Trees in tournaments

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Abstract

A digraph is said to be *n-unavoidable* if every tournament of order n contains it as a subgraph. Let $f(n)$ be the smallest integer such that every oriented tree of order n is $f(n)$-unavoidable. Sumner (see [5]) noted that $f(n) \geq 2n-2$ and conjectured that equality holds. Havet and Thomassé [4] established the upper bound $f(n) \leq \frac{7}{2}n$. Sumner's conjecture is implied by a stronger one due to Havet and Thomassé: $g(k) \leq k-1$ where $g(k)$ is the smallest integer such that every oriented tree of order n with k leaves is $(n+g(k))$-unavoidable. Häggkvist and Thomason [1] proved that $g(k) \leq 2^{512k^3}$ and Havet [2] proved $g(3) \leq 5$. A tree is *constructible* if every path from a node to another node is not directed with first and last block (maximal directed subpath) of length at least two and every path from node to a leaf has first block of length at least two. In this paper, we prove that Havet and Thomassé's conjecture is true for constructible trees.

Key words: tournament, tree, unavoidable.

1 Introduction

A *tournament* is an orientation of a complete graph. An *oriented tree* is an orientation of a tree; in particular, an *oriented path* is an orientation of a path. Throughout this paper, since we only consider oriented trees and oriented paths, we abbreviate 'oriented tree' to 'tree' and 'oriented path' to 'path'. A digraph is said to be *n-unavoidable* if every tournament of order n contains it as a subgraph. Let $f(n)$ be the smallest integer such that every (oriented) tree of order n is $f(n)$-unavoidable. Sumner (see [5]) noted that $f(n) \geq 2n-2$. Indeed, the regular tournaments of order $2n-3$ have no vertex with outdegree greater than $n-2$ and thus do not contain the oriented tree consisting of a root dominating $n-1$ leaves. He also conjectured that equality holds:

Conjecture 1 (Sumner) Every tree of order n is $(2n-2)$-unavoidable, i.e. $f(n) = 2n-2$.

Häggkvist and Thomason [1] proved the existence of a minimal function $g(k) \leq 2^{512k^3}$ such that every tree of order n with k leaves is $(n+g(k))$-unavoidable.

With Stéphan Thomassé, we conjecture the following:

Conjecture 2 (Havet et Thomassé) Every tree of order n with k leaves is $(n+k-1)$-unavoidable, i.e. $g(k) \leq k-1$.

This conjecture implies Sumner's because a tree of order n has at most $n-1$ leaves. Since trees with two leaves are paths, a result of Thomason [6] confirms this conjecture for $k=2$. In addition, Havet [2] proved that $g(3) \leq 5$. More generally, he proved that the merging of k paths is $(n+\frac{3}{2}(k^2-3k)+5)$-unavoidable.

The *degree* of a vertex x in a digraph is the sum of its outdegree and indegree. In other words, it is the degree of x in the underlying unoriented graph. In a tree, the vertices of degree 1 are called the *leaves* and the vertices of degree at least 3 *the nodes*. An *internal* (resp. *external*) *link* is a subpath of T with origin a node and terminus another node (resp. a leaf).

A tree is said to be *constructible* if:

- its external links have first block of length at least two.
- its internal links are not directed and have first (and then also last) block of length at least two.

In this paper, we give evidence for Conjecture 2, by proving it for constructible trees:

Theorem 3 *Every constructible tree with k leaves is $n+k-1$-unavoidable.*

This also gives stronger evidence for Sumner's conjecture. Indeed, the classes of arborescences and constructible trees are almost disjoint: trees in the intersection have one or two nodes, there are directed paths from the root to the nodes and each path from a node to a leaf (which is not the root) is a directed outpath. Moreover, these two classes are "opposite" in the sense that every path is directed from the root in an arborescence and that paths from one node to another are not directed in a constructible tree.

2 Sketch of the proof

The idea behind the proof of Theorem 3 is to consider the constructible trees as small pieces: *cores* that are mergings of directed paths in nodes and paths linking cores to eachother one or to a leaf. We first construct cores and then links.

Definition 4 Let $P=(x_1,\cdots,x_n)$ be a path. We say that x_1 is the *origin* of P and x_n is the *terminus* of P. If $x_1\to x_2$, P is an *outpath*, otherwise P is an *inpath*. The *directed outpath* of order n is the path $P=(x_1,\cdots,x_n)$ in which $x_i\to x_{i+1}$ for all i, $1\le i<n$; the dual notion is that of a *directed inpath*. For $n\ge 3$, the *n-outfork* F_n^+ (resp. *n-infork* F_n^-) is the tree also denoted by $[(x_1,\ldots,x_{n-2}),p_1,p_2]$ such that $(x_1,\ldots,x_{n-2},p_1)$ and $(x_1,\ldots,x_{n-2},p_2)$ are directed outpaths (resp. inpaths). The vertex x_1 is the *origin* and p_1 and p_2 are the *points* of the fork.

Let $P_1,P_2,\ldots P_k$ be forks or paths. The *merging* of $P_1,P_2,\ldots,P_k$, denoted $\bigvee_{i=1}^k P_i$, is the digraph $\bigcup_{i=1}^k P_i'$ where P_i' is isomorphic to P_i with origin x and $V(P_i')\cap V(P_j')=\{x\}$, $1\le i<j\le k$. We say that x is the *origin* of $\bigvee_{i=1}^k P_i$.

Let P be a nondirected path. If $b_1(P)\ge 2$, the *beginning* of P, denoted by $B(P)$, is defined as follows:

$B(P)=B_1(P)$, if there exists i,j such that $P=+(i,1,j)$
$B(P)=B_1(P)^*$ otherwise.

The length of the beginning of P is denoted by $b(P)$. The *remainder* $R(P)$ of P is the subpath of P with origin the end of $B(P)$ and terminus the terminus of P. Note that $R(P)$ has length at least two and has first block of length 1. If $b_1(P^{-1})\ge 2$ then the *end* of P is $E(P)=B(P^{-1})^{-1}$. If both first and last blocks of P have length at least two then the *middle* of P is its subpath $M(P)$ with origin the terminus of $B(P)$ and terminus the origin of $E(P)$. Note that $B(P)$ and $E(P)$ (when defined) are directed paths and that $M(P)$ has first and last block of length 1 and that $M(P)\neq\pm(1,1,1)$.

Let x be a node of a constructible tree A. Let $P_1,\ldots,P_k$ be the links of A with origin x. The *core tree of x in A*, denoted by $C_A(x)$ is $\bigvee_{i=1}^k F_i$ where F_i is an outfork (resp. infork) of order $b(P_i)+2$ if P_i is a nondirected outpath (resp. inpath) and $F_i{=}P_i$ if P_i is directed.

We first find the core-trees corresponding to each node with the following lemma:

Lemma 5 *A merging of order n of k forks and k' directed paths with $k+k'\ge 3$ is $(n+2k+k'-1)$-unavoidable.*

Then we construct internals links: we find each middle, with endvertices the points of the corresponding forks in the core-trees, using the following result of Thomason:

Theorem 6 (Thomason, [6]) *Let P be a nondirected path of order n with* *first and last block of length one. Let T be a tournament of order $n+2$ and X and Y be two disjoints subsets of T of order at least two.*

If $P \neq \pm(1,1,1)$, then there is a copy of P in T with origin in X and terminus in Y.

Finally, we acheive the construction of the tree by completing external links: we find the remainder of each of these with origin among the two points of the forks corresponding to its beginning, using the following theorem due to Thomason (see also [3] for a simpler proof).

Theorem 7 (Thomason, [6]) *Let P be a path of order n with first block of length one and x and y two vertices of a tournament of order $n+1$. Then x or y is an origin of P in T.*

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