Oriented Hamiltonian Cycles in Tournaments

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We prove that every tournament of order \( n \geq 68 \) contains every oriented Hamiltonian cycle except possibly the directed one when the tournament is reducible.

1. INTRODUCTION

1.1. Definitions

Definition 1.1. A tournament is an orientation of the arcs of a complete graph. An oriented path is an orientation of a path. An oriented cycle is an orientation of a cycle. From now on, we abbreviate oriented path and oriented cycle to cycle.

Let \( T \) be a tournament and let \( D \) be a digraph. We say that \( T \) contains \( D \) if \( D \) is a subgraph of \( T \). The order of a digraph \( D \), denoted by \(|D|\), is its number of vertices. Let \( x \) and \( y \) be two vertices of \( T \). We write \( x \to y \) if \((x, y)\) is an arc of \( T \). Likewise, let \( X \) and \( Y \) be two subdigraphs of \( T \). We write \( X \to Y \) if \( x \to y \) for all pairs \((x, y) \in V(X) \times V(Y)\). Let \( A_1, A_2, ..., A_k \) be a family of subdigraphs of \( T \). We denote by \( T(A_1, A_2, ..., A_k) \) the subtournament induced by \( T \) on the set of vertices \( V(T) \setminus \bigcup_{1 \leq i \leq k} V(A_i) \) and by \( T-(A_1, A_2, ..., A_k) \) the subtournament induced by \( T \) on the set of vertices \( V(T) \setminus \bigcup_{1 \leq i \leq k} V(A_i) \).

Let \( P=(x_1, ..., x_n) \) be a path. We say that \( x_1 \) is the origin of \( P \) and \( x_n \) is the terminus of \( P \). If \( x_1 \to x_2, P \) is an outpath, otherwise \( P \) is an inpath. The directed outpath of order \( n \) is the path \( P=(x_1, ..., x_n) \) in which \( x_i \to x_{i+1} \) for all \( i, 1 \leq i < n \); the dual notion is the directed inpath; both directed out- and inpaths are dipaths. The length of a path is its number of arcs. We denote the subpath \((x_1, ..., x_{n-1})\) of \( P \) by \( P^* \) and the subpath \((x_2, ..., x_n)\) of \( P \) by \( P^- \). The path \((x_n, x_{n-1}, ..., x_1)\) is denoted \( P^{-1} \).
The blocks of $P$ are the maximal subdipaths of $P$. We enumerate the blocks of $P$ from the origin to the terminus. The $i$th block of $P$ is denoted $b_i(P)$ and its length $b_i(P)$. The path $P$ is totally described by the signed sequence $\text{sgn}(P) = (b_1(P), b_2(P), ..., b_k(P))$ where $k$ is the number of blocks of $P$ and $\text{sgn}(P) = +$ if $P$ is an outpath and $\text{sgn}(P) = -$ if $P$ is an inpath.

A block is positive or negative according to the sense of its arcs.

We define $A^+(P) := |\{i, x_i \rightarrow x_{i+1}\}|$ and $A^-(P) := |\{i, x_i \leftarrow x_{i+1}\}|$.

Let $C = (x_0, ..., x_n = x_0)$ be a cycle. If for all $i, x_i \rightarrow x_{i+1}$, $C$ is said to be directed. If $C$ is not a directed cycle, the blocks of $C$ are its maximal subdipaths. Notice that such a cycle necessarily has an even number of blocks, half positive and half negative, and is totally described by the signed sequence $(b_1, b_2, ..., b_k)$ where $2k$ is the number of blocks of $C$.

The dual (sometimes called the converse) digraph of $D$ is the digraph $-D$ on the same set of vertices such that $x \rightarrow y$ is an arc of $-D$ if and only if $y \rightarrow x$ is an arc of $D$.

A tournament is strong (or strongly connected) if for any two vertices $x$ and $y$ there exists a directed outpath with origin $x$ and terminus $y$. A nonstrong tournament is said to be reducible. A tournament $T$ is reducible if and only if it admits a decomposition into two subtournaments $T_1$ and $T_2$ such that $V(T_1) \cup V(T_2) = V(T)$ and $T_1 \rightarrow T_2$; in this case, we often write $T = T_1 \rightarrow T_2$.

A tournament is $k$-strong, if $T - Y$ is strong for any set $Y$ of $k - 1$ vertices. A tournament is $(\geq k)$-strong or exactly $k$-strong, if it is $k$-strong and not $(k + 1)$-strong. A (strong) component of $T$ is a strong subtournament of $T$ which is maximal by inclusion.

Let $X$ be a set of vertices of $T$. The outsection generated by $X$ in $T$ is the set of vertices $y$ to which there exists a directed outpath from $x \in X$; we denote this set by $S^+_T(X)$ (or by $S^+_Y(x)$ if $X = \{x\}$ and $S^+_Y(x, y)$ if $X = \{x, y\}$). The dual notion, the insection, is denoted by $S^-_T(X)$. We also write $s^+_T(X)$ (resp. $s^-_T(X)$) for the number of vertices of $S^+_T(X)$ (resp. $S^-_T(X)$). An outgenerator of $T$ is a vertex $x \in T$ such that $S^+_T(x) = V(T)$; the dual notion is an ingenerator. We denote by $\text{Out}(T)$ the set of outgenerators of $T$ and by $\text{In}(T)$ the set of ingenerators of $T$. Notice that for a strong tournament $\text{Out}(T) = V(T) = \text{In}(T)$.

Conventions for drawings. All the figures in this paper represent tournaments. Subtournaments are often delimited by ellipses. If $T_1$ and $T_2$ are subtournaments such that $T_1 \rightarrow T_2$, we draw an arc from the ellipse delimiting $T_1$ and to the one delimiting $T_2$. An undrawn arc represents one that may be oriented either way.
1.2. Presentation of Results

One of the first results on cycles in tournaments is Camion's theorem [4]: a tournament has a directed Hamiltonian cycle if and only if it is strongly connected. More generally, one can seek arbitrary orientations of cycles. In 1974, Rosenfeld conjectured the following:

Rosenfeld's cycle conjecture. There is an integer \( N > 8 \) such that every tournament of order \( n \geq N \) contains every nondirected cycle of order \( n \).

An analogous conjecture of Rosenfeld for paths was recently proved by Havet and Thomassé [7]:

**Theorem 1.1** (Havet and Thomassé [7]). Let \( T \) be a tournament of order \( n \) and \( P \) a path of order \( k \leq n \). Then \( T \) contains \( P \) if and only if \((T; P)\) is not one of Grünbaum's exceptions.

Grünbaum's exceptions are \((3A; \pm (1, 1)), (5A; \pm (1, 1, 1, 1)), \) and \((7A; \pm (1, 1, 1, 1))\) where \( 3A \) is the 3-cycle, \( 5A \) the regular tournament on five vertices, and \( 7A \) the Paley tournament on seven vertices. These three tournaments are called Grünbaum's tournaments and are depicted in Fig. 1.

The existence of Grünbaum's exceptions implies the existence of tournaments that do not contain certain Hamiltonian cycles. Indeed Grünbaum's tournaments do not contain the cycle obtained from a Hamiltonian alternating path by adding an arc from its terminus to its origin (exceptions \( \mathcal{C}_1 \), \( \mathcal{C}_2 \), and \( \mathcal{C}_3 \) of Fig. 2). Moreover, the tournaments of order \( n \) that have a subtournament on \( n - 1 \) vertices isomorphic to one of Grünbaum's tournaments do not contain a Hamiltonian alternating cycle (exceptions \( \mathcal{C}_4 \), \( \mathcal{C}_5 \), and \( \mathcal{C}_6 \)).

Rosenfeld's cycle conjecture has been proved for cycles with a block of length \( n - 1 \) by Grünbaum [5], for alternating cycles (in which consecutives arcs have opposite senses) by Thomassen [12] (\( n \geq 50 \)),

![Fig. 1. Grünbaum's tournaments.](image-url)
Rosenfeld [10] \((n \geq 28)\), and Petrović [9] \((n \geq 16)\), and for cycles with just two blocks by Benhocine and Wojda [2]. Rosenfeld’s conjecture was finally proved in full generality by Thomason [11] for tournaments of order \(n \geq 2^{128}\). While Thomason made no attempt to sharpen this bound, he indicated that it should be true for tournaments of order at least 9.

We here prove that every tournament of order \(n \geq 68\) contains every oriented Hamiltonian cycle except possibly the directed one (when the tournament is reducible). We first show in Theorem 3.1 that every reducible tournament of order \(n \geq 9\) contains every nondirected Hamiltonian cycle. This allows us to restrict ourselves to strong tournaments. We next prove (Theorem 4.3) that every \((=k)\)-strong tournament of order \(n \geq 9\) contains every nondirected Hamiltonian cycle containing a block of length at most \(k + 2\). Then, we exhibit a function \(f\) depending on \(L\) and \(k\) such that every \((=k)\)-strong tournament of order at least \(f(L,k)\) contains every Hamiltonian cycle whose blocks have length at most \(L\).

From this function and Theorem 4.3, we derive a function \(g\) such that every \((=k)\)-strong tournament of order \(g(k)\) contains every Hamiltonian cycle. Alas, this function tends to infinity with \(k\). So it does not directly imply Rosenfeld’s cycle conjecture, but it allows one to prove it for \(k\)-strong tournaments, for some \(k\). Theorem 6.1 asserts that every \(8\)-strong tournament contains every Hamiltonian cycle, implying Rosenfeld’s cycle conjecture is true for \(N = 264\) (Corollary 6.2). Finally, we study Hamiltonian cycles in...
5-strong and (=4)-strong tournaments in order to sharpen this bound to \( N = 68 \) (Corollary 8.2).

2. THE TOOLS: PATHS IN TOURNAMENTS

2.1. Paths with Prescribed Origin

**Theorem 2.1 (Havet and Thomassé [7]).** Let \( T \) be a tournament of order \( n \), \( P \) an outpath of order \( k < n \), and \( x \) and \( y \) two vertices of \( T \) such that \( s^+_T(x, y) \geq b(T) + 1 \). Then \( x \) or \( y \) is an origin of \( P \) in \( T \).

**Definition 2.1.** The exception \( E_{49} \) is the pair \((8A; + (1, 1, 1, 1, 1, 1, 1))\), where \( 8A \) is the tournament depicted in Fig. 3. One can check that the vertices 1 and 2 are not origins of \(+ (1, 1, 1, 1, 1, 1)\).

The exception \( E_{50} \) is the pair \((8A; + (2, 1, 1, 1, 1, 1))\), where \( 8A \) is the tournament depicted in Fig. 3. One can check that the vertices 1 and 2 are not origins of \(+ (2, 1, 1, 1, 1, 1)\).

The exception \( E_{51} \) is the pair \((8B; +(2, 1, 2, 1, 1))\), where \( 8B \) is the tournament depicted in Fig. 3. One can check that the vertices 2 and 8 are not origins of \(+ (2, 1, 2, 1, 1)\).

We introduce here the families of exceptions \( \mathcal{F} \) of Theorem 2.2. Our notation \( \mathcal{F}(n) \) represents an exception \((F_i(n), P)\), where \( F_i(n) \) is the tournament on \( n \) vertices depicted as \( F_i \) in Fig. 4. In each case, we denote by \( S \) the set of vertices of \( F_i(n) \) which are not origins of \( P \) in \( \mathcal{F}(n) \). Finally, conditions are given on the tournament.

**FIG. 3.** The tournaments \( 8A \) and \( 8B \).
FIG. 4. The tournaments $F_i$.

$F_1(n) = (F_1(n); + (1, n - 2)); S = \{1, 2, 3\}$. Condition: $|X| \geq 1$.

$F_2(n) = (F_2(n); + (2, n - 3)); S = \{3, 4\}$. Condition: $|X| \geq 1$.

$F_3(n) = (F_3(n); + (1, n - 2)); S = \{1, 3\}$. Conditions: $N^+(3) \neq \{2\}$, and 3 is an ingenerator of $T(X)$.

$F_4(n) = (F_4(n); + (2, n - 3)); S = \{1, 4\}$. Conditions: $N^+(3) \neq \{2\}$, and 3 is an ingenerator of $T(X)$. 

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Theorem 2.2 (Havet and Thomassé [7]). Let $T$ be a tournament of order $n \geq 8$ and $P$ an outpath of order $k \leq n$. For every two vertices $x$ and $y$ of $T$ such that $\sigma^*_P(x, y) = b_1(P) + 1$, $x$ or $y$ is an origin of $P$ in $T$ if and only if $(T; P)$ is not an exception $\mathcal{E}_{ab}$, $\mathcal{E}_{sg}$, $\mathcal{E}_{sl}$, $\mathcal{F}_i(n)$, $1 \leq i \leq 4$, or $\mathcal{F}_i(n)$, $i = 8, 9, 10$.

2.2. Paths with Prescribed Origin and Terminus

**Definition 2.2.** The exception $\mathcal{P}_1(n)$ is the pair $(E_1(n); Q_1(n))$, where $E_1(n)$ is a tournament of order $n$ depicted in Fig. 5, with $T_1$ an arbitrary tournament of order $n - 3$ and $Q_1(n) = +(n - 2, 1)$.

The exception $\mathcal{P}_2$ is the pair $(E_1(4); Q_2)$, where $E_1(4)$ is the tournament of order $4$ depicted in Fig. 5 and $Q_2 = +(1, 1, 1)$. 

$\mathcal{F}_1(n) = (E_1(n); +(1, n - 3)); S = \{1, 2\}$. Condition: $|X| \geq 2$.

$\mathcal{F}_2(n) = (E_1(n); +(2, n - 2)); S = \{1, 2\}$. Conditions: $n \geq 5$, $|Y| \geq 2$, and 2 is an ingenerator of $T(Y)$.

$\mathcal{F}_3(n) = (E_1(n); +(2, n - 3)); S = \{1, 3\}$. Conditions: $|Y| \geq 2$, and 2 is an ingenerator of $T(Y)$.

$\mathcal{F}_4(n) = (E_1(n); +(1, 1, n - 3)); S = \{2, 3\}$. Conditions: $T(Y)$ is not a 3-cycle (i.e., $T(Y)$ is not isomorphic to $3A$), and $|Y| \geq 3$.

$\mathcal{F}_5(n) = (E_1(n); +(n - 2, 1, 1, 1)); S = X$. Conditions: 3A is the 3-cycle, its set of vertices is $\{1, 2, 3\}$, and $|X| \geq 2$.

$\mathcal{F}_6(n) = (E_1(n); +(n - 3, 2, 1, 1)); S = X$. Conditions: 3A is the 3-cycle, its set of vertices is $\{1, 2, 3\}$, and $|X| \geq 2$.

$\mathcal{F}_7(n) = (E_1(n); +(n - 6, 2, 1, 1, 1)); S = X$. Conditions: 5A is the 2-regular tournament, its set of vertices is $\{1, 2, 3, 4, 5\}$; we need moreover $|X| \geq 2$.

$\mathcal{F}_8(n) = (E_1(n); +(n - 6, 2, 1, 1, 1)); S = X$. Conditions: 5A is the 2-regular tournament, its set of vertices is $\{1, 2, 3, 4, 5\}$, and $|X| \geq 2$.

$\mathcal{F}_9(n) = (E_1(n); +(n - 6, 2, 1, 1, 1, 1)); S = X$. Conditions: 7A is the Paley tournament on seven vertices, its set of vertices is $\{1, 2, 3, 4, 5, 6, 7\}$, and $|X| \geq 2$.

$\mathcal{F}_{10}(n) = (E_1(n); +(n - 8, 2, 1, 1, 1, 1, 1)); S = X$. Conditions: 7A is the Paley tournament on seven vertices, its set of vertices is $\{1, 2, 3, 4, 5, 6, 7\}$, and $|X| \geq 2$.

$\mathcal{F}_{11}(n) = (E_1(n); +(1, 1, n - 3)); S = \{1, 2\}$. Condition: $|X| \geq 2$.

$\mathcal{F}_{12}(n) = (E_1(n); +(2, 1, n - 3)); S = \{1, 2\}$. Condition: $|X| \geq 2$.

$\mathcal{F}_{13}(n) = (E_1(n); +(1, 1, n - 3)); S = \{1, 2\}$. Condition: $|X| \geq 2$.

$\mathcal{F}_{14}(n) = (E_1(n); +(2, 1, n - 3)); S = \{1, 6\}$. Condition: $|X| \geq 2$.
The exception $\mathcal{P}_3$ is the pair $(E_2; Q_3)$, where $E_2$ is the tournament depicted in Fig. 5 and $Q_3 = + (2, 1, 1, 1)$.

The exception $\mathcal{P}_4$ is the pair $(E_2; Q_4)$, where $E_2$ is the tournament depicted in Fig. 5 and $Q_4 = + (1, 1, 1, 1, 1)$.

The exception $\mathcal{P}_5$ is the pair $(E_3; Q_5)$, where $E_3$ is the tournament depicted in Fig. 5 and $Q_5 = + (2, 1, 1, 1, 1, 1)$.

The exception $\mathcal{P}_6$ is the pair $(E_3; Q_6)$, where $E_3$ is the tournament depicted in Fig. 5 and $Q_6 = + (1, 1, 1, 1, 1, 1, 1)$.

**Lemma 2.1.** Let $T = T_1 \rightarrow T_2$ be a reducible tournament of order $n$ and $P$ an outpath of order $n$ such that:

1. $A^+(P) \geq |T|$;
2. $(T; P)$ is not an exception $\mathcal{P}_i(n)$ or $\mathcal{P}_i$, $2 \leq i \leq 6$.

There exists $y \in T_2$ such that for all outgenerators $x$ of $T_1$, there is a path $P$ with origin $x$ and terminus $y$.

**Proof.** Let $Q$ be the shortest initial subpath of $P$ containing $|T_1|$ positive arcs, and let $R$ be the path $P - Q$. $R$ is not empty because $A^+(P) \geq |T_1|$. Let $(t_1, j_1, t_2 - j_1, j_2 - j_1, ..., l_i - \sum_{k=1}^{i-1} b_k)$ be the type of $Q$. Let $a_i$ be an outgenerator of $T_1$; $a_i$ is an origin of a Hamiltonian directed outpath of $T_1$, $Q_1 = (a_1, ..., a_i)$.

If $(T_2; R)$ is not one of Grunbaum's exceptions, $T_2$ contains $R = (c_1, ..., c_m)$. Moreover, $T_2 - R$ contains a Hamiltonian directed inpath $Q_2 = (b_1, ..., b_l)$. We have then $(a_1, ..., a_i, b_1, ..., b_{j_1}, a_{j_1}+1, ..., a_{j_2}, b_{j_2}+1, ..., b_{j_3}, ..., a_l, c_1, c_2, ..., c_m) = P$. Thus the lemma holds with $y = c_m$.

**FIG. 5.** The tournaments $E_1(n), E_2$, and $E_3$. 
Suppose then that \((T_2; R)\) is one of Grünbaum’s exceptions:

If \(T_2 = 3A\): If \(R = + (1, 1)\), it is the exception \(A_1\). If \(R = - (1, 1)\), if \(|T_1| = 1\), it is the exception \(A_2\). If not, \(Q^*_1\) exists in \(T_1\) and \(P = (Q^*_1, 1, 2, a_k, 3)\).

If \(T_2 = 5A\): If \(R = + (1, 1, 1)\), if \(|T_1| = 1\), it is the exception \(A_1\). If not, \(Q^*_1\) exists in \(T_1\) and \(P = (Q^*_1, 1, 2, 3, a_k, 5, 4)\); If \(R = - (1, 1, 1)\), if \(|T_1| = 1\), it is the exception \(A_4\). If not, \(Q^*_1\) exists in \(T_1\) and \(P = (Q^*_1, 1, 2, a_k, 5, 3, 4)\).

If \(T_2 = 7A\): If \(R = + (1, 1, 1, 1, 1)\), if \(|T_1| = 1\) it is the exception \(A_2\). If not, \(Q^*_1\) exists in \(T_1\) and \(P = (Q^*_1, 1, 2, 3, a_k, 7, 5, 6, 4)\). If \(R = - (1, 1, 1, 1)\), if \(|T_1| = 1\) it is the exception \(A_6\). If not, \(Q^*_1\) exists in \(T_1\) and \(P = (Q^*_1, 1, 2, a_k, 7, 3, 5, 4, 6)\).

**Lemma 2.2.** Let \(T = T_1 \rightarrow T_2\) be a reducible tournament of order \(n\), \(P\) an outpath of order \(n\), and \(x\) a vertex of \(T_1\) such that:

1. \(A^+(P) \geq |T_1|\);
2. \((T, P)\) is not one of the exceptions \(A_3\), \(A_4\), or \(A_6\);
3. \(s^+_T(x) \geq b_1(P)\).

Then \(x\) is an origin of a path \(P\) in \(T\) whose terminus lies in \(T_2\).

**Proof.** We prove this in the same way as Lemma 2.1. Let \(Q\) be the shortest initial subpath of \(P\) containing \(|T_1|\) positive arcs. Let \(R\) be the path \(P - Q\). \(R\) is not empty because \(A^+(P) \geq |T_1|\). Let \((i_1, j_1, i_2 - i_1, j_2 - j_1, \ldots, \sum_{i=1}^{r}-1, i_k)\) be the type of \(Q\). Since \(s^+_T(x) \geq b_1(P)\), \(x\) is an origin of a directed outpath \(Q_1 = (a_1, \ldots, a_i)\) in \(T_1\). Let \((a_{i+1}, \ldots, a_j)\) be a Hamiltonian directed outpath of \(T_1 - Q_1\).

If \((T_2; R)\) is not one of Grünbaum’s exceptions, \(T_2\) contains \(R = (c_1, \ldots, c_m)\) and \(T_2 - R\) contains a Hamiltonian directed inpath \(Q_2 = (b_1, \ldots, b_j)\). We have then \((a_1, \ldots, a_i, b_1, \ldots, b_{j_1}, a_{i+1}, \ldots, a_{i+1}, b_{j_2}, \ldots, a_{i+r}, c_1, c_2, \ldots, c_m) = P\), giving the result.

If \((T_2; R)\) is one of Grünbaum’s exceptions, we have the same cases as in the proof of Lemma 2.1, but since \(s^+_T(x) \geq b_1(P)\), \(R\) is an inpath. So we find again the exceptions \(A_3\), \(A_4\), and \(A_6\).

**Lemma 2.3.** Let \(T\) be a reducible tournament of order \(n\) with decomposition \(T_1 \rightarrow T_2\), \(P\) an outpath of order \(n\), and \(U \subset V(T_2)\) such that:

1. \(A^+(P) \geq |T_1| \geq 2\);
2. \(b_1(P) < |T_1|\).
Lemma 2.2, in $T$. Then there exists a vertex $u \in U$ such that every outgenerator of $T_1$ is an origin of $P$ in $T$ with terminus $u$.

Proof. Let $Q$ be the shortest initial subpath of $P$ containing $|T_1|$ positive arcs, and let $R$ be the path $P - Q$. $R$ is not empty because $A^+(P) \geq |T_1|$ and $|R| < |T_2|$ because $b_i(P) < |T_1|$. Let $(i_1, j_1, i_2 - i_1, j_2 - j_1, \ldots, i_k - i_{k-1}, j_k)$ be the type of $Q$. Let $\alpha$ be an outgenerator of $T_1$, $\alpha_1$ is an origin of a Hamiltonian directed outpath of $T_1$, $Q_1 = (\alpha_1, \ldots, \alpha_m)$. Now, $T_2 - R$ contains a Hamiltonian directed inpath $Q_2 = (b_1, \ldots, b_k)$; therefore $(\alpha_1, \ldots, \alpha_m, b_1, \ldots, b_{i_k - 1}, \ldots, a_{j_k + 1}, \ldots, a_j, c_1, c_2, \ldots, c_{m-1}, u) = P$. [1]

Using Lemma 2.2, we prove the following:

Lemma 2.4. Let $T = T_1 \rightarrow T_2$ be a reducible tournament of order $n$ with $|T_1| \geq 2$ for $i \in \{1, 2\}$ and $P$ an outpath of order $n$ such that $P^{-1}$ is an inpath and $b_1(P) = b_1(P^{-1}) = 1$. For every pair of vertices $o \in T_1$ and $t \in T_2$, there is a path $P$ in $T$ with origin $o$ and terminus $t$.

Proof. Without loss of generality, we may suppose that $|T_1| \leq |T_2|$. Since $A^+(P) + A^-(P) = n - 1 = |T_1| + |T_2| - 2 \geq 2|T_1| - 1$, either $A^+(P) \geq |T_1|$ or $A^-(P) \geq |T_1|$. Suppose that $A^+(P) \geq |T_1|$. Let $u_1$ be a vertex of $T_1$. By Lemma 2.2, in $T - (t, o, u)$, there is a path $P^*$ with origin $o$ and terminus in $T_2 - t$. Then $(P^*, u, t) = P$.

Suppose now that $A^-(P) \geq |T_1|$. Let $v$ be an outgenerator of $T_1 - o$. By Lemma 2.2, in $T - (t, o)$ there is a path $(P^{-1})^*$ with origin $v$ and terminus in $T_2 - t$. In other words, there is a path $(P^*)^*$ in $T - (t, o)$ with origin in $T_2 - t$ and terminus in $v$. Then $(o, (P^*)^*, t) = P$. [1]

3. REDUCIBLE TOURNAMENTS

Theorem 3. Let $T$ be a reducible tournament of order $n$ and $C$ a non-directed cycle of order $n$. Then $T$ contains $C$ if and only if $(T; C)$ is neither one of the five exceptions $\varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7, \varepsilon_8$ nor one of their duals (Fig. 6).

Proof. $T$ is reducible; thus there exists a decomposition $T_1 \rightarrow T_2$ of $T$. By duality, it suffices to prove the theorem for $|T_1| \leq |T_2|$. We consider an
The exceptions $C_i$, $7 \leq i \leq 9$.

4. CYCLES WITH A LONG BLOCK

The aim of this section is to prove (Theorem 4.3) that every $(=k)$-strong tournament of order at least 9 contains every Hamiltonian cycle which has a block of length at least $k+2$. We divide the proof into two cases: when the cycle has a block of length at least $k+3$ (Theorem 4.1) and when the longest block of the cycle has length $k+2$ (Theorem 4.2). The proofs make use of the results established in Section 2. In order to avoid a lengthy study of particular cases generated by the numerous exceptions noted there, we have chosen to restrict our proofs to tournaments of sufficiently large order ($n \geq 20$). However, we state the full result that can be obtained by the study of these particular cases (see [6]).

**Theorem 4.1.** Let $T$ be a $(=k)$-strong $(k \geq 1)$ tournament of order $n \geq \min(k+10, 20)$ and $C$ a cycle of order $n$ that contains a block of length at least $k+3$. Then $T$ contains $C$. 
Proof. Camion’s theorem shows that every strong tournament contains a directed Hamiltonian cycle. Henceforth we suppose that $C$ is nondirected. Since $T$ is $(= k)$-strong, there exists a subtournament $X$ of $T$ such that $|X| = k$ and $T - X$ is reducible. Let $T_1 \rightarrow T_2$ be a decomposition of $T - X$ and $(x_1, \ldots, x_k)$ a directed Hamiltonian outpath of $X$. Since $T - (X \setminus \{x_1\})$ is strong, there exists an outgenerator $u$ of $T_2$ such that $u \rightarrow x_1$. Likewise, there exists an outgenerator $v$ of $T_1$ such that $x_k \rightarrow v$.

Let $B = (a_0, \ldots, a_l)$ be a block of $C$ of length $l \geq k + 3$. We orient $C$ so that the arcs of $B$ are in the positive sense and we set $C = (a_0, \ldots, a_{n - 1}, a_0)$. By duality, we may suppose $|T_1| \leq |T_2|$.

(1) Suppose first that $|T_1| \geq 2$ (so also $|T_2| \geq 2$). There exists $t \in T_1$ distinct from $v$ such that $v$ is an outgenerator of $T'_1 = T_1 - t$. We set $T'_2 := T_2 - u$.

Let $P$ be the path $(a_{k + 2}, \ldots, a_{n - 1}, a_0)$. We have $|A^+(P)| + |A^-(P)| = n - k - 2 = |T_1| + |T_2| - 2$. Thus $|T_1| - 1 \leq |A^+(P)|$ or $|T_1| - 1 \leq |A^-(P)|$.

If $|T_1| - 1 \leq |A^+(P)|$, by Lemma 2.1 $v$ is an origin of a path $P^*$ in $T'_1 \rightarrow T'_2$ with terminus in $T_2$, and $C = (P^*, t, u, x_1, \ldots, x_k, v)$.

If $|T_1| - 1 \leq |A^-(P)|$, let $b$ be an outgenerator of $T_1 - v$. By Lemma 2.1, $b$ is an origin of $(P^*)^{-1}$ with terminus $c$ in $T'_2$. In other words, there exists $c \in T'_2$, the origin of a path $P^*$ with terminus $b$. Hence $C = (v, P^* , u, x_1, \ldots, x_k, v)$.

(2) Suppose now that $|T_1| = 1$. Set $T_1 = \{ t \}$. Let $U$ be the set of vertices $y$ of $T_2$ for which there exists an outgenerator $x$ of $X$ such that $y \rightarrow x$. $U$ is not empty and contains an outgenerator of $T_2$ because $T$ is strong.

If $|U| = 1$, say $U = \{ u \}$. The tournament $T - u$ is reducible because $Out(X) \rightarrow T - (Out(X), u)$. Thus $T$ is 1-strong so $|X| = 1$, say $X = \{ x \}$. Since $|T_2| \geq 9$, taking $X = \{ u \}$ and the decomposition $T(x, t) \rightarrow T_2 - u$, we have the result by applying (1).

If $|U| \geq 2$, let $Q$ be the path $(a_1, a_0, a_{n - 1}, \ldots, a_{k + 3})$. $Q$ is an inpath and $b_1(Q) = 1$. Thus by Theorem 2.2, either (a) in $U$ there is an origin $u$ of $Q$, or (b) $(T_2; Q)$ is an exception $\mathcal{F}(n - k - 1)$, with $U$ included in the set of vertices which are not origins of $Q$ in $T_2$.

(a) Let $x \in X$ be an out generator of $X$ such that $u \rightarrow x$. $x$ is an origin of a directed Hamiltonian outpath $R$ of $X$. Hence $(R, t, Q^{-1}, x) = C$.

(b) Since $b_1(Q) = 1$ and $|T| \geq k + 10$, we need only consider the exceptions $\mathcal{F}_3(n - k - 1)$, $\mathcal{F}_3(n - k - 1)$, $\mathcal{F}_3(n - k - 1)$, and $\mathcal{F}_3(n - k - 1)$. If $(T_2; Q)$ is the dual of one of these exceptions, the path $(a_2, a_3, \ldots, a_{k + 4})$ is a directed outpath and among two vertices of $U$, there
is a vertex \( u \) that is an origin of \( Q' = (a_0, a_{n-1}, ..., a_k) \) in \( T_2 \). Let \( x \) be an outgenerator of \( X \) such that \( u \rightarrow x \) and \( R \) a directed Hamiltonian outpath of \( X \) with origin \( x \). We have \( C = (R, t, Q'^{-1}, x) \). 

**Theorem 4.2.** Let \( T \) be a \((=k)\)-strong \((k \geq 1)\) tournament of order \( n \geq \min(k+9, 20) \) and \( C \) a cycle of order \( n \) with a block of length \( k+2 \). Then \( T \) contains \( C \).

**Proof.** Let \( C = (a_0, a_1, ..., a_{n-1}, a_0) \) be a cycle such that \((a_0, a_1, ..., a_k)\) is a block. Since \( T \) is \((=k)\)-strong, there exists a subtournament \( X \) of \( T \) such that \( T' = T - X \) is reducible. By duality, we may suppose that \( |Out(T')| \leq |In(T')| \). Set \( T_1 = Out(T') \) and \( T_2 = T' - Out(T') \). We have \( T_1 \rightarrow T_2 \). Let \( O \) be a Hamiltonian directed outpath of \( X \) with origin \( x \) and terminus \( y \).

Since \( T \) is \((=k)\)-strong, there is an outgenerator \( t \) of \( T_1 \) such that \( y \rightarrow t \) and also an ingenerator \( u \) of \( T_2 \) such that \( u \rightarrow x \). Set \( U := \{ v \in T_2, v \rightarrow x \} \) and \( V := \{ v \in T_2, v \rightarrow x \} \). Note that \((U, V)\) is a partition of \( V(T_2) \). Let \( P \) be the path \((a_k, ..., a_{n-1}, a_0)\) and \( Q \) the path \((a_k, ..., a_{n-1})\). Note that \( P, P^{-1} \) and \( Q \) are outpaths, \( b_1(P) = 1 \), and \( b_1(Q) = 2 \).

1. If \( |T_1| > 1 \), since \( T_1 \) is strong, \( T_1 \) has a Hamiltonian directed cycle \( C_1 = (t, t_2, ..., t_k, t) \). Moreover, \( A^+(P) + A^-(P) = n - k - 1 = |T_1| + |T_2| - 1 \geq 2 |T_1| - 1 \), so \( A^+(P) \geq |T_1| \) or \( A^-(P) \geq |T_1| \).

   a. If \( A^+(P) \geq |T_1| \), then \( b_1(P^{-1}) \leq |T_2| \).

   Suppose first that \(|U| \geq 2 \). If \( s^*_2(U) \geq b_1(P^{-1}) + 1 \) (note that this condition is satisfied if \( U \) contains an outgenerator of \( T_2 \) or if \( b_1(P^{-1}) = 1 \)), since \( b_1(P^{-1}) = 1 < |T_1| \), we may apply Lemma 2.3. So, in \( T', t \) is an origin of \( P \) with terminus in \( U \) and \( C = (O, P, x) \). We may now suppose that \( Out(T_2) \leq \{ v \in T_2, s^*_2(v) \geq b_1(P^{-1}) + 1 \} \subseteq V \) and \( b_1(P^{-1}) \geq 2 \); in other words \( Q^{-1} \) is an outpath. If \(|V| \geq 2 \), since \( A^+(Q) = A^+(P) + 1 \geq |T_1| \), \( b_1(Q) = 1 < |T_1| \), and \( s^*_2(V) = |T_2| \geq b_1(Q^{-1}) + 1 \), we may apply the Lemma 2.3. Hence in \( T', t \) is an origin of \( Q \) with terminus in \( V \) and \( C = (O, Q, x) \). If \(|V| = 1 \), say \( V = \{ v \} \), we have \( Out(T_2) = \{ v \} \), so \( v = T_2-v \). Moreover, \( b_1(P^{-1}) = |T_2| - 1 \geq 2 \). Since \( b_1(Q) = 2 \) and \( |T_1| \geq 3 \), \((T' - V; Q^*) \) is not an exception \( \not\in \). Then by Lemma 2.1, in \( T' - v, t \) is an origin of \( Q^* \) with terminus in \( T_2 - v \) and \( C = (O, Q^*, v, x) \).

   Suppose now that \(|U| = 1 \), say \( U = \{ u \} \), so that \(|V| \geq 2 \). If \( b_1(P^{-1}) \geq 2 \), we have the result in the same way as before. So we may suppose that \( b_1(P^{-1}) = 1 \). Let \( B = (a_k, ..., a_l) \) be the block of \( C \) containing the arc \( a_k \rightarrow a_l \). Since \( A^+(P) \geq |T_1| \), we have \( a_l \neq a_0 \), and \(|B| < |T_2| \). Thus \( u \) is an origin of \( B^* \) in \( T_2 \) and \( T_2 - B^* \) is not empty. Set \( S = (a_l, ..., a_{n-1}) \). By Lemma 2.1, either \(((T' - (B^*, t, t_2)); S) \) is an exception
$P_i, i \in \{2, 4, 6\}$, and we have the result by Proposition 4.1, or in $T'-(B^*, t, t_2)$, $t_2$ is an origin of $S$ with terminus in $T_2-R$ and $C=(O, t, t_2, B^*, S, x)$.

(b) If $A^+(P) < |T_1|$, then $A^-(P) \geq |T_1|$. If $x \leftarrow t_2$, since $P$ is an inpath and $|T_1| \geq 3$, $(T'-t; (P)^{-1})$ is not an exception $\mathcal{E}_i$, so by Lemma 2.1, $t_2$ is an origin of $(P)^{-1}$ with terminus in $T_2$ and $C=(t_2, O, t, P)$. From now on, we may suppose $x \rightarrow t_2$. If $h_1(P^{-1}) \geq 2$, then $(P^*)^{-1}$ is an outpath. So by Lemma 2.1, $t_2$ is an origin of $(P^*)^{-1}$ with terminus in $T_2$ and $C=(O, t, P^*, x)$.

Henceforth, we may suppose $h_1(P^{-1}) = 1$. Let $(a_0, \ldots, a_n, a_{n-1})$ be the block of $C$ containing the arc $a_{n-2} \rightarrow a_{n-1}$, $R$ the path $(a_{l+1}, \ldots, a_{n-1}, a_0)$, and $S$ the path $(a_k, 2, \ldots, a)$. Since $A^+(P) < |T_1|$ and $|T_1| \geq |T_2|$, we have $|R| < |T_2|$. If $|U| = 2$ then by Theorem 2.2 there is a path $R^{-1}$ in $T_2$ with origin $u \in U$. Because $S$ is an inpath and $|T_1| \geq 3$, $(T'-(R, t); S^{-1})$ is not an exception $\mathcal{E}_i$; so by Lemma 2.1, in $T'-(R, t), t_2$ is an origin of $S^{-1}$ with terminus in $T_2-R$ and $C=(u, O, t, S, R)$.

Suppose now that $|U| = 1$, say $U = \{u\}$. In $T(V)$, one can find $R$, by Lemma 1.1 and by Lemma 2.1, either $(T'-(R, t, t_2); S^{-1})$ is an exception $\mathcal{E}_i, i \in \{2, 4, 6\}$, and we have the result by Proposition 4.2, or in $T-(X, R, t, t_2)$, $t_3$ is an origin of $S^{-1}$ with terminus in $T_2-R$ and $C=(O, t, t_2, S, R, x)$.

(2) If $|T_1| = 1$ then $T_3 = \{t\}$. If $k \geq 2$, we have $d_{T}^+(x) \leq k-2$, so since $T$ is $(=k)$-strong, $|U| \geq 2$. If $k \geq 1$ and $|U| = 1$, i.e., $U = \{u\}$, then $T-u$ is reducible with decomposition $x \rightarrow (T_2-u) \cup \{t\}$. Now $u \leftarrow t$ and $d_{T}^+(u) \geq 1$ because $u$ is an ingenerator of $T_2$. Therefore, we just need to consider the case $|U| = 2$.

If $s_{T}^+(U) \geq h_1(P^{-1})$, by Theorem 2.2, either there is a vertex $z \in U$ that is an origin of $(P)^{-1}$ and $C=(z, O, t, P)$ or $(T_2; (P)^{-1})$ is one of the exceptions $\mathcal{E}_i$. Thus, in $T_2-U$, there is a vertex that is an origin of $(P)^{-1}$ and $C=(O, t, P^*, x)$. So we may suppose that $s_{T}^+(U) < h_1(P^{-1})$. In particular, $h_1(P^{-1}) \geq 2, T_2$ is reducible, and $Out(T_2) \subseteq V$. If $|V| \geq 2$, $(T_2; (P)^{-1})$ could not be an exception $\mathcal{E}_i$ or $\mathcal{E}_f(n-k-1)$ with $V$ included in the set of vertices that are not origins of $(P)^{-1}$. Consequently, by Theorem 2.2, in $U$, there is an origin of $(P^*)^{-1}$ and $C=(O, t, P^*, x)$. If $|V|=1$ then $Out(T_2) = V = \{v\}$. Set $Q'=(a_{k+1}, \ldots, a_{n-2})$. By Theorem 1.1, $T_2(U)$ contains $Q'$ and $C=(O, t, Q', v, x)$.

**Proposition 4.1.** Let $T$ be a $(=k)$-strong tournament of order $n$ and $C=(a_0, a_1, \ldots, a_{n-1})$ a cycle of order $n$ with a block of length $k+2$,
(a_k, a_{k+1}, ..., a_{k+2}), such that there exists a subtournament X of order k with T' = T - X = T_1 \rightarrow T_2 with T_1 the 3-cycle (t, t_2, t_3, t), O a Hamiltonian directed path of X with origin x and terminus y such that y \rightarrow t, and u an ingenerator of T_2 such that u \rightarrow x and T_2 - u \leftarrow x. Let B = (a_{k+2}, ..., a_l) be the block of C containing the arc a_{k+2} \leftarrow a_{k+3} and S the path (a_l, ..., a_{n-1}). Let D* be a path isomorphic to B* with origin u.

If (T' - (D*, t, t_2); S) is an exception \cal F, i \in \{2, 4, 6\}, then T contains S.

Proof. Let w be a vertex of T_2 - D*. If w \rightarrow u then, by Lemma 2.1, t_2 is an origin of S in T' - (D*, t, w) and C = (O, t, w, D*, S, x). So we may suppose that T_2 - B \leftarrow u. Let z be the terminus of D* and set T_3 = T_2 - D**. It is obvious that there exists w such that T_2 - w is not 3A, 5A, 7A. Thus, by Lemma 2.1, t_3 is an origin of S in T_2 - w and C = (O, t, t_2, w, D**, S, x).

Proposition 4.2. Let T be a (=k)-strong tournament of order n such that there exists a subtournament X of order k such that T' = T - X = T_1 \rightarrow T_2 with T_1 the 3-cycle (t, t_2, t_3, t), O a Hamiltonian directed path of X with origin x and terminus y such that y \rightarrow t and x \rightarrow t_2, and u an ingenerator of T_2 such that u \rightarrow x and T_2 - u \leftarrow x. Let C = (a_0, a_1, ..., a_{n-1}) be a cycle of order n such that (a_0, a_2, ..., a_k) is a block and a_{n-2} \rightarrow a_{n-1}.

Let (a_1, ..., a_{n-2}, a_{n-1}) be the block of C containing the arc a_1 \rightarrow a_{n-1}, R the path (a_{n-1}, ..., a_{n-1}, a_0), and S the path (a_{n+1}, ..., a_l). In T_2 - u, one can find R.

If (T' - (R, t, t_2)) is the tournament of an exception \cal F, i \in \{2, 4, 6\}, then T contains S.

Proof. If (T' - (R, t, t_3)) is the tournament of \cal F (resp. \cal G, \cal H), there is a subtournament T'' of T_2 - v containing u of order 3 and nonisomorphic to the tournament of 3A (resp. 5A, 7A). Let R be a Hamiltonian directed path T_2 = T''. By Lemma 2.1, t_3 is an origin of S^{-1} with terminus in T_2 - R and C = (O, t, t_2, S, R, x).

Using the same method, one can improve the previous two theorems and prove the following more general theorem (see [6]). However, its proof requires the study of a huge number of special cases generated, on the one hand, by the exceptions \cal F (each time we use Lemma 2.1) and, on the other hand, by the 52 exceptions of order at most 8 of Theorem 2.2 (cf. [7]).

Theorem 4.3. Let T be a (=k)-strong (k \geq 1) tournament of order n and C a cycle of order n containing a block of length at most k + 2. Then T contains C if and only if (T, C) is not the exception \epsilon_9.
Corollary 4.1 (Benhocine and Wojda [2]). Let \( T \) be a tournament of order \( n \) and \( C \) a cycle of order \( n \) with exactly two blocks. Then \( T \) contains \( C \) unless \( (T, C) \) is one of the exceptions \( \mathcal{E}_1, \mathcal{E}_r, \mathcal{E}_8, \) and \( \mathcal{E}_9 \) or the dual of \( \mathcal{E}_9 \).

**Proof.** If \( T \) is reducible, Theorem 3.1 gives the result. So we may suppose that \( T \) is \( (=k)\)-strong \((k \geq 1)\). Thus \( |T| \geq 2k+1 \), so \( C \) has a block of length at least \( k+1 \). If \( C \) has a block of length at least \( k+2 \), we have the result by Theorem 4.3, so we may suppose that its maximal block is of length \( k+1 \). Thus \( C = (k+1, k) \) or, \( C = (k+1, k+1) \). If \( C = (k+1, k) \), by Theorem 1.1, either \( T \) contains the path \(+ (k, k)\), and by adding the arc between the origin and the terminus of this path we obtain \( C \) (regardless of the sense of this arc) or \( T \) is \( 3A \) and \((T; C) \) is the exception \( \mathcal{E}_1 \). If \( C = (k+1, k+1) \), let \( (x_1, x_2, x_3, ..., x_{2k+2}, x_1) \) be a Hamiltonian directed cycle of \( T \). Suppose that \( x_1 \rightarrow x_{k+2} \). If \( x_2 \rightarrow x_{k+3} \) then \( (x_{k+3}, x_2, x_3, ..., x_{k+2}, x_1, x_{2k+2}, ..., x_{k+3}) = C \). So we may suppose that \( x_2 \rightarrow x_{k+3} \) and, in general, that \( x_i \rightarrow x_{k+i+1} \). Thus we find that \( x_{k+2} \rightarrow x_1 \), contradicting the assumption \( x_1 \rightarrow x_{k+2} \). Likewise, if we suppose that \( x_1 \rightarrow x_{k+2} \), we may suppose in general that \( x_i \rightarrow x_{k+i+1} \) and we have the contradiction \( x_{k+2} \leftarrow x_1 \).

5. CYCLE WITH SMALL BLOCKS

Theorem 5.1. Let \( C \) be a nondirected cycle of order \( n \), with longest block of length \( L \geq 2 \), and \( T \) a \((=k)\)-strong tournament of order \( n \) with \( k \notin \{3, 5, 7\} \). If \( n \geq 4(k+1) \) \( L+k+1 \) then \( T \) contains \( C \).

**Proof.** Let \( X \) be a subtournament of order \( k \) such that \( T' = T - X \) is reducible with decomposition \( T_1 \rightarrow T_2 \). Without loss of generality, we may suppose that \( |T_1| \leq |T_2| \).

Let \( C = (a_0, a_1, a_2, ..., a_{n-1}, a_0) \) be such that \( a_k \rightarrow a_{k+2} \leftarrow a_{k+3} \).

Let \( R = (a_1, a_2, ..., a_k) \). By Theorem 1.1, \( X \) contains \( R \). Let \( (x_1, x_2, ..., x_k) \) be an occurrence of \( R \) in \( T \). Since \( T - R^* \) is strong, there is a vertex \( o \) of \( T_1 \) that is dominated by \( x_k \).

(A) Suppose first that \( |T_1| \geq 2 \).

Let \( i_1 \) be the greatest integer less than \( n \) such that \( a_{i_1} \leftarrow a_{i_1} \rightarrow a_{i_1} + 1 \), and let \( Q_1 \) be the path \((a_0, a_{n-1}, ..., a_{i_1+1}) \). Note that \( |Q_1| \leq 2L \).

Let \( U_1 \) be the set of vertices of \( T_2 \) which dominate \( x_1 \) and \( V_1 \) the set of vertices of \( T_2 \) which are dominated by \( x_1 \).

Suppose that \( a_1 \leftarrow a_0 \). If \( |U_1| \geq b_1(Q_1) + 1 \), then by Theorem 2.1 there is a path \( Q_1 \) in \( T_2 \) with origin in \( U_1 \) and terminus \( i_1 \). Now let \( P_1 = (a_3, a_5, ..., a_{i_1+1}) \) and \( T'_1 = T_1 \rightarrow T_2 - (Q_1^*) \). By Lemma 2.4, in \( T'_1 \) there is a
path $P_1$ with origin $o$ and terminus $t_1$. So $T$ contains $C= (x_1, x_2, x_3, x_4, P_1, Q_{-1})$. Hence, we may assume that $|U_1| < b_4(Q_1) + 1$.

Likewise, if $a_1 \rightarrow a_6$, we may suppose that $|V_1| < b_4(Q_1) + 1$.

Let us define $T_3^1$ as $T_2(U_1)$ if $a_0 \rightarrow a_1$ and $T_3(V_1)$ otherwise. We have $|T_3^1| \geq |T_2| - b_4(Q_1) \geq (2k + 1) L + 1$. Note that either $x_1$ dominates $T_3^1$ or $x_1$ is dominated by $T_3^1$.

Let $j_2$ be the greatest integer less than $n + 1$ such that $a_{j_2}$ has out-degree (resp. indegree) two in $C$, if $x_1$ dominates (resp. is dominated by) $T_3^1$, and let $i_2$ be the greatest integer less than $j_2$ such that $a_{i_2-2} \rightarrow a_{i_2-1} \rightarrow a_{j_2}$. Let $Q_2 = (a_0, a_{i_2-1}, \ldots, a_{j_2+1})$ and $Q_2 = (a_{0+1}, \ldots, a_{n-1})$. Note that $Q_2$ and $Q_1$ each have length less than $2L$.

Let $U_2$ be the set of vertices of $T_2^1$ which dominate $x_2$ and $V_2$ the set of vertices of $T_2^1$ which are dominated by $x_2$.

Suppose that $a_2 \leftrightarrow a_1$. If $|U_2| \geq b_4(Q_2) + 1$, then by Theorem 2.1, there is a path $Q_2^1$ in $T_2$ with origin in $U_2$. By Theorem 2.1, there is a path $Q_2$ with origin $t_1$ in $T_2^1 - Q_2$. Let $P_2 = (a_{i_2}, a_{i_2+1}, \ldots, a_{j_2+1})$ and $T_2 = T_2^1 - (Q_2, Q_2^1)$. By Lemma 2.4, in $T_2^1$, there is a path $P_2$ with origin $t_1$ and terminus $t_2$. So $T$ contains $C= (x_2, x_3, x_4, P_2, Q_2^1, x_2, Q_2^{-1})$. Hence, we may assume that $|U_2| < b_4(Q_2) + 1$.

Likewise, if $a_2 \rightarrow a_1$, we may assume that $|V_2| < b_4(Q_2) + 1$.

Let us define $T_2^1$ as $T_2^1(U_2)$ if $a_0 \leftrightarrow a_1$ and $T_2^1(V_2)$ otherwise. We have $|T_2^1| \geq |T_2^1| - b_4(Q_2) \geq 2kL + 1$. Note that either $x_2$ dominates $T_2^1$ or $x_2$ is dominated by $T_2^1$.

By induction, we may suppose that there is a sequence of sub-tournaments $T_2^1 \leq T_2^2 \leq \cdots \leq T_2^k = T_2$ such that, for $1 \leq i \leq k$, $T_2^i$ has order at least $(2k + 2 - i) L + 1$ and either $T_2^i \rightarrow x_i$ or $T_2^i \leftarrow x_i$.

Let $B$ be the block containing $a_{k+1}$ and $b$ its length. Let $a_{k+1}$ be the endvertex of $B$ distinct from $a_{k+2}$.

Let us define recursively $l_i$ for $i = k, k-1, \ldots, 1$ to be the greatest integer less than $l_i - 1$ such that $a_i$ has outdegree (resp. indegree) two in $C$ if $x_i$ dominates (resp. is dominated by) $T_2^i$. Let $R_i = (a_{i+1}, a_{i+2}, \ldots, a_{i+1} - 1)$ for $1 \leq i \leq k$. Note that each $R_i$ has length at most $2L - 1$ since each block of $C$ has length at most $L$.

If $|T_2^i| \geq b + 1$, let $l_i$ be a directed inpath of length $b - 1$ in $T_i$ with origin $o$, let $l_0$ be the greatest integer less than $l_i$, such that $a_{i-1} \leftrightarrow a_i \rightarrow a_{i+1}$, and let $R_0 = (a_{i+1}, a_{i+2}, \ldots, a_{i+1} - 1)$.

By Theorem 2.1, $|T_2^i| \geq (2k + 2 - i) L + 1$, $1 \leq i \leq k$. Therefore, one can find disjoint paths $R_1, R_{k-1}, \ldots, R_0$ such that $R_i$ is contained in $T_2^i$ and $R_{i-1}$ in $T_2^i - (R_k, \ldots, R_1)$, $1 \leq i \leq k$. Denote by $t$ the origin of $R_0$.

Now let $P = (a_7, a_8, \ldots, a_{i_2})$ and $T^* = T_2^i \rightarrow T_2^j$ with $T_2^i = T_1 - (\ast I)$ and $T_2^j = T_2 - (\ast I_0, R_1, \ldots, R_k)$. We have $|T_1 - \ast I| \geq 2$ because $|T_1| \geq b + 1$.
and $|T_2 - (R_1, R_2, ..., R_k)| \geq 2$, because $|T_2| \geq f(L, k)/2$. So by Lemma 2.4 in $T^*$ there is a path $P$ with origin $o$ and terminus $t$. Hence $T$ contains $C = (P, % R_0, x_1, R_1, x_2, R_2, ..., x_k, R_k, I)$.

— If $|T_1| \leq b$, let $I$ be a directed outpath of $T_1$ with origin $u$, let $j = l_{k+1} + |T_1|$ (modulo $n$), and let $P$ be the path $(a_1, a_j + 1, ..., a_{l_{k+1}})$. By Theorem 2.1, $|T_1| \geq (2k + 2 - i) L + 1$, for $1 \leq i \leq k$. Therefore, one can find disjoint paths $R_k, R_{k-1}, ..., R_0$ such that $R_k$ is contained in $T'_2$ and $R_{i-1}$ in $T'_2 - (R_i, ..., R_k)$ for $2 \leq i \leq k$. The tournament $S = T'_2 - (R_1, R_2, ..., R_k)$ has at least $L$ vertices. So by Theorem 2.2, there is a path $P$ in $T_2 - (R_1, R_2, ..., R_k)$ with terminus in $S$. Thus $T$ contains $C = (I, P, x_1, R_1, x_2, R_2, ..., x_k, R_k, U)$.

(B) Suppose now that $T_1 = \{t\}$.

— Let $U_1$ (resp. $V_1$) be the set of vertices of $T_2$ which dominate (resp. are dominated by) $x_1$. Let $Q_1 = (a_0, a_{-1}, ..., a_{k+1})$.

Suppose that $|U_1| \geq b_1(Q_1) + 1$. By Theorem 2.2, in $T_2$, there is a path $Q_1$ with origin in $U_1$. Thus $T$ contains $C = (x_1, x_2, ..., x_k, t, Q_1^{-1}, x_1)$. Hence, we may assume that $|U_1| < b_1(Q_1) + 1$. Likewise, if $a_1 \rightarrow a_0$, we may suppose that $|V_1| < b_1(Q_1) + 1$.

Let us define $T'_1$ as $T_2(U_1)$ if $a_0 \rightarrow a_1$, and $T_2(V_1)$ otherwise. We have $|T'_1| \geq |T_2| - b_1(Q_1) \geq (4k + 3) L$. Note that either $x_1$ dominates $T'_1$ or $x_1$ is dominated by $T'_1$.

— Let $l_2$ be the greatest integer less than $n + 1$ such that $a_{l_2}$ has out-degree (resp. indegree) two, if $x_1$ dominates (resp. is dominated by) $T'_1$. Let $Q'_2$ be the path $(a_0, a_{l_2}, ..., a_{l_2+1})$. Note that $Q'_2$ has length at most $2L - 1$. Let $Q_2 = (a_{k+2}, a_{k+3}, ..., a_{l_2})$. By Theorem 2.1, one can find $Q'_2$ in $T_1$. Since $T'_2 - Q'_2$ has order at least $L$, by Theorem 2.2 one can find a path $Q_2$ in $T_2 - Q'_2$ with terminus in $T'_2 - Q'_2$. Hence $T$ contains $C = (x_2, x_3, ..., x_k, t, Q_2, x_1, Q'_2^{-1}, x_2)$.

By induction, we may suppose that there is a sequence of subtournaments $T'_2 \subseteq T_2^{2^{k-1}} \subseteq \cdots \subseteq T'_2 \subseteq T'_2 = T_2$ such that, for $1 \leq i \leq k$, $T'_i$ has order at least $(4k + 4 - i) L$ and either $T'_2 \rightarrow x_i$ or $T'_2 \leftarrow x_i$.

Let $B$ be the block containing $a_{k+1}$ and let $a_{l_2+1}$ be the endvertex of $B$ distinct from $a_{k+2}$. Let us define recursively $l_i$, for $i = k, k - 1, ..., 1$, to be the greatest integer smaller than $l_{i+1} - 1$ such that $a_i$ has out degree (resp. indegree) two in $C$, if $x_i$ dominates (resp. is dominated by) $T'_2$. Let $R_i = (a_{l_i+1}, a_{l_i+2}, ..., a_{l_i})$. Note that each $R_i$ has length at most $2L - 1$ since each block of $C$ has length at most $L$. Let $P = (a_{l_{k+1} + 1}, a_{l_{k+1} + 2}, ..., a_{l_k})$. Since $|T'_2| \geq (4k + 4 - i) L$, for $1 \leq i \leq k$, one can find disjoint paths $R_k, R_{k-1}, ..., R_0$ such that $R_k$ is contained in $T'_k$ and $R_{i-1}$ in $T'_2 - (R_i, ..., R_k)$ for $2 \leq i \leq k$. The tournament $S = T'_2 - (R_1, R_2, ..., R_k)$ has at least $L$ vertices. So by Theorem 2.2, there is a path $P$ in
$T_2 = (R_1, R_2, \ldots, R_k)$ with terminus in $S$. Thus $T$ contains $C = (I, P, x_1, R_1, x_2, R_2, \ldots, x_k, R_k, u)$.}

**Theorem 5.2.** Let $C$ be a cycle of order $n$ with longest block of length $L$ and $T$ a $(=k)$-strong tournament of order $n$ with $k \in \{3, 5, 7\}$. Then $T$ contains $C$ if either:

- $L \geq 4$ and $n \geq 4(k+1) L + k + 1$, or
- $2 \leq L \leq 3$ and $n \geq 4(k+2) L + k + 2$.

**Proof.** Let $X$ be a subtournament of $T$ such that $T - X$ is reducible. Let $C = (a_0, a_1, a_2, \ldots, a_{n-1}, a_0)$ such that $a_k \rightarrow a_{k+1} \rightarrow a_{k+2} \leftarrow a_{k+3}$. Let $R = (a_1, a_2, \ldots, a_k)$. If $(X; R)$ is not one of Grünbaum's exceptions, the proof is similar to Theorem 5.1. In particular, this is the case if $L \geq 4$ because we have $a_{k-2} \rightarrow a_{k-1} \rightarrow a_k$.

If $(X; R)$ is one of Grünbaum's exceptions, let $y$ be a vertex of $T - X$ such that $T - (X, y)$ is still reducible. Let $X' = T(X, y)$ and $R' = (a_0, a_1, \ldots, a_k)$. It is easy to check that there is a vertex of $R$ that is an origin of $R'$ in $X'$. The proof proceeds as in Theorem 5.1.

Theorems 4.3, 5.1, and 5.2 yield the following corollary:

**Corollary 5.1.** Let $T$ be a $(=k)$-strong tournament of order $n \geq 4(k+1)^2 + k + 1$. Then $T$ contains every Hamiltonian cycle.

This corollary is very important: if one can prove that every $k$-strong tournament of order $n \geq \mathcal{N}(k)$ contains every Hamiltonian cycle then it implies that every tournament of order at least $\max(\mathcal{N}(k), 4k^2 + k)$ contains every Hamiltonian non-directed cycle.

### 6. Hamiltonian Cycles in 8-Strong Tournaments

#### 6.1. Useful Lemmas

Theorem 2.2 yields the following two Lemmas which will be need frequently in the proofs of our subsequent theorems.

**Lemma 6.1.** Let $T$ be a tournament of order $n \geq 8$ and $P$ a path of order $n$. Then

1. there are at least $n - b_1(P) - 1$ origins of $P$ in $T$ unless $(T; P)$ is the exception $\mathcal{F}(n)$ or its dual.
2. there are at least $n - b_1(P)$ origins of $P$ in $T$ unless $(T; P)$ is one of the exceptions $\mathcal{F}(n)$, $i \in \{1, 3, 5, 7, 8, 9, 10, 11, 13\}$, or $\mathcal{E}_{49}$, or one of their duals.
Proof. By directional duality, we may assume that $P$ is an outpath. If $(T; P)$ is one of the exceptions $\mathcal{F}_i$, $1 \leq i \leq 14$, $\mathcal{F}_{49}$, $\mathcal{F}_{50}$, or $\mathcal{F}_{51}$, it is easy to check that the statement holds. Suppose now that $(T; P)$ is not one of the exceptions $\mathcal{F}_i$, $1 \leq i \leq 14$, $\mathcal{F}_{49}$, $\mathcal{F}_{50}$, $\mathcal{F}_{51}$. If $P$ is directed the result holds trivially, so we may assume that $P$ is not directed; in particular, $b_1(P) \leq n - 2$.

Let $X$ be the set of vertices $x$ of $T$ such that $s_T^+(x) \geq b_1(P) + 1$. It is easy to see that $|X| \geq n - b_1(P)$, consider, for example, the first $n - b_1(P)$ vertices of a Hamiltonian directed outpath of $T$. Suppose, by way of contradiction, that there are fewer than $n - b_1(P)$ origins of $P$ in $T$. Then there is a vertex $x \in X$ that is not an origin of $P$. Also since $b_1(P) \leq n - 2$, there exists a vertex $y \neq x$ of $T$ which is not an origin of $P$. But this contradicts Theorem 2.2.

Lemma 6.2. Let $T$ be a tournament of order $n \geq 8$, $P$ an outpath of order $n$, and $X$ the set of origins of $P$ in $T$.

1. If $|X| \leq n - 3$ then $X \to T - X$.

2. Suppose that $(T; P)$ is not one of the exceptions $\mathcal{F}_i$, $i \in \{2, 3, 4, 5, 6, 7, 11, 12, 13, 14\}$, nor $\mathcal{E}_l$, $l \in \{49, 50, 51\}$. If $|X| \leq n - 2$ then $X \to T - X$.

Proof. If $(T; P)$ is one of the exceptions $\mathcal{F}_i$, for $1 \leq i \leq 14$, or $\mathcal{E}_l$, for $l \in \{49, 50, 51\}$, we have the result by definition. So we may assume that $(T; P)$ is not one of these exceptions. Suppose that $|X| \leq n - 2$ and suppose, by way of contradiction, that there is a vertex $y \in T - X$ that dominates a vertex $x \in X$. We have $s_T^+(y) \geq s_T^+(x) \geq b_1(P) + 1$. Let $z$ be a vertex of $T - X$ distinct from $y$. By Theorem 2.2, $y$ or $z$ is an origin of $P$ and thereby belongs to $X$. This is a contradiction.

We now present the lemma on which our proofs are based.

Lemma 6.3. Let $T$ be a $k$-strong tournament of order $n$, $C = (b_0, b_1, \ldots, b_{n-1}, b_0)$ a cycle such that $b_{n-1} \to b_0 \to b_1$, and $x$ a vertex of $T$ with both out- and in-degree greater than $7$.

Set $n^+ = |N_T^+(x)|$, $n^- = |N_T^-(x)|$, $T^+ = T(N_T^+(x))$, and $T^- = T(N_T^-(x))$ and let $P^+$ be the path $(b_0, b_1, \ldots, b_{n-1})$ and $P^-$ the path $(b_{n-1}, b_{n-2}, \ldots, b_0)$. Let $X^+$ (resp. $X^-$) be the set of origin of $P^+$ (resp. $P^-$) in $T^+$ (resp. $T^-$).

1. If $b_{n-1} \to b_{n+1}$, then $T$ contains $C$ or $X^- \to X^+$.

2. If $b_{n-1} \leftrightarrow b_{n+1}$, then $T$ contains $C$ or $X^- \leftrightarrow X^+$.

3. If $|X^+| + |X^-| \geq n - k + 1$ then $T$ contains $C$.

4. If $|X^+| + |X^-| \geq n - k$ and $b_{n+1} \to b_{n+1}$ then $T$ contains $C$. 
Then by (1) and (2), $T$ contains $C$.

(6) If $|X^-| \geq n^- + k + 4$ and $(b_{n^+}, b_{n^++1})$ is a directed path, then $T$ contains $C$.

Proof. (1) and (2) are obvious.

(3) Suppose, by way of contradiction, that $T$ does not contain $C$. Then by (1) and (2), $T(X^-, X^+)$ is reducible of order $n - k + 1$. This contradicts the hypothesis that $T$ is $k$-strong.

(4) Suppose, by way of contradiction, that $T$ does not contain $C$. Then by (1), $X^- \rightarrow X^+$. So $T(X^-, X^+, x)$ is reducible with order $n - k + 1$. This contradicts that $T$ is $k$-strong.

(5) Assume that $(b_{n^+}, b_{n^++1})$ is an outpath. Suppose, by way of contradiction, that $T$ does not contain $C$. By (1) we obtain $X^- \rightarrow X^+$. If $|X^-| \geq n^+ - 2$, then $T(X^-, X^+)$ is reducible of order $n - k + 1$. This is a contradiction. If $|X^-| < n^+ - 2$, since $P^+$ is an inpath, (the dual of) Lemma 6.2 yields that $T^+ \rightarrow X^+$ is reducible with order $n - k + 2$. This is a contradiction.

The proof of (6) is analogous to that of (5).

The basic idea of the proof of the next theorem is the following: we pick an arbitrary vertex $x$ of $T$ and seek an enumeration $(b_0, ..., b_{n^+}, b_0)$ of the vertices of $C$ such that $X^-$ and $X^+$ satisfy one of the hypotheses of Lemma 6.3. This lemma then guarantees that $T$ contains $C$.

6.1. Main result

**Theorem 6.1.** Let $T$ be a $k$-strong tournament of order $n$, where $k \geq 8$, and let $C$ be a cycle of order $n$ with blocks of length at most $L$, where $2 \leq L \leq k + 1$. Then $T$ contains $C$.

Let $B$ be a block of $C$ with maximal length $L$. Set $l = \lfloor L/2 \rfloor$. Let $C = (a_0, a_1, ..., a_{n-1}, a_0)$, where $a_0 \in B$ and $a_{n-1} \rightarrow a_{n-L+1} \rightarrow \cdots \rightarrow a_{L-1} \rightarrow a_0$. Let $x$ be a vertex of $T$, and set $n^+ = |N_T^+(x)|$ and $n^- = |N_T^-(x)|$. Let us denote by $P^+$ and $T^-$ the subtournaments $T(N_T^+(x))$ and $T(N_T^-(x))$. For every integer $i$, let $P^+_i$ be the path $(a_{n^+-i}, a_{n^+-i-1}, ..., a_{i+1})$ and $P^-_i$ the path $(a_{n^-+i}, a_{n^-+i+1}, ..., a_{i-1})$ (indices modulo $n$). Let $X^+_i$ be the set of vertices which are origins of $P^+_i$ in $T^+$ and $X^-_i$ be the set of vertices which are origins of $P^-_i$ in $T^-$. Note that $X^+_i$ and $X^-_i$ are not empty, by Theorem 1.1. Considering the paths $P^+_i$ and $P^-_i$ amounts to considering the enumeration $(b_0, b_1, ..., b_{n^+}, b_0)$ of Lemma 6.3 with $b_i = a_{j+i}$. 


Note that \((T^+, P_0^+\)) is neither the exception \(T_1\) nor its dual, otherwise \(C\) would have a block of length greater than \(k+1\).

If \(b_1(P_0^+) + b_1(P_0^-) < 4\), by Lemma 6.1.1, \(|X_0^+| \geq n^+ - b_1(P_0^+) - 1\) and \(|X_0^-| \geq n^- - b_1(P_0^-) - 1\). So, by Lemma 6.3.3, \(T\) contains \(C\). Thus we may assume that \(b_1(P_0^+) + b_1(P_0^-) \geq 5\). In particular, \(L \geq 3\).

Suppose first that \(L = 3\). Then \(2 \leq b_1(P_0^-) \leq 3\), \(2 \leq b_1(P_0^+) \leq 3\), and \((a_{n-2}, a_{n+1})\) is a block of \(C\). By symmetry, we may suppose that \(a_{n-2} \rightarrow a_{n-1}\). Let us consider \(P_1^+\) and \(P_1^-\). We have \(b_1(P_1^-) = 1\) and \(b_1(P_1^+) \leq 2\). So by Lemma 6.1, \(|X_1^-| \leq 3\) and \(|X_1^-| \leq 3\). Hence, by Lemma 6.3.3, \(T\) contains \(C\).

Suppose now that \(L \geq 4\). Then \(a_{n-2} \rightarrow a_{n-1}\) and \(a_1 \rightarrow a_2\). If \(b_1(P_0^-) = 1\) then \(b_1(P_0^+) \geq 4\). Let us consider \(P_1^+\) and \(P_1^-\). We have \(b_1(P_1^-) \leq 2\), so by Lemma 6.1.1 \(|X_1^-| \geq n^- - 3\). And since \(b_1(P_0^+) \geq 2\), \((a_{n-2}, a_{n+1}, a_{n+2})\) is a directed path, so by Lemma 6.3.5 \(T\) contains \(C\).

Similarly, the theorem holds if \(b_1(P_0^+)=1\). So we may assume that \(b_1(P_0^-) \geq 2\) and \(b_1(P_0^+) \geq 2\). If the path \((a_{n+1}, a_{n+2}, a_{n+3})\) is not directed, let us consider \(P_1^+\) and \(P_1^-\). We have \(b_1(P_1^-) = 1\); thus by Lemma 6.1, \(|X_1^-| \geq n^- - 3\). And since \(b_1(P_0^+) \geq 2\), \((a_{n+1}, a_{n+2}, a_{n+3})\) is a directed path. Hence, by Lemma 6.3.5, \(T\) contains \(C\).

Likewise, considering \(P_{n-1}^-\) and \(P_{n-1}^+\), we have the result provided that \((a_{n+1}, a_{n+2}, a_{n+3})\) is not directed.

If \((a_{n+1}, a_{n+2}, a_{n+3})\) is directed, then \(L \geq 6\) and so \(a_2 \rightarrow a_3\) and \(a_{n-2} \rightarrow a_{n-3}\). By directional duality, we may assume that \(b_1(P_0^-) \leq b_1(P_0^+)\). If \(b_1(P_0^-) = 2\), then by Lemmas 6.1 and 6.3.5 \(T\) contains \(C\). If \(b_1(P_0^-) \geq 3\), let us consider \(P_{n-2}^-\) and \(P_{n-2}^+\). By Lemma 6.1, \(|X_{n-2}^-| \geq n^- - k/2 \geq n^- - k + 4\), because \(b_1(P_0^-) \leq k/2 - 2\). Hence, by Lemma 6.3.5, \(T\) contains \(C\).

**Corollary 6.1.** Every 8-strong tournament contains every Hamiltonian cycle.

This corollary and Corollary 5.1 yield the following statement:

**Corollary 6.2.** Every tournament of order at least 264 contains every nondirected Hamiltonian cycle.

7. HAMILTONIAN CYCLES IN 5-STRONG TOURNAMENTS

We now improve the bound 8 of Corollary 6.1, for sufficiently large tournaments in order to sharpen the bound 264 of Corollary 6.2.
Theorem 7.1. Let $T$ be a $k$-strong tournament of order $n$, where $k \geq 5$, $C$ a cycle of order $n$ with blocks of length at most $L$, where $6 \leq L \leq k + 1$, and $x$ a vertex of $T$ whose in- and outdegrees are greater than $8$. Then $T$ contains $C$.

Proof. Let $B$ be a block of a of maximal length, $L$. Let $l = \lfloor L/2 \rfloor$. Let $C = (a_0, a_1, \ldots, a_{n-1}, a_0)$ be enumerated so that $a_0 \in B$ and $a_{n,l} \rightarrow a_{n-1,l} \rightarrow a_{n-2,l} \rightarrow \cdots \rightarrow a_{l-1,l} \rightarrow a_l$.

We define $n^+$, $n^-$, $T^+$, $T^-$, $P^+_i$, $P^-_i$, $X^+_i$, and $X^-_i$ in the same way as in Lemma 6.1.

By directional duality, we may assume that $b_1(P^+_i) \geq b_1(P^-_i)$. Note that $(T^+; P^+_i)$, $(T^-; P^-_i)$, $(T^+; P^+_i)$, and $(T^-; P^-_i)$ are not one of the exceptions $F_i(n^+)$ for $i \in \{1, 2, 3, 4, 5, 6, 7, 11, 12, 13, 14\}$ or one of their duals, otherwise $C$ would have a block of length at least $k + 2$. And $(T^+; P^+_i)$ is not one of the exceptions $F_i$ for $i \in \{8, 9, 10\}$ nor one of their duals, because $b_1((P^+_i)^{-1}) = 2$.

- If $b_1(P^+_i) + b_1(P^-_i) = 3$, then Lemmas 6.1.(2) and 6.1.(3) give the result.
- Suppose that $b_1(P^+_i) = b_1(P^-_i) = 2$. If $(a_n, a_{n+1}, a_{n+2})$ is directed, then Lemmas 6.1.(2) and 6.3 give the result. In the same way, if $(a_n, a_{n+1}, a_{n+2})$ is directed, $T$ contains $C$. So we may suppose that $(a_n, a_{n+1}, a_{n+2})$ is a block of $C$. Consider now $P^+_i$ and $P^-_i$. We have $b_1(P^-_i) = 1$. Recall that neither $(T^+; P^+_i)$ nor $(T^-; P^-_i)$ is an exception $F_i$, $i \in \{1, 2, 3, 4, 5, 6, 7, 11, 12, 13, 14\}$, or one of their duals. Thus, by Lemma 6.1.(1), $|X^+_i| \geq n^+ - 2$. Hence, by Lemma 6.3.(6), $T$ contains $C$.
- Suppose that $b_1(P^+_i) + b_1(P^-_i) = 3$. If $b_1(P^+_i)$ and $b_1(P^-_i)$ are contained in the same block of $C$, then $|X^+_i| \geq n^+ - k + 3$. Thus by Lemma 6.3.(5) $T$ contains $C$.

So we may suppose that $b_1(P^+_i)$ and $b_1(P^-_i)$ are not contained in the same block of $C$. Thus $b_1(P^-_i) = 1$ or $b_1(P^-_i) = 1$.

If $b_1(P^-_i) = 1$, consider $P^+_i$ and $P^-_i$. By Lemma 6.1, $|X^+_i| \geq n^+ - 2$. Thus, by Lemma 6.3.(6) $T$ contains $C$.

Suppose now that $b_1(P^+_i) = 1$. If $(T^-; P^-_i)$ is not the exception $F_i(n^+)$, then by Lemma 6.1.(1) $|X^-_i| \geq n^- - 2$. (Note that $(T^-; P^-_i) \neq F_i(n^+)$, otherwise it would have a block of length at least $k + 2$.) If $(T^+; P^+_i)$ is not one of the exceptions $F_i$ for $i \in \{2, 3, 4, 5, 6, 7, 11, 12, 13, 14\}$ or one of their duals then, by Lemma 6.3.(6), $T$ contains $C$. On the other hand, if $(T^+; P^+_i)$ is one of the exceptions $F_i$ for $i \in \{2, 3, 4, 5, 6, 7, 11, 12, 13, 14\}$ or one of their duals, then $C$ has a block of length at least $7$. So $k \geq 6$. Also, $|X^+_i| = n^+ - 2$. Hence, by Lemma 6.3.(3), $T$ contains $C$.

If $(T^-; P^-_i)$ is the exception $F_i(n^+)$, let us consider $P^+_i$ and $P^-_i$. We have $b_1(P^+_i) = 1$ and $(T^-; P^-_i)$ is not exception $F_i$, otherwise $C$ would have a block of length greater than $k + 1$. So by Lemma 6.1.(1) we have $|X^+_i| \geq n^+ - 2$. Thus by Lemma 6.3.(6) $T$ contains $C$. \]
Theorems 4.3, 5.1, 5.2. and 7.1 yield the following corollary:

**COROLLARY 7.1.** Every tournament of order at least 168 contains every nondirected Hamiltonian cycle.

**LEMMA 7.1.** Let $T$ be a $(\text{=}5)$-strong tournament of order $n = 3m \geq 15$ and $C$ a cycle of order $n$ and of type $(2, 1, 2, 1, \ldots, 2, 1)$. Then $T$ contains $C$.

**Proof.** Let $C = (a_0, a_1, \ldots, a_{n-1}, a_0)$ with $a_0 \rightarrow a_1 \rightarrow a_2$. Let $X$ be a subtournament of five vertices such that $T - X$ is reducible. By duality, we may suppose that the maximal strong component of $T - X$ is at least as big as its minimal strong component $T_1$. Set $T_2 = T - (X, T_1)$. We have $T_1 \rightarrow T_2$. Let $R = (a_2, a_3, \ldots, a_6) = -(1, 2, 1)$. It is easy to check that there are at least three origins of $R$ in $X$. So there is a vertex $x$ of $X$ that is an origin of $R$ in $X$ and such that $d^- (x) \leq 3$. Let $y$ be the terminus of the path $R$ in $T$ with origin $x$. Since $T$ is $(\text{=}5)$-strong, $y$ dominates an outgenerator $t$ of $T_1$ and $x$ is dominated by an ingenerator $u$ of $T_2$.

If $|T_1| \geq 3$, let $t'$ be the terminus of a Hamiltonian directed outpath of $T_1$ with origin $t$, and let $P = (a_7, a_8, \ldots, a_{n-1}) = +(1, 1, 2, 1, 2, \ldots, 1, 2)$. Obviously, $A^+(P) \geq |T_1 - t'|$. So by Lemma 2.1, one can find $P$ in $(T_1 - t') \rightarrow (T_2 - u)$ with origin $t$ and terminus in $T_2 - u$. Thus $T$ contains $C = (t', u, R, P, t')$.

If $T_1 = \{t\}$, then $|T_2| \geq 9$. Let $U$ be the set of vertices of $T_2$ that dominate $x$. Because $T$ is $(\text{=}5)$-strong, $d^- \left( x \right) \geq 5$ so $|U| \geq 2$. Set $Q = (a_1, a_0, a_{n-1}, \ldots, a_6) = -(1, 1, 2, 1, 2, 1, 2, 1)$. By Theorem 2.2, since $(T_2, Q)$ is not an exception $\mathcal{F}_d$, some vertex $v \in U$ is an origin of $Q$ in $T_2$. Then $T$ contains $C = (v, R, t, Q^{-1})$.

**LEMMA 7.2.** Let $T$ be a $(\text{=}5)$-strong tournament of order $n = 5m \geq 15$ and $C$ a cycle of order $n$ of type $(3, 2, 3, 2, \ldots, 3, 2)$. Then $T$ contains $C$.

**Proof.** Let $C = (a_0, a_1, \ldots, a_{n-1}, a_0)$ with $a_0 \rightarrow a_1 \rightarrow a_2 \rightarrow a_3$. Let $X$ be a subtournament of five vertices such that $T - X$ is reducible. By duality, we may suppose that the maximal strong component of $T - X$ is at least as big as its minimal strong component $T_1$. Set $T_2 = T - (X, T_1)$. We have $T_1 \rightarrow T_2$. Let $R = (a_2, a_3, \ldots, a_6) = +(1, 2, 1)$. It is easy to check that there are at least three origins of $R$ in $X$. So there is a vertex $x$ of $X$ that is an origin of $R$ in $X$ and such that $d^- \left( x \right) \leq 3$. Let $y$ be the terminus of the path $R$ in $T$ with origin $x$. Since $T$ is $(\text{=}5)$-strong, $y$ dominates an outgenerator $t$ of $T_1$ and $x$ is dominated by an ingenerator $u$ of $T_2$.

If $|T_1| \geq 3$, let $t'$ be the terminus of a Hamiltonian directed outpath of $T_1$ with origin $t$, and let $P = (a_7, a_8, \ldots, a_{n-1}) = +(1, 2, 3, 2, \ldots, 3, 2, 3, 1)$. It
is easy to check that \( A^+(P) \geq |T_1 - t'|. \) So by Lemma 2.1, in \((T_1 - t') \to (T_2 - u)\), one can find \( P \) with origin \( t \) and terminus in \( T_2 - u \). Thus \( T \) contains \( C = (t', u, R, P, t'). \)

Suppose now that \( T_1 = \{ t \} \), so that \(|T_2| \geq 9. \) Let \( U \) be the set of vertices of \( T_2 \) that dominate \( x \). Because \( T \) is \((5,5,5)-strong\), \( d_T(x) \geq 5 \) so \(|U| \geq 2. \)

Set \( Q = (a_1, a_0, a_{n-1}, ..., a_k) = -(1, 2, 3, 2, ..., 3, 2). \) By Theorem 2.2, since \((T_2; Q)\) is not one of the exceptions \( \mathcal{F}_7 \), \( 1 \leq i \leq 14 \), some vertex \( v \in U \) is an origin of \( Q \) in \( T_2. \) Then \( T \) contains \( C = (v, R, t, Q^{-1}). \)

**Lemma 7.3.** Let \( T \) be a \( k \)-strong tournament of order \( n \) where \( k \geq 5, \) \( C \) a cycle of order \( n, \) and \( B \) a block of \( C \) with maximal length \( L, \) where \( 2 \leq L \leq k + 1. \) If all blocks of \( C \) except possibly \( B \) have length at most five and if \( T \) has a vertex \( x \) with both in- and outdegree greater than \( 10, \) then \( T \) contains \( C. \)

**Proof.** Let \( B \) be a block of \( C \) of maximal length \( L, \) let \( l = \lfloor L/2 \rfloor, \) and let \( C = (a_0, a_1, ..., a_{n-1}, a_0) \) be such that \( a_0 \in B \) and \( a_{n-l} \to a_{n-l+1} \to \cdots \to a_{l-1} \to a_l. \)

We define \( n^+, n^-, T^+, T^-, P^+_1, P^-_1, X^+_1, \) and \( X^-_1 \) in the same way as in Lemma 6.1. Note that for each path \( P^+_1 \) (resp. \( P^-_1 \)), the pair \((T^+; P^+_1)\) (resp. \((T^-; P^-_1)\)) is not an exception \( \mathcal{F}_j, j \in \{1, 2, 3, 4, 5, 6, 7, 8, 11, 12, 13, 14\}; \) indeed, if it were, \( C \) would have a block of length at least \( n^+ + 4 > 5 \) (resp. \( n^- + 4 > 5 \)).

Suppose that \((T^+; P^+_1)\) is one of the exceptions \( \mathcal{F}_9(n^+), \mathcal{F}_9(n^-) \) or one of their duals and that the termini of \( P^+_1 \) and \( P^-_1 \) are in \( B. \) Since \( n^+ \geq 11 \), we have \( b_1(P^+_1) \geq 3, \) so \( L \geq 3. \)

Suppose first that \( L \geq 4. \) Because the last vertex of \( P^+_1 \) is in \( B \) and the last block of \( P^+_1 \) has length one, we have \( a_{n-l-2} \to a_{n-l-1}. \)

Suppose that \( \mathcal{F}_9(n^+) \) or \( \mathcal{F}_9(n^-) \) or one of their duals because \((T^+; P^+_1)\) was one of these. Moreover \( b_1(P^+_1) = 1, \) so \((T^-; P^-_1)\) is not an exception. Thus the result follows from Lemmas 6.1.(2) and 6.3.(6).

Suppose now that \( n^- \geq 11. \) Since \( n^- \geq 11, \) \((T^-; P^-_1)\) is \( \mathcal{F}_9(11) \) or its dual. So \( n^+ \geq 11 \) and \( n^- = n - 12 = 31. \) Thus, for every \( l, \) \((T^-; P^-_1)\) is not
one of the exceptions \( \mathcal{F} \) or one of their duals. Also, \((a_{n-r-i+1}, a_{n-r-i}, a_{n-r-i+1})\) is not directed. If \(a_{n-i-2} \rightarrow a_{n-i-1}\), let us consider \(P^+_{n-1}\) and \(P^-_{n-1}\). Since \(L = 3\), \(b_1(P^+_{n-1}) \leq 3\) and Lemmas 6.1.(2) and 6.3.(6) yield the result. If \(a_{n-i-2} \leftarrow a_{n-i-1}\), then \(a_{n-i-2} \rightarrow a_{n-i-1}\). Let us consider \(P^+_{n-1}\) and \(P^-_{n-1}\). We have \(b_1(P^-_{n-1}) \leq 3\) and \(b_1(P^+_{n-1}) = 1\). Then Lemmas 6.1.(2) and 6.3.(3) yield the result.

In the same way, the statement holds if \((T^-; P^-)\) is the exception \(\mathcal{F}_{(n^-)}\) or its dual.

In the rest of the proof, we always suppose that \(P^+\) and \(P^-\) have their termini in \(B\). So, by the previous argument, we may assume that neither \((T^+; P^+)\) nor \((T^-; P^-)\) is an exception \(\mathcal{F}_j\), \(1 \leq j \leq 14\), or one of their duals, and we may apply Lemma 6.1.(2).

If \(b_1(P^+_0) + b_1(P^-_0) \leq 3\), then \(|X^+_0| \geq n^+ - b_1(P^+_0)\) and \(|X^-_0| \geq n^- - b_1(P^-_0)\) by Lemma 6.1.(2). So by Lemma 6.3.(3) \(T\) contains \(C\) and we may assume that \(b_1(P^+_0) + b_1(P^-_0) = 4\).

(A) Suppose first that \(b_1(P^+_0) + b_1(P^-_0) = 5\). Then \(L \geq 3\). Moreover \(b_1(P^+_0)\) and \(b_1(P^-_0)\) are not contained in the same block of \(C\).

(i) Suppose that \(L \geq 4\). Then \(a_{n-2} \rightarrow a_{n-1}\) and \(a_1 \rightarrow a_2\).

If \(b_1(P^+_0) = 1\), then \(b_1(P^-_0) \geq 4\). Let us consider \(P^+\) and \(P^-\). We have \(b_1(P^-) \leq 2\), so by Lemma 6.1.(2) \(|X^-_0| \geq n^- - 2\). And since \(b_1(P^+_0) = 4\), \((a_{n-2}, a_{n-1}, a_n)\) is a directed path, so by Lemma 6.3.(6) \(T\) contains \(C\).

Similarly, we have the result if \(b_1(P^-_0) = 1\).

So we may assume that \(b_1(P^-_0) \geq 2\) and \(b_1(P^+_0) \geq 2\). If the path \((a_n, a_{n-1}, a_{n-2})\) is not directed, let us consider \(P^+\) and \(P^-\). We have \(b_1(P^-) = 1\), thus by Lemma 6.1.(2) \(|X^-_0| \geq n^- - 1\). And since \(b_1(P^+_0) = 2\), \((a_{n-2}, a_{n-1}, a_n)\) is a directed path, so by Lemma 6.3.(6) \(T\) contains \(C\).

In the same way, by considering \(P^+\) and \(P^-\), we have the result if \((a_{n-1}, a_{n-2}, a_{n-3})\) is not directed.

(ii) Suppose now that \(L = 3\).

Suppose that \(b_1(P^+_0) = b_1(P^-_0) = 3\). Since \(L = 3\), \((a_n, a_{n-1}, a_{n-2})\) is a block of \(C\). By symmetry, we may suppose that \(a_{n-2} \rightarrow a_{n-1}\). Let us consider \(P^+\) and \(P^-\). We have \(b_1(P^-) = 1\) and \(b_1(P^+) = 2\). So by Lemmas 6.1.(2) and 6.3.(3) \(T\) contains \(C\).

We may therefore assume that \(b_1(P^+_0) + b_1(P^-_0) = 5\). By symmetry, we may also suppose that \(b_1(P^+_0) = 3\) and \(b_1(P^-_0) = 2\).

Since \(L = 3\), \(a_{n-2} \rightarrow a_{n-1}\) or \(a_1 \rightarrow a_2\). If \(a_1 \rightarrow a_2\), consider \(P^+\) and \(P^-\). We have \(b_1(P^-) = b_1(P^+)= 1\). Lemmas 6.1.(2) and 6.3.(3) now yield the result. So we may assume that \(a_{n-2} \rightarrow a_{n-1}\). Since \(L = 3\), either \(a_n \rightarrow a_{n-1}\) or \(a_{n-1} \rightarrow a_{n-2}\). If \(a_n \rightarrow a_{n-1}\), Lemma 6.3.(4) applied to \(P^+\) and \(P^-\) gives the result. Similarly, if \(a_{n-1} \rightarrow a_n\), Lemma 6.3.(4) applied to \(P^+\) and \(P^-\) gives the result.
(B) Suppose now that \( b_1(P^+_0) + b_1(P^-_0) = 4 \). By symmetry, we may suppose that \( b_1(P^+_0) \geq b_1(P^-_0) \). So \( b_1(P^+_0) \geq 2 \). If \( T \) is 6-strong, then Lemmas 6.1.(2) and 6.3.(3) yield the result. Hence we may suppose that \( T \) is \((5)-strong. If \( L = 2 \), then \( b_1(P^+_0) = b_1(P^-_0) = 2 \). Moreover, \((a_{n+1}, a_{n+2})\) is a block of \( C \). Thus \((a_{n-1}, a_{n-2}, a_{n+1}, a_{n+2}, a_{n+3}) = (2, 1, 2). Now, applying this to the two blocks of length two of this path, either \( T \) contains \( C \) or \( C \) has a subpath \((2, 1, 2, 1, 2) \). By induction, we may suppose that \( C = (2, 1, 2, 1, ..., 2, 1) \), and by Lemma 7.1, \( T \) contains \( C \).

So we may suppose that \( L = 3 \).

(i) Assume that \( a_{n-2} \rightarrow a_{n-1} \) (Note that this is so when \( L \geq 4 \)). Consider \( P^+_0 \) and \( P^-_1 \).

If \( b_1(P^-_0) \leq 2 \), by Lemma 6.1.(2) \( |X^+_1| \geq n - 2 \). Also \((a_{n-2}, a_{n-1}, a_{n})\) is directed because \( b_1(P^+_0) \geq 2 \). Hence, by Lemma 6.3.(6), \( T \) contains \( C \).

If \( b_1(P^-_0) = 3 \), then \( b_1(P^+_0) = 2 \).

If the vertices \( a_{n-1} \) and \( a_{n+1} \) are not in the same block of \( C \), then either \( a_{n-1} \rightarrow a_{n+1} \) or \( a_{n-1} \rightarrow a_{n+1} \). If \( a_{n-1} \rightarrow a_{n+1} \), then Lemma 6.3.(4) applied to \( P^+_0 \) and \( P^-_0 \) gives the result.

If \( a_{n-1} \rightarrow a_{n+1} \), Lemma 6.3.(4) applied to \( P^+_1 \) and \( P^-_0 \) yields the result.

If the vertices \( a_{n-1} \) and \( a_{n+1} \) are in the same block of \( C \), then \( B' = (a_{n-2}, a_{n-1}, ..., a_{n+3}) \) is a block of \( C \) of length five. Thus \((a_{n-1}, a_{n}, a_{n+1})\) is directed. Let us consider \( P^+_0 \) and \( P^-_0 \). By Lemma 6.1.(2), \( |X^+_1| \geq n - 2 \); hence, by Lemma 6.3.(6), \( T \) contains \( C \).

(ii) Suppose that \( a_{n-2} \rightarrow a_{n-1} \). Then \( a_1 = a_2 \) and \( L = 3 \).

Assume that \( b_1(P^-_0) = b_1(P^+_0) = 2 \).

If \((a_{n-1}, a_{n}, a_{n+1})\) is not directed, then \( b_1(P^+_0) \geq 1 \). Hence, by Lemmas 6.1.(2) and 6.3.(3), \( T \) contains \( C \).

If \((a_{n-1}, a_{n}, a_{n+1})\) is directed, then by considering \( P^+_0 \) and \( P^-_0 \), Lemmas 6.1.(2) and 6.3.(6) yield the result.

Assume now that \( b_1(P^-_0) = 1 \) and \( b_1(P^+_0) = 3 \).

If \( a_{n} \rightarrow a_{n-1} \), Lemmas 6.1.(2) and 6.3.(4) give the result. So we may assume that \( a_{n} \leftarrow a_{n-1} \). Consider now \( P^+_1 \) and \( P^-_1 \). Since \( L = 3 \), \( b_1(P^+_1) = 1 \). Also \( b_1(P^-_1) \leq 3 \). If \( a_{n-1} \rightarrow a_{n-2} \), Lemmas 6.1.(2) and 6.3.(4) give the result and if \( b_1(P^-_1) \leq 2 \), Lemmas 6.1.(2) and 6.3.(4) give the result.

So \((a_{n-3}, a_{n-2}, ..., a_{n+1}) = (3, 2, 3) \). Now, applying this method to the two blocks of length 3 of this path, either \( T \) contains \( C \) or \( C \) has a subpath \((3, 2, 3, 2, 3) \). And so on, by induction, we may suppose that \( C = (3, 2, 3, 2, ..., 3, 2) \) and, by Lemma 7.2, \( T \) contains \( C \).

Theorems 7.1 and 7.3 yield:

**Corollary 7.2.** Every 5-strong tournament of order at least 43 contains every Hamiltonian cycle.
This corollary and Corollary 5.1 give the following statement:

**Corollary 7.3.** Every tournament of order at least 105 contains every nondirected Hamiltonian cycle.

8. **HAMILTONIAN CYCLES IN 4-STRONG TOURNAMENTS**

**Lemma 8.1.** Let $T$ be a $(=4)$-strong tournament of order $n \geq 19$ and $C$ a cycle of order $n$ whose maximal block is of length at most five and which contains a subpath of type $(3, 2, 3)$. Then $T$ contains $C$.

**Proof.** Let $C = (a_0, a_1, \ldots, a_{n-1}, a_0)$ with $a_0 \rightarrow a_1 \rightarrow a_2 \rightarrow a_3 \leftarrow a_4 \leftarrow a_5 \rightarrow a_6 \rightarrow a_7 \rightarrow a_8$. Let $X$ be a subtournament of four vertices such that $T - X$ is reducible. By duality, we may suppose that the maximal strong component of $T - X$ is at least as big as its minimal strong component $T_1$. Set $T_2 = T - (X, T_1)$. We have $T_1 \rightarrow T_2$. Set $R = (a_2, a_3, a_4, a_5) = +(1, 2)$. It is easy to check that there are at least two origins of $R$ in $X$. So there is a vertex $x$ of $X$ that is an origin of $R$ in $X$ and is such that $d^+_X(x) \leq 2$. Let $y$ be the terminus of the path $R$ in $T$ with origin $x$. Since $T$ is $(=4)$-strong, $y$ dominates an outgenerator $t$ of $T_1$ and $x$ is dominated by an ingenerator $u$ of $T_2$.

If $|T_1| = 1$, let $P = (a_7, a_8, \ldots, a_1)$. Let $U_x$ be the set of vertices of $T_3$ that are dominated by $x$. Since $T$ is 4-strong, $|U_x| \geq 2$. Also $|T_2| \geq 14$, so $(T_2; P)$ is not one of the exceptions $\mathcal{F}$ or one of their duals, otherwise $C$ would have a block of length greater than five. Hence, by Theorem 2.2, there is a path $P$ in $T_2$ with terminus in $U_x$. Thus $T$ contains $C = (R, T_1)$. If $|T_1| \geq 3$, let $a_i$ be the origin of the block $B$ that contains $a_0 \rightarrow a_1$. Let $I$ be an inpath of length $b - 3$ with origin $a_i$, where $b$ is the length of $B$, and let $v$ be an outgenerator of $N^+_T(I)$. Let $P$ be the path $(a_i, a_{i+1}, \ldots, a_k)$ and $Q$ the path $(a_1, a_2, \ldots, a_k)$. Either $b_1(P) \geq |T_1| - 2$ or $b_1(Q) \geq |T_1| - 2$, because $|T_1| \leq |T_2|$.

If $b_1(P) \geq |T_1| - 2$, let $w$ be the terminus of a Hamiltonian directed outpath of $T_1 - t$ with origin $v$. By Lemma 2.1, in $T'' - (t, w, I)$, there is a path $P$ with origin $v$ and terminus in $T_2 - I$. Hence $T$ contains $C = (v, f^{-1}, R, t, P, w)$.

If $b_1(Q) \geq |T_1| - 2$, let $w$ be an outgenerator of $T_1 - (t, v)$. By Lemma 2.1, in $T'' - (t, v, I)$, there is a path $Q$ with origin $w$ and terminus in $T_2 - I$. Hence $T$ contains $C = (v, f^{-1}, R, t, v, Q^{-1})$.

**Lemma 8.2.** Let $T$ be a $(=4)$-strong tournament of order $n \geq 55$ and $C$ a cycle of order $n$ whose longest block $B$ is of length four or five. Then $T$ contains $C$. 
Proof. Let $C=(a_0, a_1, ..., a_{n-1}, a_0)$, where $a_{n-2} \rightarrow a_{n-1} \rightarrow a_0 \rightarrow a_1 \rightarrow a_2$. Let $x$ be a vertex of $T$ such that $n^+ = |N^+_T(x)|$ and $n^- = |N^-_T(x)|$ are greater than 13. Such a vertex exists since $|T| \geq 55$. Let us denote by $T^+$ and $T^-$ the subtournaments $T(N^+_T(x))$ and $T(N^-_T(x))$. We define $P^+_x$, $P^-_x$, $X^+_x$, and $X^-_x$ in the same as in Lemma 6.1. Note that for each path $P^+_x$ (resp. $P^-_x$), the pair $(T^+; P^+_x)$ (resp. $(T^-; P^-_x)$) is not one of the exceptions $\tilde{\mathcal{X}}$, $1 \leq j \leq 14$ or one of their duals; indeed, if it were, $C$ would have a block of length at least $n^+ - 8 > 5$ (resp. $n^- - 8 > 5$).

If $b(P^+_x) = b(P^-_x) = 1$, then Lemmas 6.1.2 and 6.3 give the result.

So, by symmetry, we may suppose that $b(P^+_x) \geq 2$.

Suppose that $B_i(P^+_x)$ and $B_i(P^-_x)$ are contained in the same block of $C$. Then either $b_i(P^+_x) = 1$ or $b_i(P^-_x) = 1$. In both cases, by Lemmas 6.1.2 and 6.3.6, $T$ contains $C$. Thus we may assume that $B_i(P^+_x)$ and $B_i(P^-_x)$ are not contained in the same block.

If $b_i(P^+_x) = 1$, by Lemmas 6.1.2 and 6.3.6, $T$ contains $C$.

If $b_i(P^-_x) \geq 3$, consider $P^+_x$ and $P^-_{x-1}$. Then $B_i(P^+_x)$ and $B_i(P^-_{x-1})$ are contained in the same block and, as before, $T$ contains $C$. Hence we may assume that $b_i(P^-_x) = 2$ and thus that $b_i(P^+_x) = 1$.

If $a_{n-1} \rightarrow a_{n^+}$ then $P^-_x$ is an inpath. By Lemma 6.1, $|X^-_x| \geq n^+ - 2$. Suppose, by way of contradiction, that $T$ does not contain $C$. Then, by Lemma 6.3, $X^-_x \rightarrow X^+_x$. If $|X^+_x| \geq n^+ - 1$, then $T(X^-_x, X^+_x, x)$ is reducible, of order $n - 3$. This is a contradiction. If $|X^+_x| \leq n^+ - 2$, then by Lemma 6.2, $(T \Rightarrow X^+_x) \rightarrow X^+_x$, so $T(X^-_x, X^+_x, x)$ is reducible, of order $n - 2$. This is again a contradiction.

Now we may assume that $a_{n^+} \leftarrow a_{n^+ - 1} \leftarrow a_{n^+} \rightarrow a_{n^+ + 1} \rightarrow a_{n^+ + 2} \leftarrow a_{n^+ + 3}$.

If $b_i(P^+_x) = 2$, then, Lemmas 6.1 and 6.3.4 applied to $P^+_x$ and $P^-_x$ give the result.

If $b_i(P^-_{x-1}) \leq 2$, then, Lemmas 6.1 and 6.3.4 applied to $P^+_x$ and $P^-_{x-1}$ give the result.

So $C$ contains a subpath $(3, 2, 3)$ and then, by Lemma 8.1, $T$ contains $C$.

By Theorems 4.3, 5.1, and 8.2, we obtain the following corollary:

**Corollary 8.1.** Every $(=4)$-strong tournament of order $n \geq 65$ contains every Hamiltonian cycle.

This corollary, together with Corollary 5.1, yields the following:

**Corollary 8.2.** Every tournament of order at least 68 contains every nondirected Hamiltonian cycle.
The bound 68 of Corollary 8.2 is of course not sharp. The right value is very likely to be 9, as remarked by Thomason [11]. This bound could be proved by our methods with the aid of a computer, there being many cases to check. Indeed, Theorem 2.2 has 50 exceptions of order less than 8. Thus, one can prove lemmas analogous to Lemmas 6.1 and 6.2 for tournaments of order less than 8, but with numerous exceptions. By studying the many cases generated by these exceptions, one could prove that every 5-strong tournament contains every Hamiltonian cycle as in the proof of Theorem 7.1. On the other hand, using Lemmas 2.1, 2.2, and 2.3, and proofs in the same vein as those of Theorems 4.1, 4.2, and 5.1 it would be possible to prove the result for tournaments which are not 5-strong.

In [6], we prove it for (=1)-strong tournaments:

**Theorem 8.1.** Let be a 1-strong tournament of order n and C a cycle of order n. Then T contains C if and only if (T; C) is neither one of the exceptions C_i for i ∈ {1, 4, 5, 6, 9, 10, 11, 12, 13, 14, 15} (Figs. 2 and 7) nor one of their duals.

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