception and ${}^*P$ is an outpath. If $d^-(x) \geq 2$ and x is not an origin of P in T then exactly one of the following holds:*
i) The pair (T,P) is a finite exception.
ii) Every inneighbour of x is an origin of P.

Proof. We enumerate the different finite exceptions and extend them in all possible ways by a vertex x of indegree at least 2. The notations are the same as the previous proof, except that when we refer to an exception for the isomorphism, we implicitly mean its dual.
Exc 0: $-(1,1,1)$ $[\{1,2,3\}; 13x2; 21x3; 32x1]$ $[\{1,2\}; \text{Exc } 1; (3241)]$ The other possible neighbourhoods of x are equivalent.
Exc 1: $-(1,1,1,1)$ $[\{1,2\}; \text{Exc } 9; (21435)]$ $[\{1,3\}; \text{Exc } 8; (21435)]$ $[\{2,3\}; \text{Exc } 4; (15423)]$ $[\{1,2,3\}; \text{Exc } 9; (21534)]$.

Exc 2: $-(1,1,2)$ $[\{3,4\}; \text{Exc } 12; (54321)]$.
Exc 3: $-(1,2,1)$ $[\{1,2\}; 1423x; 2413x]$.
Exc 4: $-(1,1,1,1,1)$ $[\{1,2\}; 145x32; 213x54]$ $[\{1,3\}; 152x43; 312x54]$ $[\{1,2,3\}; \text{Exc } 20; (432651)]$ $[\{1,2,4\}; \text{Exc } 23; (123456)]$ $[\{1,2,3,4\}; 1453x2; 2514x3; 32x415; 43x125]$ $[\{1,2,3,4,5\}; 1453x2; 2514x3; 3125x4; 4231x5; 5342x1]$. The other possible neighbourhoods of x are equivalent.
Exc 5: $-(1,2,1,1)$ $[\{1,2,3\}; 15342x; 25341x; 35241x]$ $[\{1,2\}; 15342x; 25341x]$ $[\{1,3\}; 15243x; 35241x]$ $[\{2,3\}; 25143x; 35142x]$.
Exc 6: $-(1,1,1,2)$ $[\{4,5\}; \text{Exc } 25; (132645)]$.
Exc 7: $-(1,2,1,1)$ $[\{1,2,3,4\}; \text{Exc } 16; (132564)]$ $[\{1,2\}; 15342x; 25341x]$ $[\{1,2,4\}; 15342x; 25341x; 4123x5]$ $[\{1,4\}; 1345x2; 4123x5]$ $[\{1,2,3\}; 15342x; 25341x; 35241x]$. The other possible neighbourhoods of x are equivalent.
Exc 8: $-(1,1,1,1,1)$ $[\{2,5\}; \text{Exc } 22; (164532)]$.
Exc 9: $-(1,1,1,1,1)$ $[\{2,4,5\}; \text{Exc } 24; (136452)]$ $[\{4,5\}; \text{Exc } 30; (123456)]$ $[\{2,4\}; \text{Exc } 29; (123456)]$ $[\{2,5\}; 251x43; 54213x]$.
Exc 10: $-(1,1,2,1)$ $[\{3,5\}; \text{Exc } 15; (546132)]$.
Exc 11: $-(1,2,2)$ $[\{1,2\}; \text{Exc } 18; (215436)]$.
Exc 12: $-(1,1,1,2)$ $[\{1,2\}; \text{Exc } 19; (215436)]$.
Exc 13: $-(1,3,1,1)$ $[\{3,4\}; 361254x; 461253x]$.
Exc 14: $-(1,2,1,1,1)$ $[\{3,4\}]; 362415x; 43x5126]$.
Exc 15: $-(1,1,1,2,1)$ $[\{1,2,3,6\}; \text{Exc } 40; (3547621)]$ $[\{1,2,6\}; 145x362; 245x361; 623x415]$ $[\{1,2,3\}; 143526x; 243516x; 342516x]$ $[\{1,2\}; 143526x; 243516x]$ $[\{1,6\}; 145x362; 623x415]$. The other possible neighbourhoods of x are equivalent.
Exc 16: $-(1,1,2,1,1)$ $[\{4,5,6\}; \text{Exc } 34; (6542173)]$ $[\{4,5\}; \text{Exc } 43; (5437612)]$ $[\{4,6\}; \text{Exc } 34; (6542371)]$ $[\{5,6\}; \text{Exc } 38; (6543271)]$.
Exc 17: $-(1,2,1,1,1)$ $[\{2,4,6\}; 261435x; 423651x; 645213x]$ $[\{2,4\}; 261435x; 423651x]$. The other possible neighbourhoods of x are equivalent.
Exc 18: $-(1,1,2,2)$ $[\{2,4,6\}; 213x456; 435x612; 651x234]$ $[\{2,4\}; 213x456; 4126x35]$. The other possible neighbourhoods of x are equivalent.
Exc 19: $-(1,1,1,1,2)$ $[\{2,4,6\}; 214365x; 436521x; 652143x]$ $[\{2,4\}; 214365x; 436521x]$. The other possible neighbourhoods of x are equivalent.
Exc 20: $-(1,1,1,1,1,1)$ $[\{1,2\}; 156x423; 256x413]$.
Exc 21: $-(1,2,1,1,1)$ $[\{1,2\}; 1623x54; 2613x54]$.
Exc 22: $-(1,1,1,1,1,1)$ $[\{1,2,3\}; \text{Exc } 33; (1732456)]$ $[\{1,2\}; 14253x6; 254x361]$. The other possible neighbourhoods of x are equivalent.
Exc 23: $-(1,1,1,1,1,1)$ $[\{4,6\}; 45632x1; 65342x1]$.
Exc 24: $-(1,1,1,1,1,1)$ $[\{1,2,3,4\}; \text{Exc } 37; (1324576)]$ $[\{1,2,4\}; 13425x6; 21435x6; 41523x6]$ $[\{1,4\}; 13425x6; 41523x6]$ $[\{1,2\}; 13425x6; 21435x6]$ $[\{1,2,3\}; 13425x6; 21435x6; 32415x6]$.
Exc 25: $-(1,1,1,1,2)$ $[\{4,5\}; 435126x; 534126x]$.
Exc 26: $-(1,1,1,3)$ $[\{4,5,6\}; \text{Exc } 41; (5431276)]$ $[\{4,5\}; 412x356;$

$512x346$] [$\{4,6\}$; Exc 44; (1234567)] [$\{5,6\}$; $512x346$; $654x123$].
Exc 27: $-(1,1,3,1)$ [$\{4,6\}$; $435216x$; $65432x1$].
Exc 28: $-(1,2,1,2)$ [$\{4,5\}$; $4156x23$; $5146x23$].
Exc 29: $-(1,1,1,1,1,1)$ [$\{4,6\}$; $45632x1$; $631x524$].
Exc 30: $-(1,1,1,1,1,1)$ [$\{4,6\}$; $453x126$; $621x354$].
Exc 31: $-(1,1,2,2)$ [$\{3,4\}$; $3542x61$; $4635x12$].
Exc 32: $-(1,1,2,1,1)$ [$\{5,6\}$; Exc 47; (1234567)].
Exc 33: $-(1,1,1,1,1,1,1)$. The Paley tournament is arc transitive, so without loss of generality we may suppose that $\{1,2\} \subseteq N^-(x)$. We will prove first that 1 is an origin of $-(1,1,1,1,1,1,1)$. If $3 \to x$ then $P = 146573x2$, so we assume that $x \to 3$. If $6 \to x$ then $P = 145376x2$, and conversely if $x \to 6$ then $P = 17423x65$. Thus 1 is an origin of P in any case. Now we determine when 2 is not an origin of P: if $3 \to x$ we have $21x37564$, so $x \to 3$. If $x \to 7$ we have $213x7564$, so $7 \to x$. If $x \to 5$ we have $27163x54$, so $5 \to x$. If $4 \to x$ we have $213675x4$, so $x \to 4$. If $6 \to x$ we have $274351x6$, so $x \to 6$. Finally this extension is isomorphic to Exc 49, the isomorphism being (32876541).
Exc 34: $-(1,1,1,2,1,1)$ [$\{1,2,3\}$; $137x6245$; $237x6145$; $3621x574$]
[$\{1,2\}$; $137x6245$; $237x6145$] [$\{1,3\}$; $137x6245$; $3621x574$]
[$\{2,3\}$; $237x6145$; $361x7542$].
Exc 35: $-(1,2,1,3)$ [$\{1,2,3\}$; $1437x562$; $2417x563$; $3417x562$]
[$\{1,2\}$; $1437x562$; $2417x563$] [$\{1,3\}$; $1437x562$; $3417x562$]
[$\{2,3\}$; $2417x563$; $3417x562$].
Exc 36: $-(1,2,3,1)$ [$\{1,2\}$; $1437x256$; $2437x156$].
Exc 37: $-(1,1,1,1,1,1,1)$ [$\{4,5,6\}$; $46153x72$; $54163x72$; $65143x72$] [$\{4,5\}$; $46153x72$; $54163x72$]. The other possible neighbourhoods of x are equivalent.
Exc 38: $-(1,1,1,2,1,1)$ [$\{1,2,3\}$; $137x6245$; $217x6345$; $327x6145$] [$\{1,2\}$; $137x6245$; $217x6345$]. The other possible neighbourhoods of x are equivalent.
Exc 39: $-(1,2,1,3)$ [$\{1,2,3\}$; $1437x562$; $2417x563$; $3417x562$]
[$\{1,2\}$; $1437x562$; $2417x563$] [$\{1,3\}$; $1437x562$; $3417x562$]
[$\{2,3\}$; $2417x563$; $3417x562$].
Exc 40: $-(1,1,1,1,2,1)$ [$\{1,2\}$; $167x4325$; $267x4315$].
Exc 41: $-(1,1,1,1,3)$ [$\{6,7\}$; $6374521x$; $7364521x$].
Exc 42: $-(1,2,2,1,1)$ [$\{6,7\}$; $6457312x$; $7456312x$].
Exc 43: $-(1,1,1,2,1,1)$ [$\{2,7\}$; $245x3761$; $761x3245$].
Exc 44: $-(1,1,1,1,3)$ [$\{6,7\}$; $6173254x$; $7163254x$].
Exc 45: $-(1,2,1,2,1)$ [$\{1,7\}$; $1426x573$; $7426x513$].
Exc 46: $-(1,2,2,2)$ [$\{4,7\}$; $4562x731$; $7562x431$].
Exc 47: $-(1,1,1,2,1,1)$ [$\{4,5\}$; $425763x1$; $527463x1$] [$\{4,5,7\}$; $425763x1$; $527463x1$; $724563x1$]. The other possible neighbourhoods of x are equivalent.

Exc 48: $-(1,1,1,2,1,1)$ $[\{1,2\}; 147x3256; 234x5167]$.
Exc 49: $-(1,1,1,1,1,1,1,1)$ $[\{1,2\}; 1853426x7; 2853416x7]$.
Exc 50: $-(1,2,1,1,1,1,1)$ $[\{1,2\}; 182367x54; 281367x54]$.
Exc 51: $-(1,2,1,2,1,1)$ $[\{2,8\}; 2738451x6; 8732451x6]$. ▮

3. THE INFINITE FAMILIES OF EXCEPTIONS.

DEFINITION 3.1. - We now introduce the *infinite families of exceptions*. Our notation $E_i(n)$ represents an exception $(F_i(n), P)$, where $F_i(n)$ is the tournament on n vertices depicted as F_i in the next page. Then we note S the set of vertices of $F_i(n)$ which are not origin of P. Conditions are given on the tournament. Finally, we give the paths P with origin $x \notin S$.

Exception $E_1(n) = (F_1(n), (1, n-2))$; $S = \{1,2,3\}$. Conditions: $|X| \geq 1$. Paths: For any $u \in X$, $P = u132I_{X \setminus \{u\}}$.

Exception $E_2(n) = (F_2(n), (2, n-3))$; $S = \{3,4\}$. Conditions: $|X| \geq 1$. Paths: $P = 1234I_X$, $P = 2314I_X$ and for any $u \in X$, $P = u4321I_{X\setminus\{u\}}$.

Exception $E_3(n) = (F_3(n), (1, n-2))$; $S = \{1,3\}$. Conditions: $N^+(3) \neq \{2\}$ and 3 is an ingenerator of $T(X)$ (in particular there exists a hamiltonian directed inpath $u3R_1$ of $T(X)$ and another $3vR_2$ of $T(X)$). Paths: $P = 21u3R_1$, and for any $y \in X\setminus\{3\}$, $P = y12I_{X\setminus\{y\}}$.

Exception $E_4(n) = (F_4(n), (2, n-3))$; $S = \{1,4\}$. Conditions: $N^+(3) \neq \{2\}$ and 3 is an ingenerator of $T(X)$ (in particular there exists a hamiltonian directed inpath $u3R_1$ of $T(X)$ and another $3vR_2$ of $T(X)$). Paths: $P = 214u3R_1$, $P = 3241vR_2$ and for any $y \in X\setminus\{3\}$, $P = y142I_{X\setminus\{y\}}$.

Exception $E_5(n) = (F_5(n), (1, n-2))$; $S = \{1,2\}$. Conditions: $n \geq 5$, $|Y| \geq 2$ and 2 is an ingenerator of $T(X)$. Paths: for any $x \in X\setminus\{2\}$, $P = x1I_Y I_{X\setminus\{x\}}$ and for any $y \in Y$, $P = y1I_{Y\setminus\{y\}}I_X$.

Exception $E_6(n) = (F_6(n), (2, n-3))$; $S = \{1,3\}$. Conditions: $|Y| \geq 2$ and 2 is an ingenerator of $T(X)$. Paths: for any $x \in X$, for a given $z \in Y$, $P = xz13I_{Y\setminus\{z\}}I_{X\setminus\{x\}}$ and for any $y \in Y$,$P = y31I_{Y\setminus\{y\}}I_X$.

Exception $E_7(n) = (F_7(n), (1, 1, n-3))$, $S = \{2,3\}$. Conditions: $T(Y)$ is not a 3-cycle (i.e. $T(Y)$ is not isomorphic to 3A) and $|Y| \geq 3$. Paths: since $T(Y)$ is not the 3-cycle, there is a path $Q = -(1, n-5)$ in $T(Y)$ (this is clear if $T(Y)$ is not strongly connected; if $T(Y)$ is strongly connected with at least 4 vertices, one can find a vertex $w \in Y$ such that $T(Y)\setminus\{w\}$ is strongly connected: such a vertex w is certainly an origin of Q), we then have $P = 1Q32$. For any $y \in Y$, $P = y231O_{Y\setminus\{y\}}$.

Exception $E_8 = (F_8(n), (n-4, 1, 1, 1))$; $S = X$. Conditions: 3A is the 3-cycle, its set of vertices is $\{1,2,3\}$, we furthermore need $|X| \geq 2$. Paths: $P = 1O_{X\setminus\{u\}}2u3$, $P = 2O_{X\setminus\{u\}}1u3$ and $P = 3O_{X\setminus\{u\}}2u1$ for a given $u \in X$.

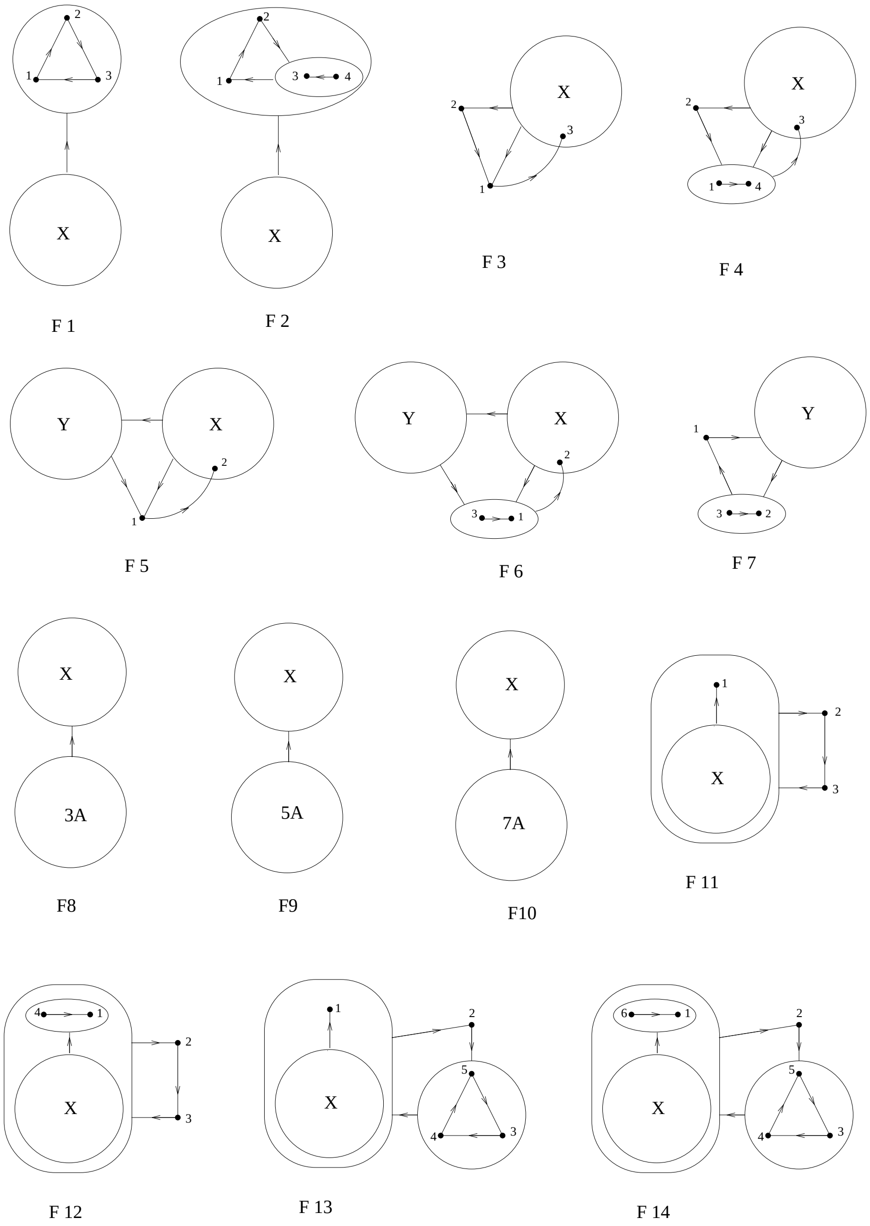

FIG. 4. The infinite families

Exception $E'_8(n) = (F_8(n),(n-4,2,1))$; $S = X$. Conditions: 3A is the 3-cycle, its set of vertices is $\{1,2,3\}$, we furthermore need $|X| \geq 2$. Paths: $P = 1O_{X\setminus\{u\}}32u$, $P = 2O_{X\setminus\{u\}}13u$ and $P = 3O_{X\setminus\{u\}}21u$ for a given $u \in X$.

Exception $E_9(n) = (F_9(n),(n-6,1,1,1,1,1))$; $S = X$. Conditions: 5A is the 2-regular tournament, its set of vertices is $\{1,2,3,4,5\}$, we furthermore need $|X| \geq 2$. Paths: $P = 1O_{X\setminus\{u\}}2u453$, $P = 2O_{X\setminus\{u\}}3u514$, $P = 3O_{X\setminus\{u\}}4u125$, $P = 4O_{X\setminus\{u\}}5u231$ and $P = 5O_{X\setminus\{u\}}1u342$ for a given $u \in X$.

Exception $E'_9(n) = (F_9(n),(n-6,2,1,1,1))$; $S = X$. Conditions: 5A is the 2-regular tournament, its set of vertices is $\{1,2,3,4,5\}$, we furthermore need $|X| \geq 2$. Paths: $P = 1O_{X\setminus\{u\}}32u45$, $P = 2O_{X\setminus\{u\}}43u51$, $P = 3O_{X\setminus\{u\}}54u12$, $P = 4O_{X\setminus\{u\}}15u23$ and $P = 5O_{X\setminus\{u\}}21u34$ for a given $u \in X$.

Exception $E_{10}(n) = (F_{10}(n),(n-8,1,1,1,1,1,1,1))$; $S = X$. Conditions: 7A is the Paley tournament on 7 vertices, its set of vertices is $\{1,2,3,4,5,6,7\}$, we furthermore need $|X| \geq 2$. Paths: let $u \in X$, $P = 1O_{X\setminus\{u\}}2u45376$, $P = 2O_{X\setminus\{u\}}3u56417$, $P = 3O_{X\setminus\{u\}}4u67521$, $P = 4O_{X\setminus\{u\}}5u71632$, $P = 5O_{X\setminus\{u\}}6u12743$, $P = 6O_{X\setminus\{u\}}7u23154$ and $P = 7O_{X\setminus\{u\}}1u34265$.

Exception $E'_{10}(n) = ((F_{10}(n),(n-8,2,1,1,1,1,1))$; $S = X$. Conditions: 7A is the Paley tournament on 7 vertices, its set of vertices is $\{1,2,3,4,5,6,7\}$, we furthermore need $|X| \geq 2$. Paths: let $u \in X$, $P = 1O_{X\setminus\{u\}}32u4657$, $P = 2O_{X\setminus\{u\}}43u5761$, $P = 3O_{X\setminus\{u\}}54u6172$, $P = 4O_{X\setminus\{u\}}65u7213$, $P = 5O_{X\setminus\{u\}}76u1324$, $P = 6O_{X\setminus\{u\}}17u2435$ and $P = 7O_{X\setminus\{u\}}21u3546$.

Exception $E_{11}(n) = (F_{11}(n),(1,1,n-3))$; $S = \{1,2\}$. Conditions: $|X| \geq 2$. Paths: $P = 31O_X2$ and for all $u \in X$, $P = u1O_{X\setminus\{u\}}23$.

Exception $E_{12}(n) = (F_{12}(n),(2,1,n-4))$; $S = \{1,4\}$. Conditions: $|X| \geq 2$. Paths: $P = 231O_X4$ and for all $u \in X$, $P = u41O_{X\setminus\{u\}}23$.

Exception $E_{13}(n) = (F_{13}(n),(1,1,n-3))$; $S = \{1,2\}$. Conditions: $|X| \geq 2$. Paths: $P = 3425O_X1$, $P = 4523O_X1$, $P = 5324O_X1$ and for all u, v in X, $P = u1v2345O_{X\setminus\{u,v\}}$.

Exception $E_{14}(n) = (F_{14}(n),(2,1,n-4))$; $S = \{1,6\}$. Conditions: $|X| \geq 2$. Paths: for every vertex u of X, $P = 24u53O_{X\setminus\{u\}}61$, $P = 3u16245O_{X\setminus\{u\}}$, $P = 4u16253O_{X\setminus\{u\}}$, $P = 5u16234O_{X\setminus\{u\}}$ and $P = u61345O_{X\setminus\{u\}}2$.

Lemma 3.1. *Let T be a tournament of order n, x a vertex of T and P an outpath of order n such that ${}^*P$ is an outpath and $(T\setminus\{x\}, {}^*P)$ belongs to one of the infinite families of exceptions. If $d^+(x) \geq 2$ and x is not an origin of P in T then exactly one of the following holds:*

i) The pair (T, P) is an exception.
ii) Every outneighbour of x is an origin of P.

Proof. Our notations for the proof are essentially the same as in the finite exceptions.
Exc $E_1(n-1)$: $P = (2, n-3)$. $[\{1,2,3\}; 123xI_X; 231xI_X; 312xI_X]$ and $[\{1,2\}; \text{Exc } E_2(n)]$. The other possible neighbourhoods are equivalent.
Exc $E_2(n-1)$: $P = (3, n-4)$. $[\{3,4\}; 3124xI_X; 4123xI_X]$.
Exc $E_3(n-1)$: $P = (2, n-3)$. $[\{1,3\}; \text{Exc } E_4(n)]$.
Exc $E_4(n-1)$: $P = (3, n-4)$. $[\{1,4\}; 13x42vR_2; 43x12vR_2]$.
Exc $E_5(n-1)$: $P = (2, n-3)$. $[\{1,2\}; \text{Exc } E_6(n)]$.
Exc $E_6(n-1)$: $P = (3, n-4)$. $[\{1,3\}; 12x3I_YI_{X\backslash\{2\}}; 32x1I_YI_{X\backslash\{2\}}]$.
Exc $E_7(n-1)$: $P = (2, 1, n-4)$. $[\{2,3\}; 21xO_Y3; 31xO_Y2]$.
Exc $E_8(n-1)$: $P = (n-4, 1, 1, 1)$. If x is not an origin of P, then $N^+(x) \subseteq X$. Thus $x \leftarrow 3A$ and we are in the exception $E_8(n)$.
Exc $E'_8(n-1)$: $P = (n-4, 2, 1)$. If x is not an origin of P, then $N^+(x) \subseteq X$. Thus $x \leftarrow 3A$ and we are in the exception $E'_8(n)$.
Exc $E_9(n-1)$: $P = (n-6, 1, 1, 1, 1, 1)$. If x is not an origin of P, then $N^+(x) \subseteq X$. Thus $x \leftarrow 5A$ and we are in the exception $E_9(n)$.
Exc $E'_9(n-1)$: $P = (n-6, 2, 1, 1, 1)$. If x is not an origin of P, then $N^+(x) \subseteq X$. Thus $x \leftarrow 5A$ and we are in the exception $E'_9(n)$.
Exc $E_{10}(n-1)$: $P = (n-8, 1, 1, 1, 1, 1, 1, 1)$. If x is not an origin of P, then $N^+(x) \subseteq X$. Thus $x \leftarrow 7A$ and we are in the exception $E_{10}(n)$.
Exc $E'_{10}(n-1)$: $P = (n-8, 2, 1, 1, 1, 1, 1)$. If x is not an origin of P, then $N^+(x) \subseteq X$. Thus $x \leftarrow 7A$ and we are in the exception $E'_{10}(n)$.
Exc $E_{11}(n-1)$: $P = (2, 1, n-4)$. $[\{1,2\}; \text{Exc } E_{12}(n)]$.
Exc $E_{12}(n-1)$: $P = (3, 1, n-5)$. $[\{1,4\}; 12x43O_X; 42x13O_X]$.
Exc $E_{13}(n-1)$: $P = (2, 1, n-4)$. $[\{1,2\}; \text{Exc } E_{14}(n)]$.
Exc $E_{14}(n-1)$: $P = (3, 1, n-5)$. $[\{1,6\}; 12x6345O_X; 62x1345O_X]$. ∎

Lemma 3.2. *Let T be a tournament of order n, x a vertex of T and P an inpath of order n such that ${}^*P$ is an outpath and $(T\backslash\{x\}, {}^*P)$ belongs to one of the infinite families of exceptions. If $d^-(x) \geq 2$ and x is not an origin of P in T then exactly one of the following holds:*
i) The pair (T, P) is an exception.
ii) Every inneighbour of x is an origin of P.

Proof.
Exc $E_1(n-1)$: $P = -(1, 1, n-3)$. $[\{1,2,3\}; 13x2I_X; 21x3I_X; 32x1I_X]$, $[\{1,2\}; 1u3x2I_{X\backslash\{u\}}; 2u3x1I_{X\backslash\{u\}}]$ for a given $u \in X$.
Exc $E_2(n-1)$: $P = -(1, 2, n-4)$. $[\{3,4\}; 3241I_Xx; 4231I_Xx]$.
Exc $E_3(n-1)$: $P = -(1, 1, n-3)$. $[\{1,3\}; 1u2x3R_1; 3v12R_2x]$.

Exc $E_4(n-1)$: $P=-(1,2,n-4)$. $[\{1,4\};1v32x4R_2;4v32x1R_2]$.

Exc $E_5(n-1)$: $P=-(1,1,n-3)$. Three cases may arise i) If $X=\{2\}$ and $T(Y)$ is not a 3-cycle, we are in the dual of exception $E_7(n)$. ii) If $X=\{2\}$ and $T(Y)$ is a 3-cycle, we are in the dual of exception 26. iii) If $|X|\geq 2$, since 2 is an ingenerator of $T(X)$, there exists a directed inpath $2Q$ of $T(X)$ with end v and u a vertex such that $u\rightarrow 2$ and we have $[\{1,2\};1vI_Yx2Q^*;2uI_Yx1I_{X\setminus\{u,2\}}]$.

Exc $E_6(n-1)$: $P=-(1,2,n-4)$. Three cases may arise. i) If $X=\{2\}$ and $|Y|\geq 3$, let $u\rightarrow v\rightarrow w$ be three vertices of Y, we have $[\{1,3\};1uvw2x3I_{Y\setminus\{u,v,w\}};3uvw2x1I_{Y\setminus\{u,v,w\}}]$. ii) If $X=\{2\}$ and $|Y|=2$, we are in the dual of exception 18. iii) If $|X|\geq 2$, since 2 is an ingenerator of $T(X)$, there exists an arc $u\rightarrow 2$ in $T(X)$ and we have $[\{1,3\};1u2I_Yx3I_{X\setminus\{2,u\}};3u2I_Yx1I_{X\setminus\{2,u\}}]$.

Exc $E_7(n-1)$: $P=-(1,1,1,n-4)$. Let $u\rightarrow v$ be an arc of $T(Y)$ (note that $Y\setminus\{u,v\}$ is not empty),

$[\{2,3\};2uvxO_{Y\setminus\{u,v\}}31;3uvxO_{Y\setminus\{u,v\}}21]$.

Exc $E_8(n-1)$: $P=-(1,n-5,1,1,1)$. If x is not an origin of P, then $N^-(x)\subseteq X$. Let $u\in X$ be an inneighbour of x, then $P=u1O_{X\setminus\{u\}}23x$.

Exc $E'_8(n-1)$: $P=-(1,n-5,2,1)$. If x is not an origin of P, then $N^-(x)\subseteq X$. Let $u\in X$ be an inneighbour of x, then $P=u1O_{X\setminus\{u\}}2x3$.

Exc $E_9(n-1)$: $P=-(1,n-7,1,1,1,1,1)$. If x is not an origin of P, then $N^-(x)\subseteq X$. Let $u\in X$ be an inneighbour of x, then $P=u1O_{X\setminus\{u\}}2435x$.

Exc $E'_9(n-1)$: $P=-(1,n-7,2,1,1,1)$. If x is not an origin of P, then $N^-(x)\subseteq X$. Let $u\in X$ be an inneighbour of x, then $P=u1O_{X\setminus\{u\}}2x435$.

Exc $E_{10}(n-1)$: $P=-(1,n-9,1,1,1,1,1,1,1)$. If x is not an origin of P, then $N^-(x)\subseteq X$. Let $u\in X$ be an inneighbour of x, then $P=u1O_{X\setminus\{u\}}243756x$.

Exc $E'_{10}(n-1)$: $P=-(1,n-9,2,1,1,1,1,1)$. If x is not an origin of P, then $N^-(x)\subseteq X$. Let $u\in X$ be an inneighbour of x, then $P=u1O_{X\setminus\{u\}}2x43756$.

Exc $E_{11}(n-1)$: $P=-(1,1,1,n-4)$. Let $u\rightarrow v$ be an arc of $T(X)$,

$[\{1,2\};13vu2xO_{X\setminus\{u,v\}};2uvx3O_{X\setminus\{u,v\}}1]$. (Note that $O_{X\setminus\{u,v\}}$ may be the empty path.)

Exc $E_{12}(n-1)$: $P=-(1,2,1,n-5)$. Let u be an element of X. $[\{1,4\};1u23xO_{X\setminus\{u\}}4;4u23xO_{X\setminus\{u\}}1]$.

Exc $E_{13}(n-1)$: $P=-(1,1,1,n-4)$. Let $u\rightarrow v$ be an arc of $T(X)$,

$[\{1,2\};1342x5O_X;2uvx345O_{X\setminus\{u,v\}}1]$. (Note that $O_{X\setminus\{u,v\}}$ may be the empty path.)

Exc $E_{14}(n-1)$: $P=-(1,2,1,n-5)$. $[\{1,6\};1345xO_X62;6345xO_X12]$. ▌

Here are some remarks on the exceptions listed in section 2 and 3.

Remark 3. 1. The sole exceptions (T,P) *where* P *is not contained at all in* T *are Grünbaum's one.*

Remark 3. 2. *If P is an outpath and (T,P) is an exception in which T has a maximal vertex, then (T,P) belongs to one of the families $E_8(n)$, $E'_8(n)$, $E_9(n)$, $E'_9(n)$, $E_{10}(n)$ or $E'_{10}(n)$.*

Remark 3. 3. *Let P be an outpath and (T,P) be an exception in which T is not strong. If X is minimal in T (i.e. $T = T(X) \to T(Y)$ and Y is not empty), then a vertex of X is an origin of P.*

Remark 3. 4. *If P is an outpath and (T,P) an exception in which T has a minimal element, then (T,P) belongs to one of the families $E_1(n)$, $E_2(n)$, $E_3(n)$, $E_4(n)$, $E_5(n)$ or $E_6(n)$.*

Remark 3. 5. *The only exceptions in which at least four vertices are not origins of an outpath with first block of length at least 2 are the exceptions 7, $E_8(n)$, $E'_8(n)$, $E_9(n)$, $E'_9(n)$, $E_{10}(n)$ and $E'_{10}(n)$.*

Remark 3. 6. *Let (T,P) be an exception on $n \geq 9$ vertices. Note that T is not 2-strong, and if T is strong then (T,P) belongs to one of the families $E_i(n)$ with $i \in \{3,4,5,6,7,11,12,13,14\}$.*

The four above lemmas guarantee that our induction hypothesis will carry through whenever it leads to one of the exceptions. From now on, we call them the *extension lemmas.*

4. ORIENTED HAMILTONIAN PATHS IN TOURNAMENTS.

THEOREM 4.1. *Let T be a tournament of order n, P an outpath of order n and x, y two distinct vertices of T such that $s^+(x,y) \geq b_1(P)+1$. Then one of the following holds:*
i) one of the vertices x, y is an origin of P;
ii) the pair (T,P) is an exception.

Proof. We proceed by induction on n, the result holding for $n = 1$. We suppose that both the theorem and its dual hold for $|T| = n-1$. Let x and y be two vertices of T such that $x \to y$ and $s^+(x,y) \geq b_1(P)+1$. We may distinguish two cases:

Case 1 : $b_1(P) \geq 2$. Assume $d^+(x) \geq 2$. If $(T\backslash\{x\},^* P)$ is an exception, by the four extension lemmas, either x or y is an origin of P, or (T, P) is an exception. If $(T\backslash\{x\},^* P)$ is not an exception, let $z \in N^+(x)$ be an outgenerator of $T(S^+(x)\backslash\{x\})$. Let $t \in N^+(x)$, t distinct from z; by definition of z, $s^+_{V\backslash\{x\}}(t, z) = s^+(x) - 1 \geq b_1(P)$. By the induction hypothesis, z or t is an origin of $^*P$ in $T\backslash\{x\}$, hence x is an origin of P in T. Assume now that $d^+(x) = 1$, in particular y is the unique outneighbour of x. Let z be an outgenerator of $T(N^+(y))$ ($N^+(y)$ is not empty since $s^+(x, y) \geq b_1(P)+1 \geq 3$). Note that $x \leftarrow z$ and $s^+_{V\backslash\{y\}}(x, z) = s^+(x, y) - 1 \geq b_1(^*P) + 1$. If $(T\backslash\{y\},^* P)$ is not an exception, by the induction hypothesis, x or z is an origin of $^*P$ in $T\backslash\{y\}$. Since $d^+_{V\backslash\{y\}}(x) = 0$, this origin is certainly z. Consequently y is an origin of P in T. If $(T\backslash\{y\},^* P)$ is an exception, since x is maximal in $T\backslash\{y\}$, by remark 3.2, the tournament $(T\backslash\{y\},^* P)$ is one of the exceptions $E_8(n-1)$, $E'_8(n-1)$, $E_9(n-1)$, $E'_9(n-1)$, $E_{10}(n-1)$ or $E'_{10}(n-1)$. If y has an outneighbour which is an origin of $^*P$ in T then y is an origin of P. If not, we suppose that $v \to y$ for all vertices v which are origins of $^*P$ in $T\backslash\{y\}$. Then (T, P) is one of the exceptions $E_8(n)$, $E'_8(n)$, $E_9(n)$, $E'_9(n)$, $E_{10}(n)$ or $E'_{10}(n)$.

Case 2 : $b_1(P) = 1$. We denote by X the set $S^-_{V\backslash\{x\}}(N^+(x))$. Consider the partition $(X, Y, \{x\})$ of V where $Y = V\backslash(X \cup \{x\})$. Then $Y \to x$, $X \to Y$ and $y \in X$.

a) Assume $d^+(x) \geq 2$ and $|X| \geq b_2(P)+1$: if $(T\backslash\{x\},^* P)$ is an exception, by the four extension lemmas, x or y is an origin of P in T. If not, let $z \in N^+(x)$ be an ingenerator of $T(X)$ and let $u \in N^+(x)$, u distinct from z. By the induction hypothesis, z or u is an origin of $^*P$ in $T\backslash\{x\}$. Hence x is an origin of P in T.

b) Assume $d^+(x) \geq 2$ and $|X| \leq b_2(P)$. These two conditions imply that $b_2(P) \geq 2$. Define A^- and A^+ as respectively $X \cap N^-(x)$ and $(X\backslash\{y\}) \cap N^+(x)$. Note that $A^+ \neq \emptyset$.

- If $P = (1, n-2)$, then $|X| \leq n-2$ and $|Y| \geq 1$. In $T\backslash\{y\}$, since $d^+_{V\backslash\{y\}}(x) > 0$, Y belongs to the maximal component of $T\backslash\{y\}$. Thus there exists a hamiltonian directed inpath P' of $T\backslash\{y\}$ whose origin is in Y, and y is an origin of $P = yP'$ in T.

- If $P = (1, n-3, 1)$, then $|X| \leq n-3$ and $|Y| \geq 2$. If A^- is not empty, let v be a vertex of Y. Then y is origin of $P = yI_{Y\backslash\{v\}}I_{A^+}xI_{A^-}v$. If A^- is empty and $|A^+| \geq 2$ let $z \in A^+$, then y is origin of $P = yI_YI_{A^+\backslash\{z\}}xz$. If A^- is empty and X consists of the arc $z \to y$ then if $|Y| = 2$ we are in exception 10. So we may assume that $|Y| > 2$. By Theorem 1.1, one can find a path $P' = -(|Y-3|, 1)$ in $T(Y)$ (with the notation $-(0, 1) = +(1)$ if $|Y| = 3$). Essentially, there exists $v \in Y$ such that $T(Y)\backslash\{v\}$ contains the path P'. Thus y is an origin of $P = yvzxP'$.

- If $P = (1, n-4, 2)$, then $|X| \leq n-4$ and $|Y| \geq 3$. Let $v \in Y$. Then y is an origin of $P = yI_{Y\setminus\{v\}}I_{X\setminus\{y\}}vx$. (Recall that $X \setminus \{y\}$ is not empty since $d^+(x) \geq 2$.

- If $P = (1, n-4, 1, 1)$, then $|X| \leq n-4$ and $|Y| \geq 3$. If A^- is not empty, let $v \to w$ be an arc of Y. Then y is origin of $P = yI_{Y\setminus\{w,v\}}I_{A^+}xI_{A^-}wv$. If A^- is empty and $|Y| \geq 4$. By Theorem 1.1, one can find a path $P' = -(|Y-4|, 1, 1)$ in $T(Y)$ (with the notation $-(0,1,1) = +(1,1)$ if $|Y| = 4$). Essentially, there exists $v \in Y$ such that $T(Y)\setminus\{v\}$ contains the path P'. Thus y is an origin of $P = yvI_{A^+}xP'$. Suppose that A^- is empty and $|Y| = 3$, say $Y = \{u, v, w\}$ with $u \to v$. If $|X| \geq 3$, let z be a vertex of X distinct from y, then y is an origin of $P = ywI_{X\setminus\{y,z\}}xuvz$. If $|X| = 2$, (T, P) is the exception 16 or the exception 32.

If P is a path distinct from the four paths above, we have $|X| \leq b_2(P) \leq n-5$ then $|Y| \geq 4$ (keep in mind that $b_2(P) \geq 2$). In $T\setminus\{y\}$, if a vertex of Y is an origin of ${}^*P$ then y is also an origin of P. This is the case if $(T\setminus\{y\}, {}^*P)$ is not an exception, as we notice that $S^-_{V\setminus\{y\}}(Y) = V\setminus\{y\}$. So we may assume that $(T\setminus\{y\}, {}^*P)$ is an exception, and that none of the vertices of Y is an origin of ${}^*P$. Hence by Remark 3.5, $(T\setminus\{y\}, {}^*P)$ is the dual of one of the exceptions 7, E_8, E'_8, E_9, E'_9, E_{10} or E'_{10}. Since in all these exceptions the indegree of any vertex inside the maximal component cannot exceed 3, this case never arises (Observe the maximal component of the tournament of $Exc7$ is the whole tournament). Indeed the vertex x belongs to the maximal component of $T\setminus\{y\}$ and has in this component indegree at least 4.

c) Assume that $d^+(x) = 1$ and $|X| \geq b_2(P) + 1$.

- If $d^+(y) = 0$, then two cases arise. Either ${}^{**}P$ is contained in $T\setminus\{x, y\}$ and x is an origin of P, or ${}^{**}P$ is not contained in $T\setminus\{x, y\}$ and by the induction hypothesis and Remark 3.1, $(T\setminus\{x, y\}, {}^{**}P)$ is one of the exceptions 0, 4 or 33. Thus (T, P) is one of the exceptions $E_8(5)$ or $E'_8(5)$, $E_9(7)$ or $E'_9(7)$, $E_{10}(9)$ or $E'_{10}(9)$ respectively.

- If $d^+(y) = 1$, let z be the unique outneighbour of y. If T has only three vertices, (T, P) is the exception 0. Otherwise T has at least four vertices. We consider two cases. i) Assume that $(T\setminus\{x, y\}, {}^{**}P)$ is not an exception. If in $T\setminus\{x, y\}$ we can find two origins of ${}^{**}P$, one is distinct from z, thus x, via y, is an origin of P. Note that if $b_1({}^{**}P) < n-3$, the induction hypothesis gives at least two origins of ${}^{**}P$. So we limit our investigation to $b_1({}^{**}P) = n-3$. If $P = (1, n-2)$, either z is maximal in $T\setminus\{x, y\}$ and (T, P) is the exception $E_1(n)$, or there exists $I_{V\setminus\{x,y\}}$ with origin distinct from z and then x is an origin of $P = xyI_{V\setminus\{x,y\}}$. If $P = (1, 1, n-3)$, we consider two cases: if $N^-(z) = \{y\}$ we are in exception $E_{11}(n)$ when $|X| > 1$ and in exception 1 when $|X| = 1$. If $N^-(z) \neq \{y\}$ we may find a path $O_{V\setminus\{x,y\}}$ whose origin is distinct from z, thus x is an origin of $P = xyO_{V\setminus\{x,y\}}$. ii) Assume now that $(T\setminus\{x, y\}, {}^{**}P)$ is an exception. If in $T\backslash\{x,y\}$ we can find two origins of $^{**}P$, one is distinct from z, thus x is an origin of P. The exceptions with at most one origin are 0, 1, 4, 7 and 33 and their duals. There is no choice extending these cases by x and y so we just specify the paths or the exceptions:

$[Exc\ 0; P=(1,1,1,1); z:=3; yz2x1]$.
$[Dual\ Exc\ 0; P=(1,2,1); z:=3; yz12x]$.
$[Exc\ 1; P=(1,1,1,1,1); z:=4; yz3x12]$.
$[Dual\ Exc\ 1; P=(1,2,1,1); z:=4; yz12x3]$.
$[Exc\ 4; P=(1,1,1,1,1,1); z:=1; yz5x243]$.
$[Dual\ Exc\ 4; P=(1,2,1,1,1); z:=1; yz23x54]$.
$[Exc\ 7; P=(1,1,2,1,1); z:=5; Exc\ 48]$.
$[Dual\ Exc\ 7; P=(1,3,1,1); z:=5; yz123x4]$.
$[Exc\ 33; P=(1,1,1,1,1,1,1,1); z:=1; yz72564x3]$.
$[Dual\ Exc\ 33; P=(1,2,1,1,1,1,1); z:=1; yz562437x]$.

Now $d^+(y)\geq 2$. Denote $S^-_{V\backslash\{y\}}(N^+(y))$ by Z and consider the partition $(Z,W,\{y\},\{x\})$ of V with $W=V\backslash(Z\cup\{y,x\})$. By construction, $W\rightarrow\{x,y\}$ and $Z\rightarrow W\cup\{x\}$.

i) If $|Z|\geq b_2(P)+1$ then if $(T\backslash\{y\},^*P)$ is not an exception, we are done by the induction hypothesis since we can find $\{z,t\}\subseteq N^+(y)$ with $s^-_{V\backslash\{y\}}(\{z,t\})\geq b_1(^*P)+1$. If $(T\backslash\{y\},^*P)$ is an exception, then by Remark 3.4, $(T\backslash\{y\},^*P)$ is the dual of one of $E_1(n-1)$, $E_2(n-1)$, $E_3(n-1)$, $E_4(n-1)$, $E_5(n-1)$ or $E_6(n-1)$. When we discuss these exceptions, we will denote by X' and Y' the sets X and Y of the exception, but the vertices will still be denoted as in the figures.

Suppose that $T\backslash\{y\}$ is the dual of $F_1(n-1)$, thus $P=(1,1,n-3)$. If y has an inneighbour in the minimal 3-cycle of $T\backslash\{y\}$ then x is an origin of P. If y has an outneighbour in X' then y is an origin of P. So $N^+(y)$ is exactly the minimal 3-cycle of $T\backslash\{y\}$ and we are in the exception $E_{13}(n)$ if $n\geq 7$ and in exception 26 if $n=6$.

Suppose that $T\backslash\{y\}$ is the dual of $F_2(n-1)$, thus $P=(1,2,n-4)$. The only possible outneighbourhood of y not containing an origin of $^*P$ is $\{3,4\}$. If $X'\backslash\{x\}$ is not empty (i.e. contains a vertex u), then x is an origin of $P=xyu2134O_{X'\backslash\{x,u\}}$. If $W=\{x\}$ then we are in exception 31.

Suppose that $T\backslash\{y\}$ is the dual of $F_3(n-1)$, thus $P=(1,1,n-3)$. The only possible outneighbourhood of y not containing an origin of $^*P$ is $\{1,3\}$. Then x is an origin of $P=xy231O_{X'\backslash\{x,3\}}$.

Suppose that $T\backslash\{y\}$ is the dual of $F_4(n-1)$, thus $P=(1,2,n-4)$. The only possible outneighbourhood of y not containing an origin of $^*P$ is $\{1,4\}$. Since 3 has an inneighbour distinct of 2, $|X'|\geq 3$. Thus there exists a vertex u of X' distinct from x and 3. Then x is an origin of $P=xyu2341O_{X'\backslash\{x,u,3\}}$.

Suppose that $T\backslash\{y\}$ is the dual of $F_5(n-1)$, thus $P=(1,1,n-3)$. The only possible outneighbourhood of y not containing an origin of $^*P$ is $\{1,2\}$. Then x is an origin of $P=xyO_{Y'}21O_{X'\backslash\{x,2\}}$.

Suppose that $T\backslash\{y\}$ is the dual of $F_6(n-1)$, thus $P=(1,2,n-4)$. The only possible outneighbourhood of y not containing an origin of $^*P$ is $\{1,3\}$. Let vu be an arc of $T(Y')$ (it exists since $|Y'|\geq 2$) then x is an origin of $P=xyuv213O_{Y'\backslash\{u,v\}}O_{X'\backslash\{x,2\}}$. Note that this path is valid even if $O_{Y'\backslash\{u,v\}}$ is empty.

ii) Suppose now that $|Z|\leq b_2(P)$. In particular $b_2(P)\geq d^+(y)\geq 2$, so $^{**}P$ is an inpath. If $|W|\geq 2$ and $(T\backslash\{x,y\},^{**}P)$ is not an exception, let u be an ingenerator of W and v a vertex of W distinct from u. We have $s^-_{T\backslash\{x,y\}}(u,v)=n-2\geq b_1(^{**}P)$, thus, by the induction hypothesis, u or v is an origin of $^{**}P$. Consequently, x, via y, is an origin of P. If $|W|\geq 2$ and $(T\backslash\{x,y\},^{**}P)$ is an exception. By Remark 3.3, x, via y, is an origin of P. If $|W|=\{u\}$ and $(T\backslash\{x,y\},^{**}P)$ is an exception we apply again the previous argument. So we may suppose that $(T\backslash\{x,y\},^{**}P)$ is not an exception. Now if y has an inneighbour v in Z, then since $^{**}P$ is an inpath we know that u or v is an origin of $^{**}P$ in $T\backslash\{x,y\}$, thus x is again an origin of P. So we may suppose that $y\to Z$. Now $b_2(P)\geq|Z|\geq n-3$, so $b_2(P)=n-2$ or $n-3$. Since $u\leftarrow Z$, if $^{***}P$ is contained in $T(Z)$, then x via y and u is an origin of P. And the only case for which $^{***}P$ is not contained in $T(Z)$ is when $P=(1,1)$ and $T(Z)$ is the 3-cycle. Thus we are in exception 27. If W is empty, $|Z|=n-2$ thus $b_2(P)=n-2$. Since $|X|\geq b_2(P)+1$, $|X|=n-1$ and y is an ingenerator of $T\backslash\{x\}$, so y is an origin of a directed inpath of $T\backslash\{x\}$. Then x is an origin of P.

d) Assume $d^+(x)=1$ and $|X|\leq b_2(P)$. Since $b_2(P)\leq n-2$, Y is not empty. If $|Y|=1$ and $b_2(P)=n-2$ then we are in the exception $E_1(n)$ if $d^+(y)=1$ or in the exception $E_3(n)$ if $d^+(y)>1$. If $|Y|\geq 2$ and $b_2(P)=n-2$ then we are in the exception $E_5(n)$ if $n\geq 5$ or in the exception 2 if $n=4$. Otherwise, if $b_2(P)<n-2$, let v be an ingenerator of Y and $w\in Y\backslash\{v\}$. In $T\backslash\{y\}$, $s^-_{V\backslash\{y\}}(v,w)=n-2\geq b_2(P)+1$. If $(T\backslash\{y\},^*P)$ is not an exception, by the induction hypothesis, v or w is an origin of $^*P$ in $T\backslash\{y\}$ so y is an origin of P in T. If $(T\backslash\{y\},^*P)$ is an exception, since x is maximal in $T\backslash\{y\}$ and $^*P$ is an inpath, by Remark 3.4 $(T\backslash\{y\},^*P)$ is the dual of one of the exceptions $E_1(n-1)$, $E_2(n-1)$, $E_3(n-1)$, $E_4(n-1)$, $E_5(n-1)$ or $E_6(n-1)$. In all these exceptions, the first block of the path is of length less than 2. Thus $b_2(P)\leq 2$ and $|X|\leq 2$. If $|X|=2$ then $T\backslash\{y\}$ has a minimal vertex, which is impossible in the duals of the exceptions $E_i(n-1)$ with $i\in\{1,2,3,4,5,6\}$. So $|X|=1$ that is $X=\{y\}$. If $(T\backslash\{y\},^*P)$ is the exception $E_i(n-1)$ for $i\in\{2,3,4,5,6\}$, $|T\backslash\{y\}|\geq 4$, in particular $|Y|\geq 3$. Since only two vertices are not origin of $^*P$ in these exceptions, there is a vertex of Y which is an origin of $^*P$ in $T\backslash\{y\}$. Then y is an origin of P. If $(T\backslash\{y\},^*P)$ is the exception $E_1(n-1)$, if $T\backslash\{y\}$ has four

vertices, it is the dual of $F_1(4)$, so (T,P) is the exception 6. If $T\backslash\{y\}$ has at least five vertices then $|Y|\geq 4$, but only three vertices are not origin of ${}^*P$ in F_1. Then some vertex of Y is an origin of ${}^*P$ in $T\backslash\{y\}$, so y is an origin of P. ▮

COROLLARY 4.1.
Let T be a tournament of order n, and P a path of order n. Then T contains P if and only if (T,P) is not a Grünbaum's exception.

Proof. Without loss of generality assume that P is an outpath. If (T,P) is an exception, the conclusion follows from Remark 3.1. If (T,P) is not an exception, pick two vertices x,y of T, x an outgenerator of T. By Theorem 4.1, one of these two is an origin of P. ▮

COROLLARY 4.2. *Let T be a 3-strong tournament on $n\geq 9$ vertices and P a path of order n. Every vertex of T is an origin of P in T.*

Proof. Without loss of generality we may assume that P is an outpath. Let x be a vertex of T. Since T is 3-strong, $d^+(x)\geq 3$ and $T\backslash\{x\}$ is 2-strong. If $(T\backslash\{x\},{}^*P)$ is an exception, it is certainly exception 49 or 50; and x is an origin of P since its outdegree is at least 3. If it is not an exception, some vertex of $N^+(x)$ is an origin of ${}^*P$ and thus x is an origin of P. ▮

Since the tournament $8A$ is 3-strong, the bound of 9 in the previous result is best possible. Moreover, let $G(n)$ be a tournament and x a vertex such that $G(n)\setminus\{x\}=F_6(n-1)$ and $\{1,3\}\to x$, $x\to X$ and $x\to Y$. The tournament $G(n)$ is 2-strong and x is not an origin of the path $-(1,2,n-4)$.

However, we have the following weaker result for 2-strong tournaments:

COROLLARY 4.3. *Let T be a 2-strong tournament on $n\geq 9$ vertices and P an outpath of order n. Every vertex of T is an origin of P or $-P$ in T.*

Proof. Let x be a vertex of T. Since T is 2-strong, $T\backslash\{x\}$ is strong. So every vertex of $T\backslash\{x\}$ is both an ingenerator and an outgenerator of $T\backslash\{x\}$. Let us prove first that if $d^+(x)\geq 3$, x is an origin of P. This is clear when $(T\backslash\{x\},{}^*P)$ is not an exception. If $(T\backslash\{x\},{}^*P)$ is an exception, since $T\backslash\{x\}$ is strong, $(T\backslash\{x\},{}^*P)$ is one of the exceptions E_3, E_4, E_5, E_6, E_7, E_{11}, E_{12}, E_{13}, E_{14}, 49, 50 and 51. But in all these cases, only two vertices of $T\backslash\{x\}$ are not an origin of ${}^*P$ in $T\backslash\{x\}$. Since $d^+(x)\geq 3$, there is a vertex of $N^+(x)$ which is an origin of ${}^*P$ in $T\backslash\{x\}$, so x is an origin of P in T. Similarly, if $d^-(x)\geq 3$, we prove that x is an origin of $-P$. By the above two assertions, since $n\geq 9$, x is an origin of P or $-P$. ▮

The bound of 9 of this corollary is best possible: $8A$ is a 2-strong tournament whose vertices 1 and 2 are not origins of the two (dual) antidirected hamiltonian paths. Moreover, let $H(2n+1)$ be the tournament which for there exists a vertex x such that $H(2n+1)\setminus\{x\}$ is the transitive tournament $(c_1\to c_2\to\cdots\to c_{2n})$, $x\to\{c_1,\dots,c_n\}$ and $x\leftarrow\{c_{n+1},\dots,c_{2n}\}$. Clearly $H(2n+1)$ is strong and for every path such that $b_1(P)=1$ and $b_2(P)>n$, the vertex x is neither an origin of P nor $-P$.

We now give a proof of an assertion that Thomason in [9] p.170 thought was possible to prove. It is a kind of analogue of corollary 1.1 for hamiltonian paths.

Corollary 4.4.
Let P be a path of order $n\geq 8$ with $b_1(P)=k$, T a tournament on n vertices and K a set of $k+2$ vertices of T. There is an origin of P in K unless (T,P) belongs to $E_1(n)$.

Proof. If (T,P) is an exception distinct from $E_1(n)$, the corollary holds. If (T,P) is not an exception, and P is an inpath (resp. outpath), let u denote an ingenerator (resp. outgenerator) of $T(K)$ and let v be any vertex of K distinct from u. By Theorem 4.1, u or v is an origin of P. ∎

We now prove a result more general than Rosenfeld's theorem of [7]:

Corollary 4.5. *Let T be a tournament on $n\geq 9$ vertices and P a path of order n such that $b_1(P)=1$ and $b_2(P)\leq\lceil\frac{n-5}{2}\rceil$. Every vertex of T is an origin of P or $-P$ in T.*

Proof. Let x be a vertex of T. We have $d^+(x)+d^-(x)=n-1$, so $d^+(x)\geq\lceil\frac{n-1}{2}\rceil$ or $d^-(x)\geq\lceil\frac{n-1}{2}\rceil$. If $d^+(x)\geq\lceil\frac{n-1}{2}\rceil$, then by the previous corollary, in $N^+(x)$, there exists a vertex which is an origin of ${}^*P$ in $T\setminus\{x\}$. Note that if $(T\setminus\{x\};{}^*P)=E_1(n-1)$, it is true because $\lceil\frac{n-1}{2}\rceil\geq 4$. Hence, x is an origin of P in T. Analogously, we prove that x is an origin of $-P$ in T, if $d^-(x)\geq\lceil\frac{n-1}{2}\rceil$. ∎

In [2], Bampis et al., proved that in every tournament of order at least 19, every vertex with outdegree at least 2 is an origin of the hamiltonian antidirected outpath. We here improve the bound 19 to the lowest possible: 11.

Corollary 4.6. *Let T be a tournament on $n\geq 11$ vertices and P the hamiltonian antidirected outpath of order n. Every vertex with outdegree at least 2 is an origin of P.*

Proof. Let x be a vertex of T with outdegree at least 2 and y and z two outneighbours of x. Since $n \geq 11$, $(T - x, {}^*P)$ is not an exception, y or z is an origin of ${}^*P$ in $T \setminus \{x\}$. Thus, x is an origin of P in T. ▮

The bound 11 of this corollary is best possible. Indeed, let T be the tournament such that $T \setminus \{x\}$ is the dual of $F_{10}(9)$ with $N^+(x) = X$. The vertex x has outdegree two and is not an origin of the hamiltonian antidirected outpath.

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