On unavoidability of trees with k leaves.

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Abstract

A digraph is said to be *n-unavoidable* if every tournament of order n contains it as a subgraph. Let $f(n)$ be the smallest integer such that every oriented tree of order n is $f(n)$-unavoidable. Sumner (see [6]) noted that $f(n) \geq 2n-2$ and conjectured that equality holds. Havet and Thomassé [4] established the upper bound $f(n) \leq \frac{7}{2}n$. Sumner's conjecture is implied by a stronger one due to Havet and Thomassé: $g(k) \leq k-1$ where $g(k)$ is the smallest integer such that every oriented tree of order n with k leaves is $(n+g(k))$-unavoidable. Häggkvist and Thomason [1] proved that $g(k) \leq 2^{512k^3}$ and Havet [2] proved $g(3) \leq 5$. A tree is *constructible* if every path from a node to another node is not directed with first and last block (maximal directed subpath) of length at least two and every path from node to a leaf has first block of length at least two. In this paper, we prove that Havet and Thomassé's conjecture is true for constructible trees.

keywords: tournament, tree, unavoidable.

1 Introduction.

A *tournament* is an orientation of a complete graph. An *oriented tree* is an orientation of a tree; in particular, an *oriented path* is an orientation of a path. Throughout this paper, since we only consider oriented trees and oriented paths, we abbreviate 'oriented tree' to 'tree' and 'oriented path' to 'path'. A digraph is said to be *n-unavoidable* if every tournament of order n contains it as a subgraph. Let $f(n)$ be the smallest integer such that every (oriented) tree of order n is $f(n)$-unavoidable. Sumner (see [6]) noted that $f(n) \geq 2n-2$. Indeed, the regular tournaments of order $2n-3$ have no vertex with outdegree greater than $n-2$ and thus do not contain the oriented tree consisting of a root dominating $n-1$ leaves. He also conjectured that equality holds:

Conjecture 1 (Sumner) Every tree of order n is $(2n-2)$-unavoidable, i.e. $f(n) = 2n-2$.

Häggkvist and Thomason [1] proved the existence of a minimal function $g(k) \leq 2^{512k^3}$ such that every tree of order n with k leaves is $(n+g(k))$-unavoidable.

With Stéphan Thomassé, we conjecture the following:

Conjecture 2 (Havet et Thomassé) Every tree of order n with k leaves is $(n+k-1)$-unavoidable, i.e. $g(k) \leq k-1$.

This conjecture implies Sumner's because a tree of order n has at most $n-1$ leaves. Since trees with two leaves are paths, a result of Thomason [7] (see Theorem 1) confirms this conjecture for $k=2$. In addition, Havet [2] proved that $g(3) \leq 5$. More generally, he proved that the merging of k paths is $(n+\frac{3}{2}(k^2-3k)+5)$-unavoidable.

The *degree* of a vertex x in a digraph is the sum of its outdegree and indegree. In other words, it is the degree of x in the underlying unoriented graph. In a tree, the vertices of degree 1 are called the *leaves* and the vertices of degree at least 3 *the nodes*. An *internal* (resp. *external*) *link* is a subpath of T with origin a node and terminus another node (resp. a leaf).

A tree is said to be *constructible* if:

- its external links have first block of length at least two.
- its internal links are not directed and have first (and then also last) block of length at least two.

In this paper, we give evidence for Conjecture 2, by proving it for constructible trees (Corollary 5). This also gives stronger evidence for Sumner's conjecture. Indeed, the classes of arborescences and constructible trees are almost disjoint: trees in the intersection have one or two nodes, there are directed paths from the root to the nodes and each path from a node to a leaf (which is not the root) is a directed outpath. Moreover, these two classes are "opposite" in the sense that every path is directed from the root in an arborescence and that paths from one node to another are not directed in a constructible tree.

The idea behind the proof of Corollary 5 is to consider the constructible trees as small pieces: *cores* that are mergings of directed paths in nodes and paths linking cores to eachother one or to a leaf. We first construct cores and then links. The following three sections are devoted to proving Lemma 4.4. This asserts the unavoidability of some trees corresponding to the cores that guarantee us afterwards the existence of links.

2 Definitions and preliminaries.

Let T be a tournament. Let x and y be two vertices of T. We write $x \to y$ if (x,y) is an arc of T. In the same way, let X and Y be two subdigraphs of T. We write $X \to Y$ if $x \to y$ for all pairs $(x,y) \in V(X) \times V(Y)$. Let $A_1, A_2, \ldots, A_k$ be a family of sets of vertices of T. We denote by $T[A_1, A_2, \ldots, A_k]$ the subtournament induced by T on the set of vertices $\bigcup_{1\leq i\leq k} V(A_i)$ and by $T-[A_1, A_2, \ldots, A_k]$ the subtournament induced by T on the set of vertices $V(T)\setminus \bigcup_{1\leq i\leq k} V(A_i)$. We often abbreviate $T-[A_1]$ to $T-A_1$ and $T-[\{x\}]$ to $T-x$.

Let $P=(x_1,\cdots,x_n)$ be a path. We say that x_1 is the *origin* of P and x_n is the *terminus* of P. If $x_1 \to x_2$, P is an *outpath*, otherwise P is an *inpath*. The *directed outpath* of order n is the path $P=(x_1,\cdots,x_n)$ in which $x_i \to x_{i+1}$ for all i, $1 \leq i < n$; the dual notion is that of a *directed inpath*. The *length* of a path is its number of edges. We denote the path $(x_2,\cdots,x_n)$ by ${}^*P$.

The *blocks* of P are the maximal subdipaths of P. We enumerate the blocks of P from the origin to the terminus. The first block of P is denoted $B_1(P)$ and its length $b_1(P)$. Likewise, the i^{th} block of P is denoted $B_i(P)$ and its length $b_i(P)$. The path P is totally described by the signed sequence $sgn(P)(b_1(P), b_2(P), \cdots, b_k(P))$ where k is the number of blocks of P and $sgn(P) = +$ if P is an outpath and $sgn(P) = -$ if P is an inpath.

Let X be a set of vertices of T. The *outsection* generated by X in T is the set of vertices to which there exists a directed outpath (possibly restricted to a single vertex) from $x \in X$; we denote this set by $S_T^+(X)$ (or by $S_T^+(x)$ if $X = \{x\}$ and $S_T^+(x,y)$ if $X = \{x,y\}$). Note that $X \subseteq S_T^+(X)$. The dual notion, the *insection*, is denoted by $S_T^-(X)$. We also write $s_T^+(X)$ (resp. $s_T^-(X)$) for the number of vertices of $S_T^+(X)$ (resp. $S_T^-(X)$). An *outgenerator* of T is a vertex x of T such that $S_T^+(x) = V(T)$; the dual notion is an *ingenerator*.

The following proposition (whose proof is omitted) is an easy consequence of the definition of an outgenerator:

Proposition 2.1 *A vertex is an origin of a hamiltonian directed outpath in a tournament T if and only if it is an outgenerator of T.*

For $n \geq 3$, the *n-outfork* F_n^+ (resp. *n-infork* F_n^-) is the tree also denoted by $[(x_1, \ldots, x_{n-2}), p_1, p_2]$ such that $(x_1, \ldots, x_{n-2}, p_1)$ and $(x_1, \ldots, x_{n-2}, p_2)$ are directed outpaths (resp. inpaths). The vertex x_1 is the *origin* and p_1 and p_2 are the *points* of the fork.

Let us denote by ST_4 the unique strong tournament of order four (see Figure 1). The *bad vertices* of ST_4 are its two vertices of outdegree 1. Let T_5 be the other tournament in Figure 1. It consists of a vertex x and a ST_4 such that x dominates the bad vertices of x and is dominated by the others. The *bad vertices* of T_5 are x and y as depicted in Figure 1.

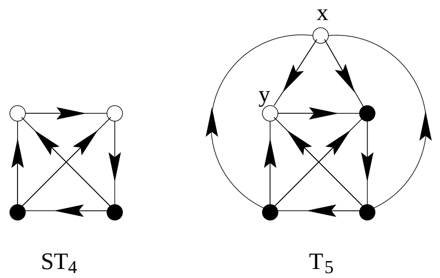

Figure 1: Tournaments ST_4 and T_5. The white vertices are the bad ones.

Lemma 2.1 *Let T be a $(n+1)$-tournament and x and y two vertices of T. If $s_T^+(x,y) = n+1$ then x or y is the origin of the n-outfork in T unless T is T_4 or T_5 and x and y are the bad vertices of T.*

Proof: By induction on n. If $n = 3$, the result is easy to check.

Suppose now that $n \geq 4$.

By duality, we may suppose that $x \to y$.

If $d^+(x) \geq 2$, by the inductive hypothesis, x is an origin of the n-outfork F_n^+ unless the outneighbours of x are the bad vertices in $ST_4 = T - x$ or $T_5 = T - x$. If $T - x = ST_4$, then $T = T_5$: it is then easy to check that every non-bad vertex of T_5 is an origin of F_4^+. If $T - x = T_5$, it is easy to check that in T every vertex distinct from x is an origin of F_5^+.

So we may assume that y is the unique outneighbour of x. Now $s^+_{T-x}(y) = s^+_T(x, y) - 1 = n$. So y is an origin of a directed outpath of order n. Thus y is an origin of F_n^+. ■

Corollary 1 *The outfork of order n is $(n + 1)$-unavoidable.*

The bound $n + 1$ in the previous corollary is tight: every reducible tournament of order n whose maximal component is a 3-circuit does not contains the n-outfork.

Corollary 2 *Let $E = \{x, y, z\}$ be a set of three vertices of a tournament and F an outfork. If $s^+_T(E) \geq |F| + 1$ then a vertex of E is an origin of F in T.*

Corollary 3 *Let F be an outfork and T a tournament. If $s^+_T(x) \geq |F| + 1$ and $d^+_T(x) \geq 3$ then x is an origin of F in $T[S^+(x)]$.*

3 Mergings of outforks and directed outpaths.

Definition 1 Let $P_1, P_2, \ldots P_k$ be forks or paths. The *merging* of $P_1, P_2, \ldots, P_k$, denoted $\bigvee_{i=1}^k P_i$, is the digraph $\bigcup_{i=1}^k P_i'$ where P_i' is isomorphic to P_i with origin x and $V(P_i') \cap V(P_j') = \{x\}$, $1 \leq i < j \leq k$. We say that x is the *origin* of $\bigvee_{i=1}^k P_i$.

A tree is *suitable* if it is a merging of k_1 forks of length two, k_2 directed outpaths of length two and k_3 directed outpaths of length one with $k_1 \leq k_3$.

In [5], Lu proved the existence of a spanning suitable tree in a tournament by proving the following lemma. See also [2], for a short elementary proof due to Bondy.

Lemma 3.1 (Lu, [5]) *Let T be a tournament. Every vertex with maximum outdegree is an origin of a spanning suitable tree.*

Lemma 3.2 *Let T be a tournament. Every origin of a suitable tree of order $n' \geq n + 2$ in T is an origin of the merging of a n-outfork and a suitable tree of order $n' - n - 2$.*

Proof: Let A be a spanning suitable tree with origin x, that is the merging of 4-outforks $F_1, F_2, \ldots, F_{k_1}$, directed outpaths $R_1, R_2, \ldots, R_{k_2}$ of length two and directed outpaths $Q_1, Q_2, \ldots, Q_{k_3}$ of legth one.

Let us define the tournament T_1 of order $m = n + 2$ as follows:

If $3k_1 + 1 \geq m$, let i be the smallest integer such that $3i + 1 \geq m$.
 If $3i + 1 = m$, set $T_1 := T[F_1, F_2, \ldots, F_i]$.
 If $3i + 1 = m + 2$, set $T_1 := T[F_1, F_2, \ldots, F_{i-1}, Q_{k_3}, y]$, with y a point of F_i.
 If $3i + 1 = m + 1$, set $T_1 := T[F_1, F_2, \ldots, F_{i-1}, Q_{k_3}]$.
If not, $3k_1 < m$.

If $3k_1 + 2k_2 + 1 \geq m$, let i be the smallest integer such that $3k_1 + 2i + 1 \geq m$.
If $3k_1 + 2i + 1 = m$, set $T_1 = T[F_1, F_2, \dots, F_{k_1}, R_1, \dots, R_i]$.
If $3k_1 + 2i + 1 = m + 1$, set $T_1 = T[F_1, F_2, \dots, F_{k_1}, R_1, \dots, R_{i-1}, z]$ with z the terminus of R_i.
If not $3k_1 + 2k_2 + 1 < m$.
Let i be the integer such that $3k_1 + 2k_2 + i + 1 = m$.
Set $T_1 = T[F_1, F_2, \dots, F_{k_1}, R_1, \dots, R_{k_2}, Q_1, \dots, Q_i]$.

We have $|T_1| = n + 2$ and $s^+_{T_1}(x) \geq n + 1$. Note that $d^+_{T_1}(x) \geq 2$ since $|T_1| \geq 5$. Let us prove that x is an origin of F^+_n in T_1. If $d^+_{T_1}(x) \geq 3$ (it is the case if $|F^+_n| \geq 6$), then by Theorem 2, x is an origin of F^+_n in T_1. Suppose now that $d^+_{T_1}(x) = 2$. If $n = 3$, x is trivially an origin of F^+_3. If $n = 4$, then $|T_1| = 6$ and so, by definition of T_1, the fork $F_1 = F^+_4$. If $n = 5$, then $T_1 = T(F_1, F_2)$. Set $F_1 = [(x, x_1), u_1, v_1]$ and $F_2 = [(x, x_2), u_2, v_2]$. Without loss of generality, we may suppose that $x_1 \to x_2$. Hence, x is an origin of $F^+_5 = [(x, x_1, x_2), u_2, v_2]$.

Now consider A', the tree induced on A by the vertices of $V(T - T_1) \cup x$. It is easy to see that A' is a suitable tree of order $n' - n - 2$. Hence, since x is an origin of $O \vee A'$, we have the result. ∎

Analogously to this lemma, one can prove the following:

Lemma 3.3 *Let T be a tournament. Every origin of a suitable tree of order $n' \geq n + 1$ in T is an origin of the merging of a directed outpath of order $n \geq 3$ and a suitable tree of order $n' - n - 1$.*

Proof: Let us define T_1 in the same way as in the proof of Lemma 3.2, but with $m = n+1$. We have $s^+_{T_1}(x) \geq n$. Then by Proposition 2.1, x is an origin of a directed n-outpath in T_1. And A', the tree induced on A by the vertices of $V(T - T_1) \cup x$, is a suitable tree of order $n' - n - 1$. ∎

Lemma 3.4 *Let T be a tournament and R the merging of order n of k_1 outforks and k'_1 directed outpaths. Every origin of a suitable tree of order $n + 2k_1 + k'_1 - 1$ in T is an origin of R in T.*

Proof: We prove the result by induction. Due to Lemma 3.2 and 3.3, we just need to prove it if R is either a directed outpath or a fork.

If R is a directed outpath O, Proposition 2.1 gives the result for $s^+_T(x) \geq |O|$.

Suppose now that R is a fork. If $d^+_T(x) \geq 3$, by Lemma 3, x is an origin of R in T. If $d^+_T(x) = 2$, then either $n = 3$ and the result hold trivially or $n = 4$. In the latter case, either x is an origin of a 4-outfork or $A = (x, u_1, u_2) \vee (x, v_1, v_2)$ with (x, u_1, u_2) and (x, v_1, v_2) two directed outpaths. Without loss of generality, we may suppose that $u_1 \to u_2$. Then x is an origin of $F^+_4 = [(x, u_1), u_2, v_1]$. ∎

Corollary 4 *The merging of order n of k_1 outforks and k'_1 directed outpaths is $(n+2k_1+k'_1-1)$-unavoidable.*

4 Mergings of forks and directed paths.

Lemma 4.1 *Let O be an outfork or a directed outpath of order $n_1 \geq 3$, I be an infork or a directed inpath of order $n_2 \geq 3$ and T be a tournament of order at least $n_1 + n_2 + 2$. If x is an origin of O in T and an origin of I in T then x is an origin of $O \vee I$ in T.*

Proof:

By duality, we may suppose that $d_T^-(x) \geq n_2 + 1$.

Set $A := N_T^+(x) \setminus V(O)$. If $|A| \leq 1$, then $d_{T-^*O}^-(x) \geq n_2$. Thus, by Corollary 3 and Proposition 2.1, x is an origin of I in $T -^* O$. Hence, x is an origin of $O \vee I$ in T.

If $|A| \geq 2$ then $d_T^+(x) \geq 3$ and $s_T^+(x) \geq n_1 + 2$. Let us consider a set S of vertices of $T - x$ of order $n_1 + 1$ such that x is an outgenerator of $T[S, x]$ and that contains an inneighbour of x if and only if $N_T^+(x) \subset S$. We have $d_{T-S}^-(x) \geq n_2 + 1$. Then by Corollary 3 and Proposition 2.1, x is an origin of O in $T[S, x]$ and an origin of I in $T - S$. Hence, x is an origin of $O \vee I$ in T. ■

Lemma 4.2 *Let R_1 be a merging of order n_1 of outforks and directed outpaths of length at least two, R_2 a merging of order n_2 of inforks and directed inpaths of length at least two and T be a tournament of order at least $n_1 + n_2 + 2$. If in T, x is an origin of R_1 and an origin of R_2 then x is an origin of $R_1 \vee R_2$.*

Proof: Set $R_1 := \bigvee_{i=1}^{l} O_i$ and $R_2 := \bigvee_{j=1}^{k} I_j$. By duality, we may suppose that $d^-(x) \geq n_2 + 1$.

We will prove by induction on k, the number of paths of R_2, that each origin x of R_1 in T such that $d^-(x) \geq n_2 + 1$ is an origin of $R_1 \vee R_2$.

- If $k = 1$ then set $R_2 = I$. Let us prove the result by induction on l, the number of paths of R_1.

  If $l = 1$, Lemma 4.1 yields the result.

  If $l > 1$, set $q := d_{T-^*O_l}^-(x)$ and $R_1' := \bigvee_{i=1}^{l-1} O_i$. If $q \geq n_2 + 1$, then x is an origin of $R_1' \vee I$ in $T -^* O_l$ by the inductive hypothesis. So x is an origin of $R_1 \vee Q$. Hence, we may assume that $q \leq n_2$. Then $d_{T-^*O_l}^+ \geq |R_1'|$ and thus, by Corollary 3 and Proposition 2.1, x is an origin of R_1' in $T[N_{T-^*O_l}^+, x]$. Now, in $T - [^*R_1']$, x has indegree $n_2 + 1$. So, by Corollary 3 and Proposition 2.1, x is an origin of I in $T - [^*R_1']$. Therefore, by Lemma 4.1, x is an origin of $O_l \vee I$ in $T - [^*R_1']$. So x is an origin of $R_1 \vee I$.

- If $k \geq 2$, set $R_2' := \bigvee_{j=1}^{k-1} I_j$. By the previous case, in T, x is an origin of $R_1 \vee I_k$. In $T -^* I_k$, x is an origin of R_1 and $d^-(x) \geq n_2 + 1 - |^*I_k| = |R_2'| + 1$. Thus, by the inductive hypothesis, x is an origin of $R_1 \vee R_2'$ in $T -^* I_k$. Therefore, x is an origin of $R_1 \vee R_2$ in T. ■

Analogously to this result, one can prove the following sharper lemma for a merging of directed paths:

Lemma 4.3 *Let R_1 be a merging of order n_1 of directed outpaths of length at least two, R_2 be a merging of order n_2 of directed inpaths of length at least two and T be a tournament of order at least $n_1 + n_2 + 1$. If in T, x is an origin of R_1 and an origin of R_2 then x is an origin of $R_1 \vee R_2$.*

Lemma 4.4 *A merging of order n of k forks and k' directed paths with $k + k' \geq 3$ is $(n + 2k + k' - 1)$-unavoidable.*

Proof: By Corollary 4 and its dual, we have the result if R is a merging of outforks and directed outpaths or inforks and directed inpaths. So we may suppose that R contains an infork or directed inpath and an outfork or directed outpaths. Let $R = R_1 \vee R_2$ with R_1 a merging of k_1 outforks and k'_1 directed outpaths and R_2 a merging of $k_2 \geq 1$ inforks and k'_2 directed inpaths. Let n_i be the order of R_i. Note that $n_1 + n_2 = n + 1$ so T has $n_1 + n_2 + 2(k_1 + k_2) + k'_1 + k'_2 - 2$ vertices.

Since R_1 is $(n_1 + 2k_1 + k'_1 - 1)$-unavoidable, there are $n_2 + 2k_2 + k'_2$ origins of R_1 in T. Identically, there are $n_1 + 2k_1 + k'_1$ origins of R_2 in T. Thus there is a vertex x which is an origin of R_1 and an origin of R_2 in T. Now, T has at least $n + 2$ vertices and at least $n + 3$ if R contains a fork. Hence, by Lemmas 4.3 and 4.2, x is an origin of $R_1 \vee R_2$ in T. ■

5 Unavoidability of constructible trees.

Definition 2 Let P be a nondirected path. If $b_1(P) \geq 2$, the *beginning* of P, denoted by $B(P)$, is defined as follows:

$B(P) = B_1(P)$, if there exists i, j such that $P = +(i, 1, j)$

$B(P) = B_1(P)^*$ otherwise.

The length of the beginning of P is denoted by $b(P)$. The *remainder* $R(P)$ of P is the subpath of P with origin the end of $B(P)$ and terminus the terminus of P. Note that $R(P)$ has length at least two and has first block of length 1.
If $b_1(P^{-1}) \geq 2$ then the *end* of P is $E(P) = B(P^{-1})^{-1}$. If both first and last blocks of P have length at least two then the *middle* of P is its subpath $M(P)$ with origin the terminus of $B(P)$ and terminus the origin of $E(P)$. Note that $B(P)$ and $E(P)$ (when defined) are directed paths and that $M(P)$ has first and last block of length 1 and that $M(P) \neq \pm(1, 1, 1)$.

Let x be a node of a constructible tree A. Let $P_1, \ldots, P_k$ be the links of A with origin x. The *core tree of x in A*, denoted by $C_A(x)$ is $\bigvee_{i=1}^{k} F_i$ where F_i is an outfork (resp. infork) of order $b(P_i) + 2$ if P_i is a nondirected outpath (resp. inpath) and $F_i{=}P_i$ if P_i is directed.

In order to prove Theorem 3, we need the following two results due to Thomason [7]. (See also [3] for a simpler proof of Theorem 1).

Theorem 1 (Thomason, [7]) *Let P be a path of order n with first block of length one and x and y two vertices of a tournament of order $n + 1$. Then x or y is an origin of P in T.*

Considering Theorems 3 and 5 of [7] for paths with first and last block of length one, we obtain the following:

Theorem 2 (Thomason, [7]) *Let P be a nondirected path of order n with first and last block of length one. Let T be a tournament of order $n + 2$ and X and Y be two disjoints subsets of T of order at least two.*

If $P \neq \pm(1, 1, 1)$, then there is a copy of P in T with origin in X and terminus in Y.

Theorem 3 *Every constructible tree with p internal links with node of degree at most $d(\geq 3)$ is $(n+p+d-1)$-unavoidable.*

Proof: Let T be a tournament of order $n+p+d+1$ and A a tree of order n with p internal links $P_1, P_2, \ldots, P_p$ and nodes of degree at most d. Let F_0 be the forest which is the union of the core-trees of the nodes of A. Let M_i $1 \leq i \leq p$ be the middle of P_i and m_i its order. Let $Q_1, Q_2, \ldots, Q_e$ the non-directed external links and $R_1, R_2, \ldots, R_e$ their remainder with respectives orders $r_1, r_2, \ldots r_e$.

The forest F_0 has order at most $n+p-e$. Therefore, by Lemma 4.4, F_0 is $(n+p+d-1)$ unavoidable.

For $1 \leq i \leq p$, let F_i be the forest obtained from F_{i-1} by replacing the forks B_i and E_i corresponding to the beginning and the end of P_i in F_{i-1} by P_i itself. Let us prove, by induction, that T contains F_i. It is easy to check that $n+p+d-1 \geq |F_i|+m_i-2$. Thus, there are at least m_i-2 vertices in $T' = T - F_i$. Let b_1 and b_2 (resp. e_1 and e_2) be the points of the fork B_i (resp. E_i). Now $T' + (b_1, b_2, e_1, e_2)$ has at least m_i+2 vertices. Thus, by Theorem 2, in T', there is a path M_i with origin in $\{b_1, b_2\}$ and terminus in $\{e_1, e_2\}$. Hence T contains F_i.

Let $A_0 = F_p$. Clearly F_p is a tree. For $1 \leq i \leq e$, let A_i be the tree obtained from A_{i-1} by replacing the forks B'_i corresponding to the beginning of Q_i in A_{i-1}. Since $n+p+d-1 \geq |A_i|+r_i-1$, in the same way as previously, by Theorem 1, we prove by induction that T contains A_i, for $0 \leq i \leq e$.

Since $A_e = A$, we have the result. ■

From this result, it follows immediately that Conjecture 2 holds for constructible trees:

Corollary 5 *Every constructible tree with k leaves is $n+k-1$-unavoidable.*

Proof: A tree with degree d and p internal links has at least k leaves. ■

Remark 1 Note that for most of the trees, we may obtain a better bound than $n+p+d-1$ in Theorem 3. For example, if A is a constructible tree with no directed external path and such that the middle (resp. remainder) of each internal (resp. external) link has order at least three, then A is $(n+2)$-unavoidable. Indeed F_0 has order at least $n-2e$ and therefore is n-unavoidable; analogously to the end of proof of Theorem 3, we obtain the $(n+2)$-unavoidability of A.

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