Journal of Combinatorial Theory, Series B **80**, 1–31 (2000)
doi:10.1006/jctb.2000.1959, available online at http://www.idealibrary.com on IDEAL®

Oriented Hamiltonian Cycles in Tournaments

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Received July 20, 1999

We prove that every tournament of order $n \geqslant 68$ contains every oriented Hamiltonian cycle except possibly the directed one when the tournament is reducible. © 2000 Academic Press

1. INTRODUCTION

1.1. *Definitions*

DEFINITION 1.1. A *tournament* is an orientation of the arcs of a complete graph. An *oriented path* is an orientation of a path. An *oriented cycle* is an orientation of a cycle. From now on, we abbreviate oriented *path* and oriented cycle to *cycle*.

— Let T be a tournament and let D be a digraph. We say that T *contains* D if D is a subgraph of T. The *order* of a digraph D, denoted by $|D|$, is its number of vertices. Let x and y be two vertices of T. We write $x \rightarrow y$ if (x, y) is an arc of T. Likewise, let X and Y be two subdigraphs of T. We write $X \rightarrow Y$ if $x \rightarrow y$ for all pairs $(x, y) \in V(X) \times V(Y)$. Let $A_1, A_2, ..., A_k$ be a family of subdigraphs of T. We denote by $T(A_1, A_2, ..., A_k)$ the subtournament induced by T on the set of vertices $\bigcup_{1 \leqslant i \leqslant k} V(A_i)$ and by $T - (A_1, A_2, ..., A_k)$ the subtournament induced by T on the set of vertices $V(T) \backslash \bigcup_{1 \leqslant i \leqslant k} V(A_i)$.

— Let $P = (x_1, ..., x_n)$ be a path. We say that x_1 is the *origin* of P and x_n is the *terminus* of P. If $x_1 \rightarrow x_2$, P is an *outpath*, otherwise P is an inpath. The *directed outpath* of order n is the path $P = (x_1, ..., x_n)$ in which $x_i \rightarrow x_{i+1}$ for all i, $1 \leqslant i < n$; the dual notion is the *directed inpath*; both directed out- and inpaths are *dipaths*. The *length* of a path is its number of arcs. We denote the subpath $(x_1, ..., x_{n-1})$ of P by P^* and the subpath $(x_2, ..., x_n)$ of P by $^*P$. The path $(x_n, x_{n-1}, ..., x_1)$ is denoted P^{-1}.

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The *blocks* of P are the maximal subdipaths of P. We enumerate the blocks of P from the origin to the terminus. The ith block of P is denoted $B_i(P)$ and its length $b_i(P)$. The path P is totally described by the signed sequence $sgn(P)(b_1(P), b_2(P), ..., b_k(P))$ where k is the number of blocks of P and $sgn(P) = +$ if P is an outpath and $sgn(P) = -$ if P is an inpath. A block is *positive* or *negative* according to the sense of its arcs.

We define $A^+(P) := |\{i, x_i \to x_{i+1}\}|$ and $A^-(P) := |\{i, x_i \leftarrow x_{i+1}\}|$.

— Let $C = (x_0, ..., x_n = x_0)$ be a cycle. If for all i, $x_i \to x_{i+1}$, C is said to be *directed*. If C is not a directed cycle, the *blocks* of C are its maximal subdipaths. Notice that such a cycle necessarily has an even number of blocks, half positive and half negative, and is totally described by the signed sequence $(b_1, b_2, ..., b_{2k})$ where $2k$ is the number of blocks of C.

— The *dual* (sometimes called the *converse*) digraph of D is the digraph $-D$ on the same set of vertices such that $x \to y$ is an arc of $-D$ if and only if $y \to x$ is an arc of D.

— A tournament is *strong* (or *strongly connected*) if for any two vertices x and y there exists a directed outpath with origin x and terminus y. A nonstrong tournament is said to be *reducible*. A tournament T is reducible if and only if it admits a *decomposition* into two subtournaments T_1 and T_2 such that $V(T_1) \cup V(T_2) = V(T)$ and $T_1 \to T_2$; in this case, we often write $T = T_1 \to T_2$.

A tournament is *k-strong*, if $T - Y$ is strong for any set Y of $k-1$ vertices. A tournament is *(=k)-strong* or *exactly k-strong*, if it is k-strong and not $(k+1)$-strong. A (strong) *component* of T is a strong subtournament of T which is maximal by inclusion.

— Let X be a set of vertices of T. The *outsection* generated by X in T is the set of vertices y to which there exists a directed outpath from $x \in X$; we denote this set by $S_T^+(X)$ (or by $S_T^+(x)$ if $X = \{x\}$ and $S_T^+(x, y)$ if $X = \{x, y\}$). The dual notion, the *insection*, is denoted by $S_T^-(X)$. We also write $s_T^+(X)$ (resp. $s_T^-(X)$) for the number of vertices of $S_T^+(X)$ (resp. $S_T^-(X)$). An *outgenerator* of T is a vertex $x \in T$ such that $S_T^+(x) = V(T)$; the dual notion is an *ingenerator*. We denote by $Out(T)$ the set of outgenerators of T and by $In(T)$ the set of ingenerators of T. Notice that for a strong tournament $\mathrm{Out}(T) = V(T) = In(T)$.

Conventions for drawings. All the figures in this paper represent tournaments. Subtournaments are often delimited by ellipses. If T_1 and T_2 are subtournaments such that $T_1 \to T_2$, we draw an arc from the ellipse delimiting T_1 and to the one delimiting T_2. An undrawn arc represents one that may be oriented either way.

1.2. *Presentation of Results*

One of the first results on cycles in tournaments is Camion's theorem [4]: a tournament has a directed Hamiltonian cycle if and only if it is strongly connected. More generally, one can seek arbitrary orientations of cycles. In 1974, Rosenfeld conjectured the following:

Rosenfeld's cycle conjecture. There is an integer $N > 8$ such that every tournament of order $n \geqslant N$ contains every nondirected cycle of order n.

An analogous conjecture of Rosenfeld for paths was recently proved by Havet and Thomassé [7]:

THEOREM 1.1 (Havet and Thomassé [7]). *Let T be a tournament of order n and P a path of order $k \leqslant n$. Then T contains P if and only if $(T; P)$ is not one of Grünbaum's exceptions.*

Grünbaum's exceptions are $(3A; \pm(1, 1))$, $(5A; \pm(1, 1, 1, 1))$, and $(7A; \pm(1, 1, 1, 1))$ where $3A$ is the 3-cycle, $5A$ the regular tournament on five vertices, and $7A$ the Paley tournament on seven vertices. These three tournaments are called *Grünbaum's tournaments* and are depicted in Fig. 1.

The existence of Grünbaum's exceptions implies the existence of tournaments that do not contain certain Hamiltonian cycles. Indeed Grünbaum's tournaments do not contain the cycle obtained from a Hamiltonian alternating path by adding an arc from its terminus to its origin (exceptions $\mathscr{C}_1$, $\mathscr{C}_2$, and $\mathscr{C}_3$ of Fig. 2). Moreover, the tournaments of order n that have a subtournament on $n-1$ vertices isomorphic to one of Grünbaum's tournaments do not contain a Hamiltonian alternating cycle (exceptions $\mathscr{C}_4$, $\mathscr{C}_5$, and $\mathscr{C}_6$).

Rosenfeld's cycle conjecture has been proved for cycles with a block of length $n-1$ by Grünbaum [5], for alternating cycles (in which consecutives arcs have opposite senses) by Thomassen [12] ($n \geqslant 50$),

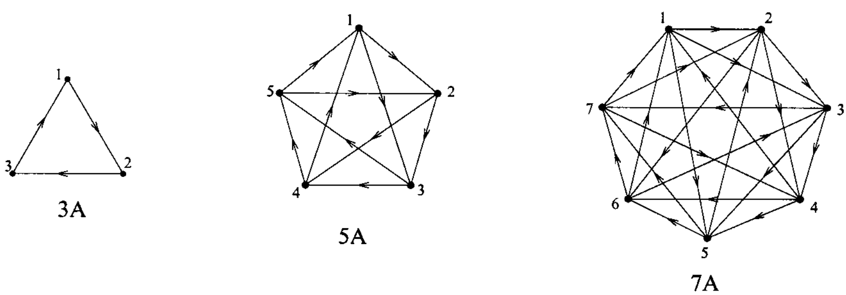

FIG. 1. Grünbaum's tournaments.

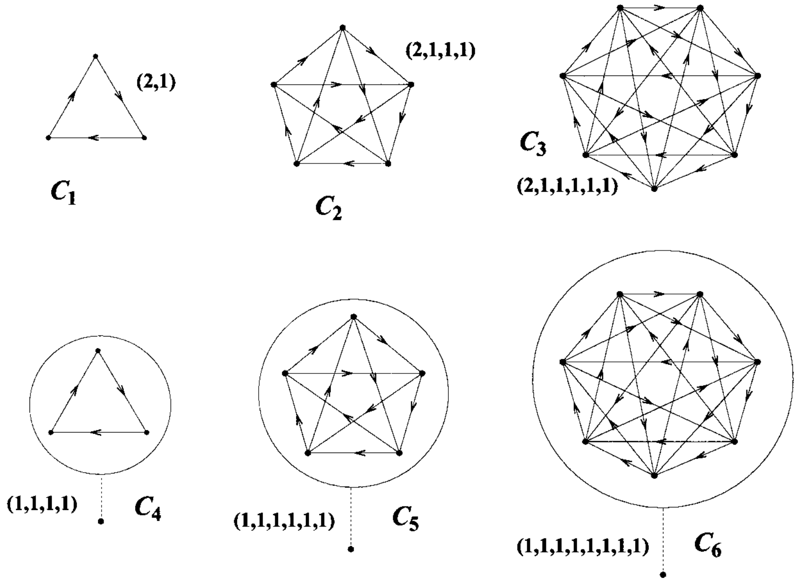

FIG. 2. The exceptions C_i, $1 \leqslant i \leqslant 6$.

Rosenfeld [10] ($n \geqslant 28$), and Petrović [9] ($n \geqslant 16$), and for cycles with just two blocks by Benhocine and Wojda [2]. Rosenfeld's conjecture was finally proved in full generality by Thomason [11] for tournaments of order $n \geqslant 2^{128}$. While Thomason made no attempt to sharpen this bound, he indicated that it should be true for tournaments of order at least 9.

We here prove that every tournament of order $n \geqslant 68$ contains every oriented Hamiltonian cycle except possibly the directed one (when the tournament is reducible). We first show in Theorem 3.1 that every reducible tournament of order $n \geqslant 9$ contains every nondirected Hamiltonian cycle. This allows us to restrict ourselves to strong tournaments. We next prove (Theorem 4.3) that every $(=k)$-strong tournament of order $n \geqslant 9$ contains every nondirected Hamiltonian cycle containing a block of length at most $k+2$. Then, we exhibit a function f depending on L and k such that every $(=k)$-strong tournament of order at least $f(L, k)$ contains every Hamiltonian cycle whose blocks have length at most L. From this function and Theorem 4.3, we derive a function g such that every $(=k)$-strong tournament of order $g(k)$ contains every Hamiltonian cycle. Alas, this function tends to infinity with k. So it does not directly imply Rosenfeld's cycle conjecture, but it allows one to prove it for k-strong tournaments, for some k. Theorem 6.1 asserts that every 8-strong tournament contains every Hamiltonian cycle, implying Rosenfeld's cycle conjecture is true for $N=264$ (Corollary 6.2). Finally, we study Hamiltonian cycles in

5-strong and $(=4)$-strong tournaments in order to sharpen this bound to $N=68$ (Corollary 8.2).

2. THE TOOLS: PATHS IN TOURNAMENTS

2.1. *Paths with Prescribed Origin*

THEOREM 2.1 (Havet and Thomassé [7]). *Let T be a tournament of order n, P an outpath of order $k<n$, and x and y two vertices of T such that $s_T^+(x, y) \geqslant b_1(P)+1$. Then x or y is an origin of P in T.*

DEFINITION 2.1. The *exception* $\mathscr{E}_{49}$ is the pair $(8A;\ +(1, 1, 1, 1, 1, 1, 1))$, where $8A$ is the tournament depicted in Fig. 3. One can check that the vertices 1 and 2 are not origins of $+(1, 1, 1, 1, 1, 1, 1)$.

The *exeption* $\mathscr{E}_{50}$ is the pair $(8A;\ +(2, 1, 1, 1, 1, 1))$, where $8A$ is the tournament depicted in Fig. 3. One can check that the vertices 1 and 2 are not origins of $+(2, 1, 1, 1, 1, 1)$.

The *exeption* $\mathscr{E}_{51}$ is the pair $(8b;\ +(2, 1, 2, 1, 1))$, where $8B$ is the tournament depicted in Fig. 3. One can check that the vertices 2 and 8 are not origins of $+(2, 1, 2, 1, 1)$.

We introduce here the *families of exceptions* $\mathscr{F}_i$ of Theorem 2.2. Our notation $\mathscr{F}_i(n)$ represents an exception $(F_i(n), P)$, where $F_i(n)$ is the tournament on n vertices depicted as F_i in Fig. 4. In each case, we denote by S the set of vertices of $F_i(n)$ which are not origins of P in $\mathscr{F}_i(n)$. Finally, conditions are given on the tournament.

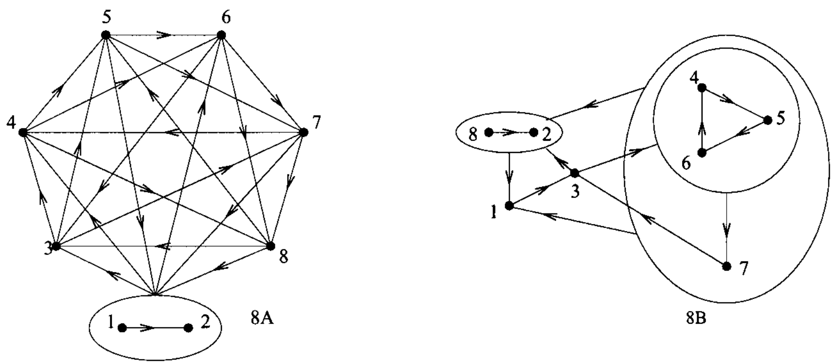

FIG. 3. The tournaments $8A$ and $8B$.

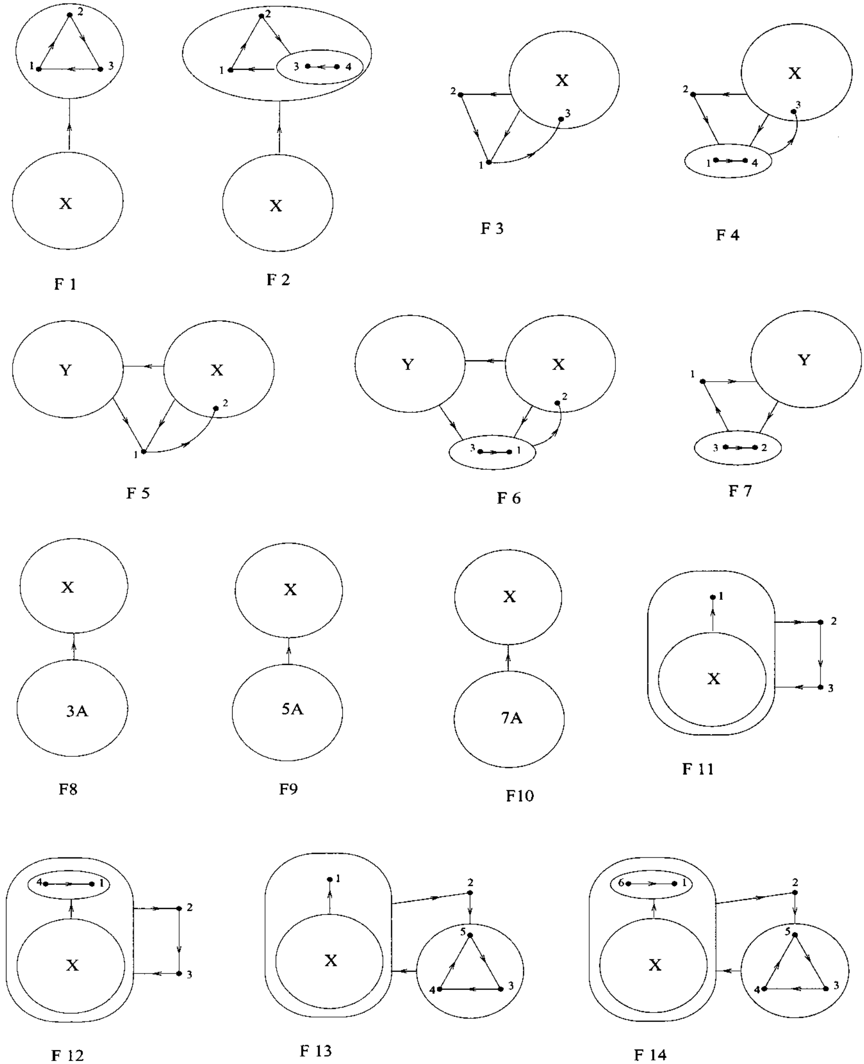

FIG. 4. The tournaments F_i.

$\mathscr{F}_1(n)=(F_1(n);\ +(1, n-2))$; $S=\{1, 2, 3\}$. Condition: $|X| \geqslant 1$.

$\mathscr{F}_2(n)=(F_2(n);\ +(2, n-3))$; $S=\{3, 4\}$. Condition: $|X| \geqslant 1$.

$\mathscr{F}_3(n)=(F_3(n);\ +(1, n-2))$; $S=\{1, 3\}$. Conditions: $N^+(3) \neq \{2\}$, and 3 is an ingenerator of $T(X)$.

$\mathscr{F}_4(n)=(F_4(n);\ +(2, n-3))$; $S=\{1, 4\}$. Conditions: $N^+(3) \neq \{2\}$, and 3 is an ingenerator of $T(X)$.

$\mathscr{F}_5(n) = (F_5(n);\ +(1, n-2))$; $S = \{1, 2\}$. Conditions: $n \geqslant 5$, $|Y| \geqslant 2$, and 2 is an ingenerator of $T(X)$.

$\mathscr{F}_6(n) = (F_6(n);\ +(2, n-3))$; $S = \{1, 3\}$. Conditions: $|Y| \geqslant 2$, and 2 is an ingenerator of $T(X)$.

$\mathscr{F}_7(n) = (F_7(n);\ +(1, 1, n-3))$, $S = \{2, 3\}$. Conditions: $T(Y)$ is not a 3-cycle (i.e., $T(Y)$) is not isomorphic to 3A), and $|Y| \geqslant 3$.)

$\mathscr{F}_8(n) = (F_8(n);\ +(n-4, 1, 1, 1))$; $S = X$. Conditions: 3A is the 3-cycle, its set of vertices is $\{1, 2, 3\}$, and $|X| \geqslant 2$.

$\mathscr{F}'_8(n) = (F_8(n);\ +(n-4, 2, 1))$; $S = X$. Conditions: 3A is the 3-cycle, its set of vertices is $\{1, 2, 3\}$, and $|X| \geqslant 2$.

$\mathscr{F}_9(n) = (F_9(n);\ +(n-6, 1, 1, 1, 1, 1))$; $S = X$. Conditions: 5A is the 2-regular tournament, its set of vertices is $\{1, 2, 3, 4, 5\}$; we need moreover $|X| \geqslant 2$.

$\mathscr{F}'_9(n) = (F_9(n);\ +(n-6, 2, 1, 1, 1))$; $S = X$. Conditions: 5A is the 2-regular tournament, its set of vertices is $\{1, 2, 3, 4, 5\}$, and $|X| \geqslant 2$.

$\mathscr{F}_{10}(n) = (F_{10}(n);\ +(n-8, 1, 1, 1, 1, 1, 1, 1))$; $S = X$. Conditions: 7A is the Paley tournament on seven vertices, its set of vertices is $\{1, 2, 3, 4, 5, 6, 7\}$, and $|X| \geqslant 2$.

$\mathscr{F}'_{10}(n) = (F_{10}(n);\ +(n-8, 2, 1, 1, 1, 1, 1))$; $S = X$. Conditions: 7A is the Paley tournament on seven vertices, its set of vertices is $\{1, 2, 3, 4, 5, 6, 7\}$, and $|X| \geqslant 2$.

$\mathscr{F}_{11}(n) = (F_{11}(n);\ +(1, 1, n-3))$; $S = \{1, 2\}$. Condition: $|X| \geqslant 2$.

$\mathscr{F}_{12}(n) = (F_{12}(n);\ +(2, 1, n-4))$; $S = \{1, 4\}$. Condition: $|X| \geqslant 2$.

$\mathscr{F}_{13}(n) = (F_{13}(n);\ +(1, 1, n-3))$; $S = \{1, 2\}$. Condition: $|X| \geqslant 2$.

$\mathscr{F}_{14}(n) = (F_{14}(n);\ +(2, 1, n-4))$; $S = \{1, 6\}$. Condition: $|X| \geqslant 2$.

THEOREM 2.2 (Havet and Thomassé [7]). *Let T be a tournament of order $n \geqslant 8$ and P an outpath of order $k \leqslant n$. For every two vertices x and y of T such that $s_T^+(x, y) \geqslant b_1(P) + 1$, x or y is an origin of P in T if and only if $(T; P)$ is not an exception $\mathscr{E}_{49}$, $\mathscr{E}_{50}$, $\mathscr{E}_{51}$, $\mathscr{F}_i(n)$, $1 \leqslant i \leqslant 4$, or $\mathscr{F}'_i(n)$, $i = 8, 9, 10$.*

2.2. *Paths with Prescribed Origin and Terminus*

DEFINITION 2.2. The *exception* $\mathscr{P}_1(n)$ is the pair $(E_1(n); Q_1(n))$, where $E_1(n)$ is a tournament of order n depicted in Fig. 5, with T_1 an arbitrary tournament of order $n-3$ and $Q_1(n) = +(n-2, 1)$.

The *exception* $\mathscr{P}_2$ is the pair $(E_1(4); Q_2)$, where $E_1(4)$ is the tournament of order 4 depicted in Fig. 5 and $Q_2 = +(1, 1, 1)$.

The *exception* $\mathscr{P}_3$ is the pair $(E_2; Q_3)$, where E_2 is the tournament depicted in Fig. 5 and $Q_3 = +(2, 1, 1, 1)$.

The *exception* $\mathscr{P}_4$ is the pair $(E_2; Q_4)$, where E_2 is the tournament depicted in Fig. 5 and $Q_4 = +(1, 1, 1, 1, 1)$.

The *exception* $\mathscr{P}_5$ is the pair $(E_3; Q_5)$, where E_3 is the tournament depicted in Fig. 5 and $Q_5 = +(2, 1, 1, 1, 1, 1)$.

The *exception* $\mathscr{P}_6$ is the pair $(E_3; Q_6)$, where E_3 is the tournament depicted in Fig. 5 and $Q_6 = +(1, 1, 1, 1, 1, 1, 1)$.

LEMMA 2.1. *Let $T = T_1 \rightarrow T_2$ be a reducible tournament of order n and P an outpath of order n such that*:

(1) $A^+(P) \geqslant |T|$;

(2) *$(T; P)$ is not an exception $\mathscr{P}_1(n)$ or $\mathscr{P}_i$, $2 \leqslant i \leqslant 6$.*

There exists $y \in T_2$ such that for all outgenerators x of T_1, there is a path P with origin x and terminus y.

Proof. Let Q be the shortest initial subpath of P containing $|T_1|$ positive arcs, and let R be the path $P - Q$. R is not empty because $A^+(P) \geqslant |T_1|$. Let $(i_1, j_1, i_2 - i_1, j_2 - j_1, ..., i_r - \sum_{k=1}^{r-1} i_k)$ be the type of Q. Let a_1 be an outgenerator of T_1; a_1 is an origin of a Hamiltonian directed outpath of T_1, $Q_1 = (a_1, ..., a_{i_r})$.

If $(T_2; R)$ is not one of Gruñbaum's exceptions, T_2 contains $R = (c_1, ..., c_m)$. Moreover, $T_2 - R$ contains a Hamiltonian directed inpath $Q_2 = (b_1, ..., b_1)$. We have then $(a_1, ..., a_{i_1}, b_1, ..., b_{j_1}, a_{i_1+1}, ..., a_{i_2}, b_{j_1+1}, ..., b_{j_2}, ..., a_{i_r}, c_1, c_2, ..., c_m) = P$. Thus the lemma holds with $y = c_m$.

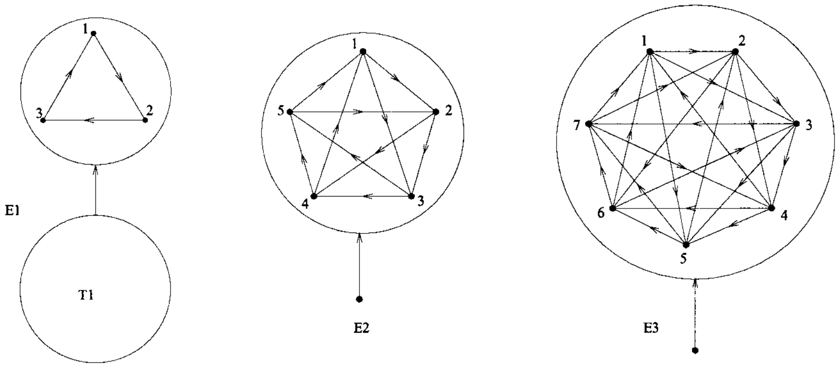

FIG. 5. The tournaments $E_1(n)$, E_2, and E_3.

Suppose then that $(T_2; R)$ is one of Grünbaum's exceptions:

If $T_2 = 3A$: If $R = +(1, 1)$, it is the exception $\mathscr{P}_1$. If $R = -(1, 1)$, if $|T_1| = 1$, it is the exception $\mathscr{P}_2$. If not, Q_1^* exists in T_1 and $P = (Q_1^*, 1, 2, a_k, 3)$.

If $T_2 = 5A$: If $R = +(1, 1, 1, 1)$, if $|T_1| = 1$, it is the exception $\mathscr{P}_3$. If not, Q_1^* exists in T_1 and $P = (Q_1^*, 1, 2, 3, a_k, 5, 4)$; If $R = -(1, 1, 1, 1)$, if $|T_1| = 1$, it is the exception $\mathscr{P}_4$. If not, Q_1^* exists in T_1 and $P = (Q_1^*, 1, 2, a_k, 5, 3, 4)$.

If $T_2 = 7A$: If $R = +(1, 1, 1, 1, 1, 1)$, if $|T_1| = 1$ it is the exception $\mathscr{P}_5$. If not, Q_1^* exists in T_1 and $P = (Q_1^*, 1, 2, 3, a_k, 7, 5, 6, 4)$. If $R = -(1, 1, 1, 1)$, if $|T_1| = 1$, it is the exception $\mathscr{P}_6$. If not, Q_1^* exists in T_1 and $P = (Q_1^*, 1, 2, a_k, 7, 3, 5, 4, 6)$. ∎

LEMMA 2.2. *Let $T = T_1 \to T_2$ be a reducible tournament of order n, P an outpath of order n, and x a vertex of T_1 such that*:

(1) $A^+(P) \geqslant |T_1|$;

(2) $(T; P)$ *is not one of the exceptions* $\mathscr{P}_2$, $\mathscr{P}_4$, *or* $\mathscr{P}_6$;

(3) $s^+_{T_1}(x) \geqslant b_1(P)$.

Then x is an origin of a path P in T whose terminus lies in T_2.

Proof. We prove this in the same way as Lemma 2.1. Let Q be the shortest initial subpath of P containing $|T_1|$ positive arcs. Let R be the path $P - Q$. R is not empty because $A^+(P) \geqslant |T_1|$. Let $(i_1, j_1, i_2 - i_1, j_2 - j_1, ..., i_r - \sum_{k=1}^{r-1} i_k)$ be the type of Q. Since $s^+_{T_1}(x) \geqslant b_1(P)$, x is an origin of a directed outpath $Q_1 = (a_1, ..., a_{i_1})$ in T_1. Let $(a_{i_1+1}, ..., a_{i_r})$ be a Hamiltonian directed outpath of $T_1 - Q_1$.

If $(T_2; R)$ is not one of Grünbaum's exceptions, T_2 contains $R = (c_1, ..., c_m)$ and $T_2 - R$ contains a Hamiltonian directed inpath $Q_2 = (b_1, ..., b_l)$. We have then $(a_1, ..., a_{i_1}, b_1, ..., b_{j_1}, a_{i_1+1} ..., a_{i_2}, b_{j_2}, ..., a_{i_r}, c_1, c_2, ..., c_m) = P$, giving the result.

If $(T_2; R)$ is one of Grünbaum's exceptions, we have the same cases as in the proof of Lemma 2.1, but since $s^+_{T_1}(x) \geqslant b_1(P)$, R is an inpath. So we find again the exceptions $\mathscr{P}_2$, $\mathscr{P}_4$, and $\mathscr{P}_6$. ∎

LEMMA 2.3. *Let T be a reducible tournament of order n with decomposition $T_1 \to T_2$, P an outpath of order n, and $U \subset V(T_2)$ such that*:

(1) $A^+(P) \geqslant |T_1| \geqslant 2$;

(2) $b_1(P) < |T_1|$;

(3) P^{-1} *is an outpath*;

(4) $|u| \geqslant 2$ *and* $s^+_{T_2}(U) \geqslant b_1(P^{-1})+1$;

Then there exists a vertex $u \in U$ *such that every outgenerator of* T_1 *is an origin of* P *in* T *with terminus* u.

Proof. Let Q be the shortest initial subpath of P containing $|T_1|$ positive arcs, and let R be the path $P-Q$. R is not empty because $A^+(P) \geqslant |T_1|$ and $|R| < |T_2|$ because $b_1(P) < |T_1|$. Let $(i_1, j_1, i_2 - i_1, j_2 - j_1, ..., i_r - \sum_{k=1}^{r-1} i_k)$ be the type of Q. Let a_1 be an outgenerator of T_1; a_1 is an origin of a Hamiltonian directed outpath of T_1, $Q_1 = (a_1, ..., a_{i_r})$.

Since $b_1(P) < |T_1|$, $(T_2; R^{-1})$ is not one of the exceptions $\mathscr{F}_j$ or $\mathscr{F}'_l$. So, by Theorem 2.2, there exists $u \in U$ that is an origin of R^{-1} in T_2. In other words, u is the terminus of $R = (c_1, ..., c_{m-1}, u)$. Now, $T_2 - R$ contains a Hamiltonian directed inpath $Q_2 = (b_1, ..., b_l)$; therefore $(a_1, ..., a_{i_1}, b_1, ..., b_{j_1}, a_{i_1+1}, ..., a_{i_2}, b_{j_1+1}, ..., b_{j_2}, ..., a_{i_r}, c_1, c_2, ..., c_{m-1}, u) = P$. ∎

Using Lemma 2.2, we prove the following:

LEMMA 2.4. *Let* $T = T_1 \to T_2$ *be a reducible tournament of order* n *with* $|T_i| \geqslant 2$ *for* $i \in \{1, 2\}$ *and* P *an outpath of order* n *such that* P^{-1} *is an inpath and* $b_1(P) = b_1(P^{-1}) = 1$. *For every pair of vertices* $o \in T_1$ *and* $t \in T_2$, *there is a path* P *in* T *with origin* o *and terminus* t.

Proof. Without loss of generality, we may suppose that $|T_1| \leqslant |T_2|$. Since $A^+(P) + A^-(P) = n - 1 = |T_1| + |T_2| - 1 \geqslant 2|T_1| - 1$, either $A^+(P) \geqslant |T_1|$ or $A^-(P) \geqslant |T_1|$.

Suppose that $A^+(P) \geqslant |T_1|$. Let u_1 be a vertex of T_1. By Lemma 2.2, in $T - (t, o, u)$, there is a path P^{**} with origin o and terminus in $T_2 - t$. Then $(P^{**}, u, t) = P$.

Suppose now that $A-(P) \geqslant |T_1|$. Let v be an outgenerator of $T_1 - o$. By Lemma 2.2, in $T - (t, o)$ there is a path $^*(P^{-1})^*$ with origin v and terminus in $T_2 - t$. In other words, there is a path $^*P^*$ in $T - (t, o)$ with origin in $T_2 - t$ and terminus in v. Then $(o, ^*P^*, t) = P$. ∎

3. REDUCIBLE TOURNAMENTS

THEOREM 3. *Let* T *be a reducible tournament of order* n *and* C *a non-directed cycle of order* n. *Then* T *contains* C *if and only if* $(T; C)$ *is neither one of the five exceptions* $\mathscr{C}_4, \mathscr{C}_5, \mathscr{C}_6, \mathscr{C}_7, \mathscr{C}_8$ *nor one of their duals* (*Fig.* 6).

Proof. T is reducible; thus there exists a decomposition $T_1 \to T_2$ of T. By duality, it suffices to prove the theorem for $|T_1| \leqslant |T_2|$. We consider an

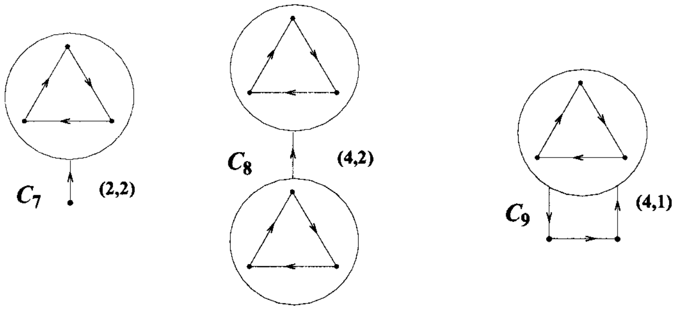

FIG. 6. The exceptions C_i, $7 \leqslant i \leqslant 9$.

orientation of C for which there are at least as many positive as negative arcs. Let $B_1 = (x_1, ..., x_k)$ be a smallest positive block of C, where $C = (x_1, ..., x_n, x_1)$. We thus have $x_1 \to x_n$. Let P be the outpath $(x_1, ..., x_n)$. Since $A^+(P) \geqslant \frac{n}{2} \geqslant |T_1|$, by Lemma 2.1 either there exist $a \in T_1$ and $b \in T_2$ such that a is the origin and b the terminus of P in T, and by adding the arc $a \to b$ we obtain C, or $(T; P)$ is an exception $\mathscr{P}_i$. Since B_1 is a smallest positive block of C, $(T; P)$ can only be one of the exceptions $\mathscr{P}_1(n)$, $\mathscr{P}_2$, $\mathscr{P}_4$, and $\mathscr{P}_6$. If $(T; P) = \mathscr{P}_1(n)$, $T = E_1(n)$ and $C = (n-2, 2)$. If $|T_1| = 1$, then $(T; C)$ is the exception $\mathscr{C}_7$. If $|T_1| \geqslant 2$ and T_1 is not 3A then, by Theorem 1.1, T_1 contains $Q = -(1, n-5)$. Hence T contains $C = (3, Q, 1, 2, 3)$. If T_1 is 3A, it is the exception $\mathscr{C}_8$. If $(T; P)$ is one of the exceptions $\mathscr{P}_2$, $\mathscr{P}_4$, and $\mathscr{P}_6$, then $(T; C)$ is respectively the exception $\mathscr{C}_4$, $\mathscr{C}_5$ or $\mathscr{C}_6$. ∎.

4. CYCLES WITH A LONG BLOCK

The aim of this section is to prove (Theorem 4.3) that every $(=k)$-strong tournament of order at least 9 contains every Hamiltonian cycle which has a block of length at least $k+2$. We divide the proof into two cases: when the cycle has a block of length at least $k+3$ (Theorem 4.1) and when the longest block of the cycle has length $k+2$ (Theorem 4.2). The proofs make use of the results established in Section 2. In order to avoid a lengthy study of particular cases generated by the numerous exceptions noted there, we have chosen to restrict our proofs to tournaments of sufficiently large order $(n \geqslant 20)$. However, we state the full result that can be obtained by the study of these particular cases (see [6]).

THEOREM 4.1. *Let T be a $(=k)$-strong $(k \geqslant 1)$ tournament of order $n \geqslant \min(k+10, 20)$ and C a cycle of order n that contains a block of length at least $k+3$. Then T contains C.*

Proof. Camion's theorem shows that every strong tournament contains a directed Hamiltonian cycle. Henceforth we suppose that C is nondirected. Since T is $(=k)$-strong, there exists a subtournament X of T such that $|X|=k$ and $T-X$ is reducible. Let $T_1 \to T_2$ be a decomposition of $T-X$ and $(x_1, ..., x_k)$ a directed Hamiltonian outpath of X. Since $T-(X\backslash\{x_1\})$ is strong, there exists an ingenerator u of T_2 such that $u \to x_1$. Likewise, there exists an outgenerator v of T_1 such that $x_k \to v$.

Let $B=(a_0, ..., a_l)$ be a block of C of length $l \geqslant k+3$. We orient C so that the arcs of B are in the positive sense and we set $C=(a_0, ..., a_{n-1}, a_0)$. By duality, we may suppose $|T_1| \leqslant |T_2|$.

(1) Suppose first that $|T_1| \geqslant 2$ (so also $|T_2| \geqslant 2$). There exists $t \in T_1$ distinct from v such that v is an outgenerator of $T'_1 = T_1 - t$. We set $T'_2 := T_2 - u$.

Let P be the path $(a_{k+2}, ..., a_{n-1}, a_0)$. We have $|A^+(P)| + |A^-(P)| = n-k-2 = |T_1| + |T_2| - 2$. Thus $|T_1| - 1 \leqslant |A^+(P)|$ or $|T_1| - 1 \leqslant |A^-(P)|$.

— If $|T_1| - 1 \leqslant |A^+(P)|$, by Lemma 2.1 v is an origin of a path P^* in $T'_1 \to T'_2$ with terminus in T'_2, and $C=(P^*, t, u, x_1, ..., x_k, v)$.

— If $|T_1| - 1 \leqslant |A^-(P)|$, let b be an outgenerator of $T_1 - v$. By Lemma 2.1, b is an origin of $(P^*)^{-1}$ with terminus c in T'_2. In other words, there exists $c \in T'_2$, the origin of a path $^*P$ with terminus b. Hence $C=(v, {}^*P, u, x_1, ..., x_k, v)$.

(2) Suppose now that $|T_1|=1$. Set $T_1=\{t\}$. Let U be the set of vertices y of T_2 for which there exists an outgenerator x of X such that $y \to x$. U is not empty and contains an ingenerator of T_2 because T is strong.

If $|U|=1$, say $U=\{u\}$. The tournament $T-u$ is reducible because $Out(X) \to T-(Out(X), u)$. Thus T is 1-strong so $|X|=1$, say $X=\{x\}$. Since $|T_2| \geqslant 9$, taking $X=\{u\}$ and the decomposition $T(x, t) \to T_2 - u$, we have the result by applying (1).

If $|U| \geqslant 2$, let Q be the path $(a_1, a_0, a_{n-1}, ..., a_{k+3})$. Q is an inpath and $b_1(Q)=1$. Thus by Theorem 2.2, either (a) in U there is an origin u of Q, or (b) $(T_2; Q)$ is an exception $\mathscr{F}_i(n-k-1)$, with U included in the set of vertices which are not origins of Q in T_2.

(a) Let $x \in X$ be an out generator of X such that $u \to x$. x is an origin of a directed Hamiltonian outpath R of X. Hence $(R, t, Q^{-1}, x) = C$.

(b) Since $b_1(Q)=1$ and $|T| \geqslant k+10$, we need only consider the exceptions $\mathscr{F}_3(n-k-1)$, $\mathscr{F}_5(n-k-1)$, $\mathscr{F}_7(n-k-1)$, $\mathscr{F}_{11}(n-k-1)$, and $\mathscr{F}_{13}(n-k-1)$. If $(T_2; Q)$ is the dual of one of these exceptions, the path $(a_2, a_3, ..., a_{k+4})$ is a directed outpath and among two vertices of U, there is a vertex u that is an origin of $Q' = (a_0, a_{n-1}, ..., a_{k+2})$ in T_2. Let x be an outgenerator of X such that $u \to x$ and R a directed Hamiltonian outpath of X with origin x. We have $C = (R, t, Q'^{-1}, x)$. ∎

THEOREM 4.2. *Let T be a $(=k)$-strong $(k \geqslant 1)$ tournament of order $n \geqslant \min(k+9, 20)$ and C a cycle of order n with a block of length $k+2$. Then T contains C.*

Proof. Let $C = (a_0, a_1, ..., a_{n-1}, a_0)$ be a cycle such that $(a_0, a_1, ..., a_{k+2})$ is a block. Since T is $(=k)$-strong, there exists a subtournament X of T such that $T' = T - X$ is reducible. By duality, we may suppose that $|Out(T')| \leqslant |In(T')|$. Set $T_1 = Out(T')$ and $T_2 = T' - Out(T')$. We have $T_1 \to T_2$. Let O be a Hamiltonian directed outpath of X with origin x and terminus y.

Since T is $(=k)$-strong, there is an outgenerator t of T_1 such that $y \to t$ and also an ingenerator u of T_2 such that $u \to x$. Set $U := \{v \in T_2, v \to x\}$ and $V := \{v \in T_2, v \leftarrow x\}$. Note that (U, V) is a partition of $V(T_2)$. Let P be the path $(a_{k+1}, ..., a_{n-1}, a_0)$ and Q the path $(a_k, ..., a_{n-1})$. Note that P, P^{-1} and Q are outpaths, $b_1(P) = 1$, and $b_1(Q) = 2$.

(1) If $|T_1| > 1$, since T_1 is strong, T_1 has a Hamiltonian directed cycle $C_1 = (t, t_2, ..., t_k, t)$. Moreover, $A^+(P) + A^-(P) = n - k - 1 = |T_1| + |T_2| - 1 \geqslant 2\,|T_1| - 1$, so $A^+(P) \geqslant |T_1|$ or $A^-(P) \geqslant |T_1|$.

(a) If $A^+(P) \geqslant |T_1|$, then $b_1(P^{-1}) \leqslant |T_2|$.

Suppose first that $|U| \geqslant 2$. If $s^+_{T_2}(U) \geqslant b_1(P^{-1}) + 1$ (note that this condition is satisfied if U contains an outgenerator of T_2 or if $b_1(P^{-1}) = 1$), since $b_1(P) = 1 < |T_1|$, we may apply Lemma 2.3. So, in T', t is an origin of P with terminus in U and $C = (O, P, x)$. So we may now suppose that $Out(T_2) \subseteq \{v \in T_2, s^+_{T_2}(v) \geqslant b_1(P^{-1}) + 1\} \subseteq V$ and $b_1(P^{-1}) \geqslant 2$; in other words Q^{-1} is an outpath. If $|V| \geqslant 2$, since $A^+(Q) = A^+(P) + 1 > |T_1|$, $b_1(P) = 1 < |T_1|$, and $s^+_{T_2}(V) = |T_2| \geqslant b_1(Q^{-1}) + 1$, we may apply the Lemma 2.3. Hence in T', t is an origin of Q with terminus in V and $C = (O, Q, x)$. If $|V| = 1$, say $V = \{v\}$, we have $Out(T_2) = \{v\}$, so $v \to T_2 - v$. Moreover, $b_1(P^{-1}) = |T_2| - 1 \geqslant 2$. Since $b_1(Q) = 2$ and $|T_1| \geqslant 3$, $(T' - V; Q^*)$ is not an exception $\mathscr{P}_i$. Then by Lemma 2.1, in $T' - v$, t is an origin of Q^* with terminus in $T_2 - v$ and $C = (O, Q^*, v, x)$.

Suppose now that $|U| = 1$, say $U = \{u\}$, so that $|V| \geqslant 2$. If $b_1(P^{-1}) \geqslant 2$, we have the result in the same way as before. So we may suppose that $b_1(P^{-1}) = 1$. Let $B = (a_{k+2}, ..., a_l)$ be the block of C containing the arc $a_{k+2} \leftarrow a_{k+3}$. Since $A^+(P) \geqslant |T_1| \geqslant 3$, we have $a_l \neq a_0$, and $|B| < |T_2|$. Thus u is an origin of B^* in T_2 and $T_2 - B^*$ is not empty. Set $S = (a_l, ..., a_{n-1})$. By Lemma 2.1, either $((T' - (B^*, t, t_2); S)$ is an exception

$\mathscr{P}_i$, $i \in \{2, 4, 6\}$, and we have the result by Proposition 4.1, or in $T' - (B^*, t, t_2)$, t_3 is an origin of S with terminus in $T_2 - B$ and $C = (O, t, t_2, B^*, S, x)$.

(b) If $A^+(P) < |T_1|$, then $A^-(P) \geqslant |T_1|$.

If $x \leftarrow t_2$, since $*P$ is an inpath and $|T_1| \geqslant 3$, $(T' - t; (*P)^{-1})$ is not an exception $\mathscr{P}_i$, so by Lemma 2.1, t_2 is an origin of $(*P)^{-1}$ with terminus in T_2 and $C = (t_2, O, t, *P)$.

From now on, we may suppose $x \rightarrow t_2$. If $b_1(P^{-1}) \geqslant 2$, then $(P^*)^{-1}$ is an outpath. So by Lemma 2.1, t_2 is an origin of $(P^*)^{-1}$ with terminus in T_2 and $C = (O, t, P^*, x)$.

Henceforth, we may suppose $b_1(P^{-1}) = 1$. Let $(a_l, ..., a_{n-2}, a_{n-1})$ be the block of C containing the arc $a_{n-2} \rightarrow a_{n-1}$, R the path $(a_{l+1}, ..., a_{n-1}, a_0)$, and S the path $(a_{k+2}, ..., a_l)$. Since $A^+(P) < |T_1|$ and $|T_1| \geqslant |T_2|$, we have $|R| < |T_2|$. If $|U| = 2$ then by Theorem 2.2 there is a path R^{-1} in T_2 with origin $u \in U$. Because S is an inpath and $|T_1| \geqslant 3$, $(T' - (R, t); S^{-1})$ is not an exception $\mathscr{P}_i$; so by Lemma 2.1, in $T' - (R, t)$, t_2 is an origin of S^{-1} with terminus in $T_2 - R$ and $C = (u, O, t, S, R)$.

Suppose now that $|U| = 1$, say $U = \{u\}$. In $T(V)$, one can find R, by Lemma 1.1. and by Lemma 2.1, either $(T' - (R, t, t_2); S^{-1})$ is an exception $\mathscr{P}_i$, $i \in \{2, 4, 6\}$, and we have the result by Proposition 4.2, or in $T - (X, R, t, t_2)$, t_3 is an origin of S^{-1} with terminus in $T_2 - R$ and $C = (O, t, t_2, S, R, x)$.

(2) If $|T_1| = 1$ then $T_1 = \{t\}$.

If $k \geqslant 2$, we have $d^-_X(x) \leqslant k - 2$, so since T is $(=k)$-strong, $|U| \geqslant 2$. If $k = 1$ and $|U| = 1$, i.e., $U = \{u\}$, then $T - u$ is reducible with decomposition $x \rightarrow (T_2 - u) \cup \{t\}$. Now $u \leftarrow t$ and $d^-_{T_2}(u) \geqslant 1$ because u is an ingenerator of T_2. Therefore, we just need to consider the case $|U| = 2$.

If $s^+_{T_2}(U) \geqslant b_1(P^{-1})$, by Theorem 2.2, either there is a vertex z in U that is an origin of $(*P)^{-1}$ and $C = (z, O, t, *P)$ or $(T_2; (*P)^{-1})$ is one of the exceptions $\mathscr{F}_j(n - k - 1)$. Thus, in $T_2 - U$, there is a vertex that is an origin of $(*P)^{-1}$ and $C = (O, t, P^*, x)$. So we may suppose that $s^+_{T_2}(U) < b_1(P^{-1})$. In particular, $b_1(P^{-1}) \geqslant 2$, T_2 is reducible, and $Out(T_2) \subseteq V$. If $|V| \geqslant 2$, $(T_2; (P^*)^{-1})$ could not be an exception $\mathscr{E}_i$ or $\mathscr{F}_j(n - k - 1)$ with V included in the set of vertices that are not origins of $(P^*)^{-1}$. Consequently, by Theorem 2.2, in U, there is an origin of $(P^*)^{-1}$ and $C = (O, t, P^*, x)$. If $|V| = 1$ then $|Out(T_2)| = 1$, say $Out(T_2) = V = \{v\}$. Set $Q' = (a_{k+1}, ..., a_{n-2})$. By Theorem 1.1, $T_2(U)$ contains Q' and $C = (O, t, Q', v, x)$. ∎

Proposition 4.1. *Let T be a $(=k)$-strong tournament of order n and $C = (a_0, a_1, ..., a_{n-1})$ a cycle of order n with a block of length $k+2$,*

$(a_0, a_1, ..., a_{k+2})$, *such that there exists a subtournament* X *of order* k *with* $T' = T - X = T_1 \rightarrow T_2$ *where* T_1 *is the* 3-*cycle* (t, t_2, t_3, t), O *a Hamiltonian directed outpath of* X *with origin* x *and terminus* y *such that* $y \rightarrow t$, *and* u *an ingenerator of* T_2 *such that* $u \rightarrow x$ *and* $T_2 - u \leftarrow x$. *Let* $B = (a_{k+2}, ..., a_l)$ *be the block of* C *containing the arc* $a_{k+2} \leftarrow a_{k+3}$ *and* S *the path* $(a_l, ..., a_{n-1})$. *Let* D^* *be a path isomorphic to* B^* *with origin* u.

If $(T' - (D^*, t, t_2); S)$ *is an exception* $\mathscr{P}_i$, $i \in \{2, 4, 6\}$, *then* T *contains* S.

Proof. Let w be a vertex of $T_2 - D^*$. If $w \rightarrow u$ then, by Lemma 2.1, t_2 is an origin of S in $T' - (D^*, t, w)$ and $C = (O, t, w, D^*, S, x)$. So we may suppose that $T_2 - B \leftarrow u$. Let z be the terminus of D^* and set $T'_2 = T_2 - D^{**}$. It is obvious that there exists w such that $T'_2 - w$ is not 3A, 5A, 7A. Thus, by Lemma 2.1, t_3 is an origin of S in $T'_2 - w$ and $C = (O, t, t_2, w, D^{**}, S, x)$. ∎

Proposition 4.2. *Let* T *be a* $(=k)$-*strong tournament of order* n *such that there exists a subtournament* X *of order* k *such that* $T' = T - X = T_1 \rightarrow T_2$ *with* T_1 *the* 3-*cycle* (t, t_2, t_3, t), O *a Hamiltonian directed outpath of* X *with origin* x *and terminus* y *such that* $y \rightarrow t$ *and* $x \rightarrow t_2$, *and* u *an ingenerator of* T_2 *such that* $u \rightarrow x$ *and* $T_2 - u \leftarrow x$. *Let* $C = (a_0, a_1, ..., a_{n-1})$ *be a cycle of order* n *such that* $(a_0, a_1, ..., a_{k+2})$ *is a block and* $a_{n-2} \rightarrow a_{n-1}$. *Let* $(a_l, ..., a_{n-2}, a_{n-1})$ *be the block of* C *containing the arc* $a_{n-2} \rightarrow a_{n-1}$, R *the path* $(a_{l+1}, ..., a_{n-1}, a_0)$, *and* S *the path* $(a_{k+2}, ..., a_l)$. *In* $T_2 - u$, *one can find* R.

If $(T' - (R, t, t_2))$ *is the tournament of an exception* $\mathscr{P}_i$, $i \in \{2, 4, 6\}$, *then* T *contains* S.

Proof. If $(T' - (R, t, t_2))$ is the tournament of $\mathscr{P}_2$ (resp. $\mathscr{P}_4$, $\mathscr{P}_6$), there is a subtournament T'' of $T_2 - v$ containing u of order 3 and nonisomorphic to the tournament of 3A (resp. 5A, 7A). Let R be a Hamiltonian directed inpath $T_2 - T''$. By Lemma 2.1, t_3 is an origin of S^{-1} with terminus in $T_2 - R$ and $C = (O, t, t_2, S, R, x)$. ∎

Using the same method, one can improve the previous two theorems and prove the following more general theorem (see [6]). However, its proof requires the study of a huge number of special cases generated, on the one hand, by the exceptions $\mathscr{P}_i$ (each time we use Lemma 2.1) and, on the other hand, by the 52 exceptions of order at most 8 of Theorem 2.2 (cf. [7]).

Theorem 4.3. *Let* T *be a* $(=k)$-*strong* $(k \geqslant 1)$ *tournament of order* n *and* C *a cycle of order* n *containing a block of length at most* $k+2$. *Then* T *contains* C *if and only if* $(T; C)$ *is not the exception* $\mathscr{C}_9$.

COROLLARY 4.1 (Benhocine and Wojda [2]). *Let T be a tournament of order n and C a cycle of order n with exactly two blocks. Then T contains C unless $(T; C)$ is one of the exceptions $\mathscr{C}_1, \mathscr{C}_7, \mathscr{C}_8$, and $\mathscr{C}_9$ or the dual of $\mathscr{C}_8$.*

Proof. If T is reducible, Theorem 3.1 gives the result. So we may suppose that T is $(=k)$-strong $(k \geqslant 1)$. Thus $|T| \geqslant 2k+1$, so C has a block of length at least $k+1$. If C has a block of length at least $k+2$, we have the result by Theorem 4.3, so we may suppose that its maximal block is of length $k+1$. Thus $C=(k+1, k)$ or, $C=(k+1, k+1)$. If $C=(k+1, k)$, by Theorem 1.1, either T contains the path $+(k, k)$, and by adding the arc between the origin and the terminus of this path we obtain C (regardless of the sense of this arc) or T is 3A and $(T; C)$ is the exception $\mathscr{C}_1$. If $C=(k+1, k+1)$, let $(x_1, x_2, x_3, ..., x_{2k+2}, x_1)$ be a Hamiltonian directed cycle of T. Suppose that $x_1 \rightarrow x_{k+2}$. If $x_2 \rightarrow x_{k+3}$ then $(x_{k+3}, x_2, x_3, ..., x_{k+2}, x_1, x_{2k+2}, ..., x_{k+3})=C$. So we may suppose that $x_2 \rightarrow x_{k+3}$ and, in general, that $x_i \rightarrow x_{k+i+1}$. Thus we find that $x_{k+2} \rightarrow x_1$, contradicting the assumption $x_1 \rightarrow x_{k+2}$. Likewise, if we suppose that $x_1 \leftarrow x_{k+2}$, we may suppose in general that $x_i \leftarrow x_{k+i+1}$ and we have the contradiction $x_{k+2} \leftarrow x_1$. ∎

5. CYCLE WITH SMALL BLOCKS

THEOREM 5.1. *Let C be a nondirected cycle of order n, with longest block of length $L \geqslant 2$, and T a $(=k)$-strong tournament of order n with $k \notin \{3, 5, 7\}$. If $n \geqslant 4(k+1) L+k+1$ then T contains C.*

Proof. Let X be a subtournament of order k such that $T'=T-X$ is reducible with decomposition $T_1 \rightarrow T_2$. Without loss of generality, we may suppose that $|T_1| \leqslant |T_2|$.

Let $C=(a_0, a_1, a_2, ..., a_{n-1}, a_0)$ be such that $a_k \rightarrow a_{k+2} \leftarrow a_{k+3}$.

Let $R=(a_1, a_2, ..., a_k)$. By Theorem 1.1, X contains R. Let $(x_1, x_2, ..., x_k)$ be an occurrence of R in T. Since $T-R^*$ is strong, there is a vertex o of T_1 that is dominated by x_k.

(A) Suppose first that $|T_1| \geqslant 2$.

Let i_1 be the greatest integer less than n such that $a_{i_1} \leftarrow a_{i_1} \rightarrow a_{i_1+1}$, and let Q_1 be the path $(a_0, a_{n-1}, ..., a_{i_1+1})$. Note that $|Q_1| \leqslant 2L$.

— Let U_1 be the set of vertices of T_2 which dominate x_1 and V_1 the set of vertices of T_2 which are dominated by x_1.

Suppose that $a_1 \leftarrow a_0$. If $|U_1| \geqslant b_1(Q_1)+1$, then by Theorem 2.1 there is a path Q_1 in T_2 with origin in U_1 and terminus t_1. Now let $P_1=(a_5, a_6, ..., a_{i_1+1})$ and $T_1'=T_1 \rightarrow T_2-(Q_1^*)$. By Lemma 2.4, in T_1' there is a path P_1 with origin o and terminus t_1. So T contains $C=(x_1, x_2, x_3, x_4, P_1, Q_1^{-1})$. Hence, we may assume that $|U_1|<b_1(Q_1)+1$.

Likewise, if $a_1 \to a_0$, we may suppose that $|V_1|<b_1(Q_1)+1$.

Let us define T_2^1 as $T_2(U_1)$ if $a_0 \leftarrow a_1$ and $T_2(V_1)$ otherwise. We have $|T_1^1| \geqslant |T_2| - b_1(Q_1) \geqslant (2k+1)L+1$. Note that either x_1 dominates T_2^1 or x_1 is dominated by T_2^1.

— Let j_2 be the greatest integer less than $n+1$ such that a_{j_2} has outdegree (resp. indegree) two in C, if x_1 dominates (resp. is dominated by) T_2^1, and let i_2 be the greatest integer less than j_2 such that $a_{i_2-2} \leftarrow a_{i_2-1} \to a_{i_2}$. Let $Q_2=(a_0, a_{n-1}, ..., a_{j_2+1})$ and $Q_2=(a_{i_2+1}, ..., a_{j_2-1})$. Note that Q_2 and Q_1' each have length less than $2L$.

Let U_2 be the set of vertices of T_2^1 which dominate x_2 and V_2 the set of vertices of T_2^1 which are dominated by x_2.

Suppose that $a_2 \leftarrow a_1$. If $|U_2| \geqslant b_1(Q_2)+1$, then by Theorem 2.1, there is a path Q_2' in T_2 with origin in U_2. By Theorem 2.1, there is a path Q_2' with origin t_2 in $T_2^1 - Q_2$. Let $P_2=(a_5, a_6, ..., a_{i_2+1})$ and $T_2'=T_1 \to T_2-(Q_2, Q_2')$. By Lemma 2.4, in T_2', there is a path P_2 with origin o and terminus t_2. So T contains $C=(x_2, x_3, x_4, P_2, Q_2', x_1, Q_2^{-1})$. Hence, we may assume that $|U_2|<b_1(Q_2')+1$.

Likewise, if $a_2 \to a_1$, we may asume that $|V_2|<b_1(Q_2)+1$.

Let us define T_2^2 as $T_2^1(U_1)$ if $a_0 \leftarrow a_1$ and $T_2^1(V_1)$ otherwise. We have $|T_2^2| \geqslant |T_2^1| - b_1(Q_2) \geqslant 2kL+1$. Note that either x_2 dominates T_2^2 or x_2 is dominated by T_2^2.

— By induction, we may suppose that there is a sequence of subtournaments $T_2^k \subseteq T\,2^{k-1} \subseteq \cdots \subseteq T_2^1 \subseteq T_2^0 = T_2$ such that, for $1 \leqslant i \leqslant k$, T_2^i has order at least $(2k+2-i)L+1$ and either $T_2^i \to x_i$ or $T_2^i \leftarrow x_i$.

Let B be the block containing a_{k+1} and b its length. Let $a_{l_{k+1}}$ be the endvertex of B distinct from a_{k+2}.

Let us define recursively l_i for $i=k, k-1, ..., 1$ to be the greatest integer less than $l_{i+1}-1$ such that a_{l_i} has outdegree (resp. indegree) two in C if x_i dominates (resp. is dominated by) T_2^k. Let $R_i=(a_{l_i+1}, a_{l_i+2}, ..., a_{l_i+1}-1)$ for $1 \leqslant i \leqslant k$. Note that each R_i has length at most $2L-1$ since each block of C has length at most L.

— If $|T_1| \geqslant b+1$, let I be a directed inpath of length $b-1$ in T_1 with origin o, let l_0 be the greatest integer less than l_i such that $a_{l_0-1} \leftarrow a_{l_0} \to a_{l_0+1}$, and let $R_0=(a_{l_0+1}, a_{l_0+2}, ..., a_{l_1-1})$.

By Theorem 2.1, $|T_2^i| \geqslant (2k+2-i)L+1$, $1 \leqslant i \leqslant k$. Therefore, one can find disjoint paths R_k, R_{k-1}, ..., R_0 such that R_k is contained in T_2^k and R_{i-1} in $T_2^i - (R_k, ..., R_i)$, $1 \leqslant i \leqslant k$. Denote by t the origin of R_0.

Now let $P=(a_7, a_8, ..., a_{l_0})$ and $T''=T_1'' \to T_2''$ with $T_1''=T_1-(*I)$ and $T_2''=T_2-(*R_0, R_1, ..., R_k)$. We have $|T_1 - *I| \geqslant 2$ because $|T_1| \geqslant b+1$

and $|T_2-(R_1, R_2, ..., R_4)| \geqslant 2$, because $|T_2| \geqslant f(L, k)/2$. So by Lemma 2.4 in T'' there is a path P with origin o and terminus t. Hence T contains $C=(P, {}^*R_0, x_1, R_1, x_2, R_2, ..., x_k, R_k, I)$.

— If $|T_1| \leqslant b$, let I be a directed outpath of T_1 with origin u, let $j=l_{k+1}+|T_1|$ (modulo n), and let P be the path $(a_j, a_{j+1}, ..., a_{l_1-1})$. By Theorem 2.1, $|T_2^i| \geqslant (2k+2-i)\, L+1$, for $1 \leqslant i \leqslant k$. Therefore, one can find disjoint paths $R_k, R_{k-1}, ..., R_0$ such that R_k is contained in T_2^k and R_{i-1} in $T_2^i-(R_k, ..., R_i)$ for $2 \leqslant i \leqslant k$. The tournament $S=T_2^1-(R_1 R_2, ..., R_k)$ has at least L vertices. So by Theorem 2.2, there is a path P in $T_2-(R_1, R_2, ..., R_k)$ with terminus in S. Thus T contains $C=(I, P, x_1, R_1, x_2, R_2, ..., x_k, R_k, U)$.

(B) Suppose now that $T_1=\{t\}$.

— Let U_1 (resp. V_1) be the set of vertices of T_2 which dominate (resp. are dominated by) x_1. Let $Q_1=(a_0, a_{n-1}, ..., a_{k+2})$.

Suppose that $|U_1| \geqslant b_1(Q_1)+1$. By Theorem 2.2, in T_2, there is a path Q_1 with origin in U_1. Thus T contains $C=(x_1, x_2, ..., x_k, t, Q_1^{-1}, x_1)$. Hence, we may assume that $|U_1|<b_1(Q_1)+1$. Likewise, if $a_1 \rightarrow a_0$, we may suppose that $|V_1|<b_1(Q_1)+1$.

Let us define T_2^1 as $T_2(U_1)$ if $a_0 \leftarrow a_1$, and $T_2(V_1)$ otherwise. We have $|T_2^1| \geqslant |T_2|-b_1(Q_1) \geqslant (4k+3)\, L$. Note that either x_1 dominates T_2^1 or x_1 is dominated by T_2^1.

— Let j_2 be the greatest integer less than $n+1$ such that a_{j_2} has outdegree (resp. indegree) two, if x_1 dominates (resp. is dominated by) T_2^1. Let Q_2' be the path $(a_0, a_{n-1}, ..., a_{j_2+1})$. Note that Q_2' has length at most $2L-1$. Let $Q_2=(a_{k+2}, a_{k+3}, ..., a_{j_2-1})$. By Theorem 2.1, one can find Q_2' in T_1. Since $T_2^1-Q_2'$ has order at least L, by Theorem 2.2 one can find a path Q_2 in T_2-Q_2' with terminus in $T_2^1-Q_2'$. Hence T contains $C=(x_2, x_3, ..., x_k, t, Q_2, x_1, Q_2'^{-1}, x_2)$.

By induction, we may suppose that there is a sequence of subtournaments $T_2^k \subseteq T2^{k-1} \subseteq \cdots \subseteq T_2^1 \subseteq T_2^0=T_2$ such that, for $1 \leqslant i \leqslant k$, T_2^i has order at least $(4k+4-i)\, L$ and either $T_2^i \rightarrow x_i$ or $T_2^i \leftarrow x_i$.

Let B be the block containing a_{k+1} and let $a_{l_{k+1}}$ be the endvertex of B distinct from a_{k+2}. Let us define recursively l_i, for $i=k, k-1, ..., 1$, to be the greatest integer smaller than $l_{i+1}-1$ such that a_{l_i} has out degree (resp. indegree) two in C, if x_i dominates (resp. is dominated by) T_2^k. Let $R_i=(a_{l_i+1}, a_{l_i+2}, ...a_{l_{i+1}-1})$ for $1 \leqslant i \leqslant k$. Note that each R_i has length at most $2L-1$ since each block of C has length at most L. Let $P=(a_{l_{k+1}+1}, a_{l_{k+1}+2}, ..., a_{l_1-1})$. Since $|T_2^i| \geqslant (4k+4-i)\, L$, for $1 \leqslant i \leqslant k$, one can find disjoint paths R_k, $R_{k-1}, ..., R_0$ such that R_k is contained in T_2^k and R_{i-1} in $T_2^i-(R_k, ..., R_i)$ for $2 \leqslant i \leqslant k$. The tournament $S=T_2^1-(R_1 R_2, ..., R_k)$ has at least L vertices. So by Theorem 2.2, there is a path P in

$T_2-(R_1, R_2, ..., R_k)$ with terminus in S. Thus T contains $C=(I, P, x_1, R_1, x_2, R_2, ..., x_k, R_k, u)$. ∎

THEOREM 5.2. *Let C be a cycle of order n with longest block of length L and T a $(=k)$-strong tournament of order n with $k \in \{3, 5, 7\}$. Then T contains C if either:*

$L \geqslant 4$ *and* $n \geqslant 4(k+1)L+k+1$, *or*

$2 \leqslant L \leqslant 3$ *and* $n \geqslant 4(k+2)L+k+2$.

Proof. Let X be a subtournament of T such that $T-X$ is reducible. Let $C=(a_0, a_1, a_2, ..., a_{n-1}, a_0)$ such that $a_k \rightarrow a_{k+1} \rightarrow a_{k+2} \leftarrow a_{k+3}$. Let $R=(a_1, a_2, ..., a_k)$. If $(X; R)$ is not one of Grünbaum's exceptions, the proof is similar to Theorem 5.1. In particular, this is the case if $L \geqslant 4$ because we have $a_{k-2} \rightarrow a_{k-3} \rightarrow a_k$.

If $(X; R)$ is one of Grünbaum's exceptions, let y be a vertex of $T-X$ such that $T-(X, y)$ is still reducible. Let $X'=T(X, y)$ and $R'=(a_0, a_1, ..., a_k)$. It is easy to check that there is a vertex of R that is an origin of R' in X'. The proof proceeds as in Theorem 5.1. ∎

Theorems 4.3, 5.1, and 5.2 yield the following corollary:

COROLLARY 5.1. *Let T be a $(=k)$-strong tournament of order $n \geqslant 4(k+1)^2+k+1$. Then T contains every Hamiltonian cycle.*

This corollary is very important: if one can prove that every k-strong tournament of order $n \geqslant N(k)$ contains every Hamiltonian cycle then it implies that every tournament of order at least $\max(N(k), 4k^2+k)$ contains every Hamiltonian nondirected cycle.

6. HAMILTONIAN CYCLES IN 8-STRONG TOURNAMENTS

6.1. *Useful Lemmas*

Theorem 2.2 yields the following two Lemmas which will be need frequently in the proofs of our subsequent theorems.

LEMMA 6.1. *Let T be a tournament of order $n \geqslant 8$ and P a path of order n. Then*

(1) *there are at least $n-b_1(P)-1$ origins of P in T unless $(T; P)$ is the exception $\mathscr{F}_1(n)$ or its dual.*

(2) *there are at least $n-b_1(P)$ origins of P in T unless $(T; P)$ is one of the exceptions $\mathscr{F}_i(n)$, $i \in \{1, 3, 5, 7, 8, 9, 10, 11, 13\}$, or $\mathscr{E}_{49}$, or one of their duals.*

Proof. By directional duality, we may assume that P is an outpath. If $(T; P)$ is one of the exceptions $\mathscr{F}_i$, $1 \leqslant i \leqslant 14$, $\mathscr{E}_{49}$, $\mathscr{E}_{50}$, or $\mathscr{E}_{51}$, it is easy to check that the statement holds. Suppose now that $(T; P)$ is not one of the exceptions $\mathscr{F}_i$, $1 \leqslant i \leqslant 14$, $\mathscr{E}_{49}$, $\mathscr{E}_{50}$, $\mathscr{E}_{51}$. If P is directed the result holds trivially, so we may assume that P is not directed; in particular, $b_1(P) \leqslant n-2$. Let X be the set of vertices x of T such that $s_T^+(x) \geqslant b_1(P)+1$. It is easy to see that $|X| \geqslant n - b_1(P)$: consider, for example, the first $n - b_1(P)$ vertices of a Hamiltonian directed outpath of T. Suppose, by way of contradiction, that there are fewer than $n - b_1(P)$ origins of P in T. Then there is a vertex $x \in X$ that is not an origin of P. Also since $b_1(P) \leqslant n-2$, there exists a vertex $y \neq x$ of T which is not an origin of P. But this contradicts Theorem 2.2. ∎

Lemma 6.2. *Let T be a tournament of order $n \geqslant 8$, P an outpath of order n, and X the set of origins of P in T.*

(1) *If $|X| \leqslant n-3$ then $X \to T - X$.*

(2) *Suppose that $(T; P)$ is not one of the exceptions $\mathscr{F}_i$, $i \in \{2, 3, 4, 5, 6, 7, 11, 12, 13, 14\}$, nor $\mathscr{E}_l$, $l \in \{49, 50, 51\}$. If $|X| \leqslant n-2$ then $X \to T - X$.*

Proof. If $(T; P)$ is one of the exceptions $\mathscr{F}_i$, for $1 \leqslant i \leqslant 14$, or $\mathscr{E}_l$, for $l \in \{49, 50, 51\}$, we have the result by definition. So we may assume that $(T; P)$ is not one of these exceptions. Suppose that $|X| \leqslant n-2$ and suppose, by way of contradiction, that there is a vertex $y \in T - X$ that dominates a vertex $x \in X$. We have $s_T^+(y) \geqslant s_T^+(x) \geqslant b_1(P)+1$. Let z be a vertex of $T - X$ distinct from y. By Theorem 2.2, y or z is an origin of P and thereby belongs to X. This is a contradiction. ∎

We now present the lemma on which our proofs are based.

Lemma 6.3. *Let T be a k-strong tournament of order n, $C = (b_0, b_1, ..., b_{n-1}, b_0)$ a cycle such that $b_{n-1} \to b_0 \to b_1$, and x a vertex of T with both out- and in degree greater than 7.*

Set $n^+ = |N_T^+(x)|$, $n^- = |N_T^-(x)|$, $T^+ = T(N_T^+(x))$, and $T^- = T(N_T^-(x))$ and let P^+ be the path $(b_{n^+}, b_{n^+-1}, ..., b_1)$ and P^- the path $(b_{n^++1}, b_{n^++2}, ..., b_{n-1})$. Let X^+ (resp. X^-) be the set of origin of P^+ (resp. P^-) in T^+ (resp. T^-).

(1) *If $b_{n^+} \to b_{n^++1}$, then T contains C or $X^- \to X^+$.*

(2) *If $b_{n^+} \leftarrow b_{n^++1}$, then T contains C or $X^- \leftarrow X^+$.*

(3) *If $|X^+| + |X^-| \geqslant n-k+1$ then T contains C.*

(4) *If $|X^+| + |X^-| \geqslant n-k$ and $b_{n^+} \to b_{n^++1}$ then T contains C.*

(5) *If* $|X^-| \geqslant n^- - k + 4$ *and* $(b_{n^+-1}, b_{n^+}, b_{n^++1})$ *is a directed path, then T contains C.*

(6) *If* $|X^-| \geqslant n^- - k + 3$, (T^+, P^+) *is not one of the exceptions* $\mathscr{F}_i$, $i \in \{2, 3, 4, 5, 6, 7, 11, 12, 13, 14\}$, *or* $\mathscr{E}_l$, $l \in \{49, 50, 51\}$, *or one of their duals, and* $(b_{n^+-1}, b_{n^+}, b_{n^++1})$ *is a directed path, then T contains C.*

Proof. (1) and (2) are obvious.

(3) Suppose, by way of contradiction, that T does not contain C. Then by (1) and (2), $T(X^-, X^+)$ is reducible of order $n-k+1$. This contradicts the hypothesis that T is k-strong.

(4) Suppose, by way of contradiction, that T does not contains C. Then by (1), $X^- \to X^+$. So $T(X^-, X^+, x)$ is reducible with order $n-k+1$. This contradicts that T is k-strong.

(5) Assume that $(b_{n^+-1}, b_{n^+}, b_{n^++1})$ is an outpath. Suppose, by way of contradiction, that T does not contain C. By (1) we obtain $X^- \to X^+$. If $|X^+| \geqslant n^+ - 2$, then $T(X^-, X^+)$ is reducible of order $n-k+1$. This is a contradiction. If $|X^+| < n^+ - 2$, since P^+ is an inpath, (the dual of) Lemma 6.2 yields that $T^+ - X^+ \to X^+$. Thus $T(T^+, X^-)$ is reducible with order $n-k+2$. This is a contradiction.

The proof of (6) is analogous to that of (5). ∎

The basic idea of the proof of the next theorem is the following: we pick an arbitrary vertex x of T and seek an enumeration $(b_0, ..., b_{n-1}, b_0)$ of the vertices of C such that X^- and X^+ satisfy one of the hypotheses of Lemma 6.3. This lemma then guarantees that T contains C.

6.1. *Main result*

THEOREM 6.1. *Let T be a k-strong tournament of order n, where* $k \geqslant 8$, *and let C be a cycle of order n with blocks of length at most L, where* $2 \leqslant L \leqslant k+1$. *Then T contains C.*

Let B be a block of C with maximal length L. Set $l = \lfloor L/2 \rfloor$. Let $C = (a_0, a_1, ..., a_{n-1}, a_0)$, where $a_0 \in B$ and $a_{n-1} \to a_{n-l+1} \to \cdots \to a_{l-1} \to a_l$. Let x be a vertex of T, and set $n^+ = |N_T^+(x)|$ and $n^- = |N_T^-(x)|$. Let us denote by T^+ and T^- the subtournaments $T(N_T^+(x))$ and $T(N_T^-(x))$. For every integer i, let P_i^+ be the path $(a_{n^+-i}, a_{n^+-i-1}, ..., a_{1-i})$ and P_i^- the path $(a_{n^+-i+1}, a_{n^+-i+2}, ..., a_{n-i-1})$ (indices modulo n). Let X_i^+ be the set of vertices which are origins of P_i^+ in T^+ and X_i^- be the set of vertices which are origins of P_i^- in T^-. Note that X_i^+ and X_i^- are not empty, by Theorem 1.1. Considering the paths P_i^+ and P_i^- amounts to considering the enumeration $(b_0, b_1, ..., b_{n-1}, b_0)$ of Lemma 6.3 with $b_j = a_{j+i}$.

Note that (T^+, P_0^+) is neither the exception $\mathscr{F}_1$ nor its dual, otherwise C would have a block of length greater than $k+1$.

If $b_1(P_0^+)+b_1(P_0^-)\leqslant 4$, by Lemma 6.1.(1), $|X_0^+|\geqslant n^+ - b_1(P_0^+)-1$ and $|X_0^-|\geqslant n^- - b_1(P_0^-)-1$. So, by Lemma 6.3.(3), T contains C. Thus we may assume that $b_1(P_0^+)+b_1(P_0^-)\geqslant 5$. In particular, $L\geqslant 3$.

Suppose first that $L=3$. Then $2\leqslant b_1(P_0^+)\leqslant 3$, $2\leqslant b_1(P_0^-)\leqslant 3$, and (a_{n^+}, a_{n^++1}) is a block of C. By symmetry, we may suppose that $a_{n-2}\to a_{n-1}$. Let us consider P_1^+ and P_1^-. We have $b_1(P_1^-)=1$ and $b_1(P_1^+)\leqslant 2$. So by Lemma 6.1, $|X_1^-|\leqslant 3$ and $|X_1^-|\leqslant 3$. Hence, by Lemma 6.3.(3), T contains C.

Suppose now that $L\geqslant 4$. Then $a_{n-2}\to a_{n-1}$ and $a_1\to a_2$. If $b_1(P_0^-)=1$ then $b_1(P_0^+)\geqslant 4$. Let us consider P_1^+ and P_1^-. We have $b_1(P_1^-)\leqslant 2$, so by Lemma 6.1.(1) $|X_1^-|\geqslant n^- - 3$. And since $b_1(P_0^+)\geqslant 2$, $(a_{n^+-2}, a_{n^+-1}, a_{n^+})$ is a directed path, so by Lemma 6.3.(5) T contains C.

Similarly, the theorem holds if $b_1(P_0^+)=1$. So we may assume that $b_1(P_0^-)\geqslant 2$ and $b_1(P_0^+)\geqslant 2$. If the path $(a_{n^+}, a_{n^++1}, a_{n^++2})$ is not directed, let us consider P_1^+ and P_1^-. We have $b_1(P_1^-)=1$; thus by Lemma 6.1, $|X_1^-|\geqslant n^- - 3$. And since $b_1(P_0^+)\geqslant 2$, $(a_{n^+-2}, a_{n^+-1}, a_{n^+})$ is a directed path. Hence, by Lemma 6.3.(5), T contains C.

Likewise, considering P_{-1}^+ and P_{-1}^-, we have the result provided that $(a_{n^+-1}, a_{n^+}, a_{n^++1})$ is not directed.

If $(a_{n^+-1}, a_{n^+}, a_{n^++1}, a_{n^++2})$ is directed, then $L\geqslant 6$ and so $a_2\to a_3$ and $a_{n-2}\to a_{n-3}$. By directional duality, we may assume that $b_1(P_0^-)\leqslant b_1(P_0^+)$. If $b_1(P_0^-)=2$, then by Lemmas 6.1 and 6.3.(5) T contains C. If $b_1(P_0^-)\geqslant 3$, let us consider P_{-2}^+ and P_{-2}^-. By Lemma 6.1, $|X_2^-|\geqslant n^- - k/2\geqslant n^- - k+4$, because $b_1(P_0^-)\leqslant k/2-2$. Hence, by Lemma 6.3.(5). T contains C. ∎

Corollary 6.1. *Every 8-strong tournament contains every Hamiltonian cycle.*

This corollary and Corollary 5.1 yield the following statement:

Corollary 6.2. *Every tournament of order at least 264 contains every nondirected Hamiltonian cycle.*

7. HAMILTONIAN CYCLES IN 5-STRONG TOURNAMENTS

We now improve the bound 8 of Corollary 6.1, for sufficiently large tournaments in order to sharpen the bound 264 of Corollary 6.2.

THEOREM 7.1. *Let T be a k-strong tournament of order n, where $k \geqslant 5$, C a cycle of order n with blocks of length at most L, where $6 \leqslant L \leqslant k+1$, and x a vertex of T whose in- and outdegrees are greater than* 8. *Then T contains C.*

Proof. Let B be a block of a of maximal length, L. Let $l = \lfloor L/2 \rfloor$. Let $C = (a_0, a_1, ..., a_{n-1}, a_0)$ be enumerated so that $a_0 \in B$ and $a_{n-l} \to a_{n-l+1} \to \cdots \to a_{l-1} \to a_l$.

We define n^+, n^-, T^+, T^-, P_i^+, P_i^- X_i^+, and X_i^- in the same way as in Lemma 6.1.

By directional duality, we may assume that $b_1(P_0^+) \geqslant b_1(P_0^-)$. Note that $(T^+; P_0^+)$, $(T^-; P_0^-)$, $(T^+; P_1^+)$, and $(T^-; P_1^-)$ are not one of the exceptions $\mathscr{F}_i(n^+)$ for $i \in \{1, 2, 3, 4, 5, 6, 7, 11, 12, 13, 14\}$ or one of their duals, otherwise C would have a block of length at least $k+2$. And $(T^+; P_0^+)$ is not one of the exceptions $\mathscr{F}_i$ for $i \in \{8, 9, 10\}$ nor one of their duals, because $b_1((P_0^+)^{-1}) = 2$.

- If $b_1(P_0^+) + b_1(P_0^-) = 3$, then Lemmas 6.1.(2) and 6.1.(3) give the result.

- Suppose that $b_1(P_0^-) = b_1(P_0^+) = 2$. If $(a_{n^+-1}, a_{n^+}, a_{n^++1})$ is directed, then Lemmas 6.1.(2) and 6.3 give the result. In the same way, if $(a_{n^+}, a_{n^++1}, a_{n^++2})$ is directed, T contains C. So we may suppose that (a_{n^+}, a_{n^++1}) is a block of C. Consider now P_1^+ and P_1^-. We have $b_1(P_1^-) = 1$. Recall that neither $(T^+; P_1^+)$ nor $(T^-; P_1^-)$ is an exception $\mathscr{F}_i$, $i \in \{1, 2, 3, 4, 5, 6, 7, 11, 12, 13, 14\}$, or one of their duals. Thus, by Lemma 6.1.(1), $|X_1^-| \geqslant n^- - 2$. Hence, by Lemma 6.3.(6), T contains C.

- Suppose that $b_1(P_0^+) \geqslant 3$. If $B_1(P_0^+)$ and $B_1(P_0^-)$ are contained in the same block of C, then $|X_0^-| \geqslant n^- - k + 3$. Thus by Lemma 6.3.(5) T contains C.

So we may suppose that $B_1(P_0^+)$ and $B_1(P_0^-)$ are not contained in the same block of C. Thus $b_1(P_1^-) = 1$ or $b_1(P_2^-) = 1$.

If $b_1(P_1^-) = 1$, consider P_1^+ and P_1^-. By Lemma 6.1, $|X_1^-| \geqslant n^- - 2$. Thus, by Lemma 6.3.(6) T contains C.

Suppose now that $b_1(P_2^-) = 1$. If $(T_2^-; P_2^-)$ is not the exception $\mathscr{F}_1(n^-)$, then by Lemma 6.1.(1) $|X_2^-| \geqslant n^- - 2$. (Note that $(T_2^-; P_2^-) \neq -\mathscr{F}_1(n^-)$, otherwise it would have a block of length at least $k+2$.) If $(T^+; P_2^+)$ is not one of the exceptions $\mathscr{F}_i$ for $i \in \{2, 3, 4, 5, 6, 7, 11, 12, 13, 14\}$ nor one of their duals then, by Lemma 6.3.(6), T contains C. On the other hand, if $(T^+; P_2^+)$ is one of the exceptions $\mathscr{F}_i$ for $i \in \{2, 3, 4, 5, 6, 7, 11, 12, 13, 14\}$ or one of their duals, then C has a block of length at least 7. So $k \geqslant 6$. Also, $|X_0^+| = n^+ - 2$. Hence, by Lemma 6.3.(3), T contains C.

If $(T_2^-; P_2^-)$ is the exception $\mathscr{F}_1(n^-)$, let us consider P_{-1}^+ and P_{-1}^+. We have $b_1(P_{-1}^+) = 1$ and $(T^+; P_{-1}^+)$ is not exception $\mathscr{F}_1$, otherwise C would have a block of length greater than $k+1$. So by Lemma 6.1.(1) we have $|X_{-1}^+| \geqslant n^+ - 2$. Thus by Lemma 6.3.(6) T contains C. ∎

Theorems 4.3, 5.1, 5.2. and 7.1 yield the following corollary:

Corollary 7.1. *Every tournament of order at least* 168 *contains every nondirected Hamiltonian cycle.*

Lemma 7.1. *Let T be a* $(=5)$*-strong tournament of order* $n=3m\geqslant 15$ *and C a cycle of order n and of type* $(2, 1, 2, 1, ..., 2, 1)$. *Then T contains C.*

Proof. Let $C=(a_0, a_1, ..., a_{n-1}, a_0)$ with $a_0\rightarrow a_1\rightarrow a_2$. Let X be a subtournament of five vertices such that $T-X$ is reducible. By duality, we may suppose that the maximal strong component of $T-X$ is at least as big as its minimal strong component T_1. Set $T_2=T-(X, T_1)$. We have $T_1\rightarrow T_2$. Let $R=(a_2, a_3, ..., a_6)=-(1, 2, 1)$. It is easy to check that there are at least three origins of R in X. So there is a vertex x of X that is an origin of R in X and such that $d^-(x)\leqslant 3$. Let y be the terminus of the path R in T with origin x. Since T is $(=5)$-strong, y dominates an outgenerator t of T_1 and x is dominated by an ingenerator u of T_2.

If $|T_1|\geqslant 3$, let t' be the terminus of a Hamiltonian directed outpath of T_1 with origin t, and let $P=(a_7, a_8, ..., a_{n-1})=+(1, 1, 2, 1, 2, ..., 1, 2)$. Obviously, $A^+(P)\geqslant |T_1-t'|$. So by Lemma 2.1, one can find P in $(T_1-t')\rightarrow(T_2-u)$ with origin t and terminus in T_2-u. Thus T contains $C=(t', u, R, P, t')$.

If $T_1=\{t\}$, then $|T_2|\geqslant 9$. Let U be the set of vertices of T_2 that dominate x. Because T is $(=5)$-strong, $d^-_T(x)\geqslant 5$ so $|U|\geqslant 2$. Set $Q=(a_1, a_0, a_{n-1}, ..., a_8)=-(1, 1, 2, 1, ..., 2, 1)$. By Theorem 2.2, since $(T_2; Q)$ is not an exception $\mathscr{F}_i$, some vertex $v\in U$ is an origin of Q in T_2. Then T contains $C=(v, R, t, Q^{-1})$. ∎

Lemma 7.2. *Let T be a* $(=5)$*-strong tournament of order* $n=5m\geqslant 15$ *and C a cycle of order n of type* $(3, 2, 3, 2, ..., 3, 2)$. *Then T contains C.*

Proof. Let $C=(a_0, a_1, ..., a_{n-1}, a_0)$ with $a_0\rightarrow a_1\rightarrow a_2\rightarrow a_3$. Let X be a subtournament of five vertices such that $T-X$ is reducible. By duality, we may suppose that the maximal strong component of $T-X$ is at least as big as its minimal strong component T_1. Set $T_2=T-(X, T_1)$. We have $T_1\rightarrow T_2$. Let $R=(a_2, a_3, ..., a_6)=+(1, 2, 1)$. It is easy to check that there are at least three origins of R in X. So there is a vertex x of X that is an origin of R in X and such that $d^-_X(x)\leqslant 3$. Let y be the terminus of the path R in T with origin x. Since T is $(=5)$-strong, y dominates an outgenerator t of T_1 and x is dominated by an ingenerator u of T_2.

If $|T_1|\geqslant 3$, let t' be the terminus of a Hamiltonian directed outpath of T_1 with origin t, and let $P=(a_7, a_8, ..., a_{n-1})=+(1, 2, 3, 2, ..., 3, 2, 3, 1)$. It is easy to check that $A^+(P) \geqslant |T_1 - t'|$. So by Lemma 2.1, in $(T_1 - t') \rightarrow (T_2 - u)$, one can find P with origin t and terminus in $T_2 - u$. Thus T contains $C = (t', u, R, P, t')$.

Suppose now that $T_1 = \{t\}$, so that $|T_2| \geqslant 9$. Let U be the set of vertices of T_2 that dominate x. Because T is $(=5)$-strong, $d_T^-(x) \geqslant 5$ so $|U| \geqslant 2$. Set $Q = (a_1, a_0, a_{n-1}, ..., a_8) = -(1, 2, 3, 2, ..., 3, 2)$. By Theorem 2.2, since $(T_2; Q)$ is not one of the exceptions $\mathscr{F}_i$, $1 \leqslant i \leqslant 14$, some vertex $v \in U$ is an origin of Q in T_2. Then T contains $C = (v, R, t, Q^{-1})$. ∎

LEMMA 7.3. *Let T be a k-strong tournament of order n where $k \geqslant 5$, C a cycle of order n, and B a block of C with maximal length L, where $2 \leqslant L \leqslant k+1$. If all blocks of C except possibly B have length at most five and if T has a vertex x with both in- and outdegree greater than* 10, *then T contains C.*

Proof. Let B be a block of C of maximal length L, let $l = \lfloor L/2 \rfloor$, and let $C = (a_0, a_1, ..., a_{n-1}, a_0)$ be such that $a_0 \in B$ and $a_{n-l} \rightarrow a_{n-l+1} \rightarrow \cdots \rightarrow a_{l-1} \rightarrow a_l$.

We define n^+, n^-, T^+, T^-, P_i^+, P_i^- X_i^+, and X_i^- in the same way as in Lemma 6.1. Note that for each path P_i^+ (resp. P_i^-), the pair $(T^+; P_i^+)$ (resp. $(T^-; P_i^-)$) is not an exception $\mathscr{F}_j$, $j \in \{1, 2, 3, 4, 5, 6, 7, 8, 11, 12, 13, 14\}$; indeed, if it were, C would have a block of length at least $n^+ - 4 > 5$ (resp. $n^- - 4 > 5$).

Suppose that $(T^+; P_i^+)$ is one of the exceptions $\mathscr{F}_9(n^+)$, $\mathscr{F}_{10}(n^+)$ or one of their duals and that the termini of P_i^+ and P_i^- are in B. Since $n^+ \geqslant 11$, we have $b_1(P_i^+) \geqslant 3$, so $L \geqslant 3$.

— Suppose first that $L \geqslant 4$. Because the last vertex of P_i^+ is in B and the last block of P_i^+ has length one, we have $a_{n-i-2} \rightarrow a_{n-i-1}$.

Suppose that $a_{n-i-3} \rightarrow a_{n-i-2}$. (Note that this is the case if $L = 5$). Let us consider P_{i+2}^+ and P_{i+2}^-. The pair $(T^+; P_{i+2}^+)$ is not one of the exceptions $\mathscr{F}_9(n^+)$, $\mathscr{F}_{10}(n^+)$ nor one of their duals because $(T^+; P_i^+)$ was one of these. Moreover $b_1(P_{i+2}^-) = 1$, so (T^-, P_{i+2}^-) is not an exception. Thus the result follows from Lemmas 6.1.(2) and 6.3.(6).

Suppose now that $a_{n-i-3} \leftarrow a_{n-i-2}$. Then $L = 4$ and $a_{1-i} \rightarrow a_{2-i}$. Let us consider P_{i+1}^- and P_{i+1}^-. Since $(T^+; P_i^+)$ is one of the exceptions, $(T^+; P_{i+1}^+)$ is not. If $b_1(P_{i+1}^-) \leqslant 2$, then $(T^-; P_{i+1}^-)$ is not an exception. Thus Lemmas 6.1.(2) and 6.3.(6) give the result. If $b_1(P_{i+1}^-) \geqslant 3$, consider P_{i-1}^+ and P_{i-1}^-. We have $b_1(P_{i-1}^-) \leqslant 2$ and $b_1(P_{i-1}^+) = 1$. So $(T^+; P_{i-1}^+)$ and $(T^-; P_{i-1}^-)$ are not exceptions. Thus the result follows Lemmas 6.1.(2) and 6.3.(3).

— Suppose now that $L = 3$. Since $n^+ \geqslant 11$, $(T^+; P_i^+)$ is $\mathscr{F}_{10}(11)$ or its dual. So $n^+ = 11$ and $n^- = n - 12 = 31$. Thus, for every l, $(T^-; P_l^-)$ is not one of the exceptions $\mathscr{F}_j$ nor one of their duals. Also, $(a_{n^+-i-1}, a_{n^+-i}, a_{n^+-i+1})$ is not directed. If $a_{n-i-2} \to a_{n-i-1}$, let us consider P_{i+1}^+ and P_{i+1}^-. Since $L=3$, $b_1(P_{i+1}^-) \leqslant 3$ and Lemmas 6.1.(2) and 6.3.(6) yield the result. If $a_{n-i-2} \leftarrow a_{n-i-1}$, then $a_{1-i} \to a_{2-i}$. Let us consider P_{i-1}^+ and P_{i-1}^-. We have $b_1(P_{i-1}^-) \leqslant 3$ and $b_1(P_{i-1}^+) = 1$. Then Lemmas 6.1.(2) and 6.3.(3) yield the result.

In the same way, the statement holds if $(T^-; P_i^-)$ is the exception $\mathscr{F}_{10}(n^-)$ or its dual.

In the rest of the proof, we always suppose that P_i^+ and P_i^- have their termini in B. So, by the previous argument, we may assume that neither $(T^+; P_i^+)$ nor $(T^-; P_i^-)$ is an exception $\mathscr{F}_j$, $1 \leqslant j \leqslant 14$, or one of their duals, and we may apply Lemma 6.1.(2).

If $b_1(P_0^+) + b_1(P_0^-) \leqslant 3$, then $|X_0^+| \geqslant n^+ - b_1(P_0^+)$ and $|X_0^-| \geqslant n^- - b_1(P_0^-)$ by Lemma 6.1.(2). So by Lemma 6.3.(3) T contains C and we may assume that $b_1(P_0^+) + b_1(P_0^-) \geqslant 4$.

(A) Suppose first that $b_1(P_0^+) + b_1(P_0^-) \geqslant 5$. Then $L \geqslant 3$. Moreover $B_1(P_0^+)$ and $B_1(P_0^-)$ are not contained in the same block of C.

(i) Suppose that $L \geqslant 4$. Then $a_{n-2} \to a_{n-1}$ and $a_1 \to a_2$.

If $b_1(P_0^-) = 1$, then $b_1(P_0^+) \geqslant 4$. Let us consider P_1^+ and P_1^-. We have $b_1(P_1^-) \leqslant 2$, so by Lemma 6.1.(2) $|X_1^-| \geqslant n^- - 2$. And since $b_1(P_0^+) \geqslant 4$, $(a_{n^+-2}, a_{n^+-1}, a_{n^+})$ is a directed path, so by Lemma 6.3.(6) T contains C.

Similarly, we have the result if $b_1(P_0^+) = 1$.

So we may assume that $b_1(P_0^-) \geqslant 2$ and $b_1(P_0^+) \geqslant 2$. If the path $(a_{n^+}, a_{n^++1}, a_{n^++2})$ is not directed, let us consider P_1^+ and P_1^-. We have $b_1(P_1^-) = 1$, thus by Lemma 6.1.(2) $|X_1^-| \geqslant n^- - 1$. And since $b_1(P_0^+) \geqslant 2$, $(a_{n^+-2}, a_{n^+-1}, a_{n^+})$ is a directed path, so by Lemma 6.3.(6) T contains C.

In the same way, by considering P_{-1}^+ and P_{-1}^-, we have the result if $(a_{n^+-1}, a_{n^+}, a_{n^++1})$ is not directed.

(ii) Suppose now that $L = 3$.

Suppose that $b_1(P_0^-) = b_1(P_0^+) = 3$. Since $L = 3$, (a_{n^+}, a_{n^++1}) is a block of C. By symmetry, we may suppose that $a_{n-2} \to a_{n-1}$. Let us consider P_1^+ and P_1^-. We have $b_1(P_1^-) = 1$ and $b_1(P_1^+) = 2$. So by Lemmas 6.1.(2) and 6.3.(3) T contains C.

We may therefore assume that $b_1(P_0^-) + b_1(P_0^+) = 5$. By symmetry, we may also suppose that $b_1(P_0^+) = 3$ and $b_1(P_0^-) = 2$.

Since $L = 3$, $a_{n-2} \to a_{n-1}$ or $a_1 \to a_2$. If $a_1 \to a_2$, consider P_{-1}^- and P_{-1}^+. We have $b_1(P_{-1}^+) = b_1(P_{-1}^+) = 1$. Lemmas 6.1.(2) and 6.3.(3) now yield the result. So we may assume that $a_{n-2} \to a_{n-1}$. Since $L = 3$, either $a_{n^+} \to a_{n^++1}$ or $a_{n^+-1} \to a_{n^+}$. If $a_{n^+} \to a_{n^++1}$, Lemma 6.3.(4) applied to P_0^+ and P_0^+ gives the result. Similarly, if $a_{n^+-1} \to a_{n^+}$, Lemma 6.3.(4) applied to P_1^+ and P_1^- gives the result.

(B) Suppose now that $b_1(P_0^+)+b_1(P_0^-)=4$. By symmetry, we may suppose that $b_1(P_0^+)\geqslant b_1(P_0^-)$. So $b_1(P_0^+)\geqslant 2$. If T is 6-strong, then Lemmas 6.1.(2) and 6.3.(3) yield the result. Hence we may suppose that T is $(=5)$-strong.

If $L=2$, then $b_1(P_0^+)=b_1(P_0^-)=2$. Moreover, (a_{n^+}, a_{n^++1}) is a block of C. Thus $(a_{n^+-2}, a_{n^+-1}, a_{n^+}, a_{n^++1}, a_{n^++2})=\pm(2, 1, 2)$. Now, applying this to the two blocks of length two of this path, either T contains C or C has a subpath $\pm(2, 1, 2, 1, 2)$. By induction, we may suppose that $C=(2, 1, 2, 1, ..., 2, 1)$, and by Lemma 7.1, T contains C.

So we may suppose that $L=3$.

(i) Assume that $a_{n-2}\rightarrow a_{n-1}$ (Note that this is so when $L\geqslant 4$). Consider P_1^- and P_1^+.

If $b_1(P_1^-)\leqslant 2$, by Lemma 6.1.(2) $|X_1^+|\geqslant n^- -2$. Also $(a_{n^+-2}, a_{n^+-1}, a_{n^+})$ is directed because $b_1(P_0^+)\geqslant 2$. Hence, by Lemma 6.3.(6), T contains C.

If $b_1(P_1^-)=3$, then $b_1(P_0^+)=2$.

If the vertices a_{n^+-1} and a_{n^++1} are not in the same block of C, either $a_{n^+}\rightarrow a_{n^++1}$ or $a_{n^+-1}\rightarrow a_{n^+}$. If $a_{n^+}\rightarrow a_{n^++1}$, then Lemma 6.3.(4) applied to P_0^+ and P_0^- gives the result.

If $a_{n^+-1}\rightarrow a_{n^+}$, Lemma 6.3.(4) applied to P_1^+ and P_1^- yields the result.

If the vertices a_{n^+-1} and a_{n^++1} are in the same block of C, then $B'=(a_{n^+-2}, a_{n^+-1}, ..., a_{n^++3})$ is a block of C of length five. Thus $(a_{n^+-1}, a_{n^+}, a_{n^++1})$ is directed. Let us consider P_0^+ and P_0^-. By Lemma 6.1.(2), $|X_0^-|\geqslant n^- -2$; hence, by Lemma 6.3.(6) T contains C.

(ii) Suppose that $a_{n-2}\rightarrow a_{n-1}$. Then $a_1\rightarrow a_2$ and $L=3$.

Assume that $b_1(P_0^-)=b_1(P_0^+)=2$.

If $(a_{n^+-1}, a_{n^+}, a_{n^++1})$ is not directed, then $b_1(P_{-1}^+)=b_1(P_{-1}^-)=1$. Hence, by Lemmas 6.1.(2) and 6.3.(3), T contains C.

If $(a_{n^+-1}, a_{n^+}, a_{n^++1})$ is directed, then by considering P_0^+ and P_0^-, Lemmas 6.1.(2) and 6.3.(6) yield the result.

Assume now that $b_1(P_0^-)=1$ and $b_1(P_0^+)=3$.

If $a_{n^+}\rightarrow a_{n^+-1}$, Lemmas 6.1.(2) and 6.3.(4) give the result. So we may assume that $a_{n^+}\leftarrow a_{n^+-1}$. Consider now P_{-1}^+ and P_{-1}^-. Since $L=3$, $b_1(P_{-1}^+)=1$. Also $b_1(P_{-1}^-)\leqslant 3$. If $a_{n^+-1}\rightarrow a_{n^+-2}$, Lemmas 6.1.(2) and 6.3.(4) give the result and if $b_1(P_{-1}^-)\leqslant 2$, Lemmas 6.1.(2) and 6.3.(4) give the result.

So $(a_{n^+-3}, a_{n^+-2}, ..., a_{n^++5})=+(3, 2, 3)$. Now, applying this method to the two blocks of length 3 of this path, either T contains C or C has a subpath $+(3, 2, 3, 2, 3)$. And so on, by induction, we may suppose that $C=(3, 2, 3, 2, ..., 3, 2)$ and, by Lemma 7.2, T contains C. ∎

Theorems 7.1 and 7.3 yield:

COROLLARY 7.2. *Every* 5-*strong tournament of order at least* 43 *contains every Hamiltonian cycle.*

This corollary and Corollary 5.1 give the following statement:

COROLLARY 7.3. *Every tournament of order at least* 105 *contains every nondirected Hamiltonian cycle.*

8. HAMILTONIAN CYCLES IN 4-STRONG TOURNAMENTS

LEMMA 8.1. *Let T be a* $(=4)$-*strong tournament of order $n \geqslant 19$ and C a cycle of order n whose maximal block is of length at most five and which contains a subpath of type* $(3, 2, 3)$. *Then T contains C.*

Proof. Let $C = (a_0, a_1, ..., a_{n-1}, a_0)$ with $a_0 \rightarrow a_1 \rightarrow a_2 \rightarrow a_3 \leftarrow a_4 \leftarrow a_5 \rightarrow a_6 \rightarrow a_7 \rightarrow a_8$. Let X be a subtournament of four vertices such that $T - X$ is reducible. By duality, we may suppose that the maximal strong component of $T - X$ is at least as big as its minimal strong component T_1. Set $T_2 = T - (X, T_1)$. We have $T_1 \rightarrow T_2$. Set $R = (a_2, a_3, a_4, a_5) = +(1, 2)$. It is easy to check that there are at least two origins of R in X. So there is a vertex x of X that is an origin of R in X and is such that $d^-_X(x) \leqslant 2$. Let y be the terminus of the path R in T with origin x. Since T is $(=4)$-strong, y dominates an outgenerator t of T_1 and x is dominated by an ingenerator u of T_2.

If $|T_1| = 1$, let $P = (a_7, a_8, ..., a_1)$. Let U_x be the set of vertices of T_2 that are dominated by x. Since T is 4-strong, $|U_x| \geqslant 2$. Also $|T_2| \geqslant 14$, so $(T_2; P)$ is not one of the exceptions $\mathscr{F}_i$ or one of their duals, otherwise C would have a block of length greater than five. Hence, by Theorem 2.2, there is a path P in T_2 with terminus in U_x. Thus T contains $C = (R, T, P)$.

If $|T_1| \geqslant 3$, let a_l be the origin of the block B that contains $a_0 \rightarrow a_1$. Let I be an inpath of length $b - 3$ with origin u, where b is the length of B, and let v be an outgenerator of $N^+_{T_1}(t)$. Let P be the path $(a_7, ..., a_{l-1})$ and Q the path $(a_l, a_{l-1}, ..., a_8)$. Either $b_1(P) \geqslant |T_1| - 2$ or $b_1(Q) \geqslant |T_1| - 2$, because $|T_1| \leqslant |T_2|$.

If $b_1(P) \geqslant |T_1| - 2$, let w be the terminus of a Hamiltonian directed outpath of $T_1 - t$ with origin v. By Lemma 2.1, in $T' - (t, w, I)$, there is a path P with origin v and terminus in $T_2 - I$. Hence T contains $C = (w, I^{-1}, R, t, P, w)$.

If $b_1(Q) \geqslant |T_1| - 2$, let w be an outgenerator of $T_1 - (t, v)$. By Lemma 2.1, in $T' - (t, v, I)$, there is a path Q with origin w and terminus in $T_2 - I$. Hence T contains $C = (v, I^{-1}, R, t, v, Q^{-1})$. ∎

LEMMA 8.2. *Let T be a* $(=4)$-*strong tournament of order $n \geqslant 55$ and C a cycle of order n whose longest block B is of length four or five. Then T contains C.*

Proof. Let $C=(a_0, a_1, ..., a_{n-1}, a_0)$, where $a_{n-2} \to a_{n-1} \to a_0 \to a_1 \to a_2$. Let x be a vertex of T such that $n^+ = |N_T^+(x)|$ and $n^- = |N_T^-(x)|$ are greater than 13. Such a vertex exists since $|T| \geqslant 55$. Let us denote by T^+ and T^- the subtournaments $T(N_T^+(x))$ and $T(N_T^-(x))$. We define P_i^+, P_i^-, X_i^+, and X_i^- in the same as in Lemma 6.1. Note that for each path P_i^+ (resp. P_i^-), the pair $(T^+; P_i^+)$ (resp. $(T^-; P_i^-)$) is not one of the exceptions $\mathscr{F}_j$, $1 \leqslant j \leqslant 14$ or one of their duals; indeed, if it were, C would have a block of length at least $n^+ - 8 > 5$ (resp. $n^- - 8 > 5$).

If $b_1(P_0^+) = b_1(P_0^-) = 1$, then Lemmas 6.1.(2) and 6.3.(3) give the result. So, by symmetry, we may suppose that $b_1(P_0^+) \geqslant 2$.

Suppose that $B_1(P_0^+)$ and $B_1(P_0^-)$ are contained in the same block of C. Then either $b_1(P_0^-) = 1$ or $b_1(P_{-1}^-) = 1$. In both cases, by Lemmas 6.1.(2) and 6.3.(6), T contains C. Thus we may assume that $B_1(P_0^+)$ and $B_1(P_0^-)$ are not contained in the same block.

If $b_1(P_1^-) = 1$, by Lemmas 6.1.(2) and 6.3.(6), T contains C.

If $b_1(P_1^-) \geqslant 3$, consider P_{-1}^+ and P_{-1}^+. Then $B_1(P_{-1}^+)$ and $B_1(P_{-1}^-)$ are contained in the same block and, as before, T contains C. Hence we may assume that $b_1(P_1^-) = 2$ and thus that $b_1(P_0^-) = 1$.

If $a_{n^+-1} \to a_{n^+}$ then P_1^+ is an inpath. By Lemma 6.1, $|X_1^-| \geqslant n^- - 2$. Suppose, by way of contradiction, that T does not contain C. Then, by Lemma 6.3, $X_1^- \to X_1^+$. If $|X_1^+ = | \geqslant n-1$, then $T(X_1^-, X_1^+, x)$ is reducible, of order $n-3$. This is a contradiction. If $|X_1^+| \leqslant n-2$, then by Lemma 6.2, $(T - X_1^+) \to X_1^+$, so $T(X_1^-, N_T^+, x)$ is reducible, of order $n-2$. This is again a contradiction.

Now we may assume that $a_{n^+-2} \leftarrow a_{n^+-1} \leftarrow a_{n^+} \to a_{n^++1} \to a_{n^++2} \leftarrow a_{n^++3}$.

If $b_1(P_0^+) = 2$, then, Lemmas 6.1 and 6.3.(4) applied to P_0^+ and P_0^- give the result.

If $b_1(P_{-1}^+) \leqslant 2$, then, Lemmas 6.1 and 6.3.(4) applied to P_{-1}^+ and P_{-1}^- give the result.

So C contains a subpath $+(3, 2, 3)$ and then, by Lemma 8.1, T contains C. ∎

By Theorems 4.3, 5.1, and 8.2, we obtain the following corollary:

Corollary 8.1. *Every $(=4)$-strong tournament of order $n \geqslant 65$ contains every Hamiltonian cycle.*

This corollary, together with Corollary 5.1, yields the following:

Corollary 8.2. *Every tournament of order at least 68 contains every nondirected Hamiltonian cycle.*

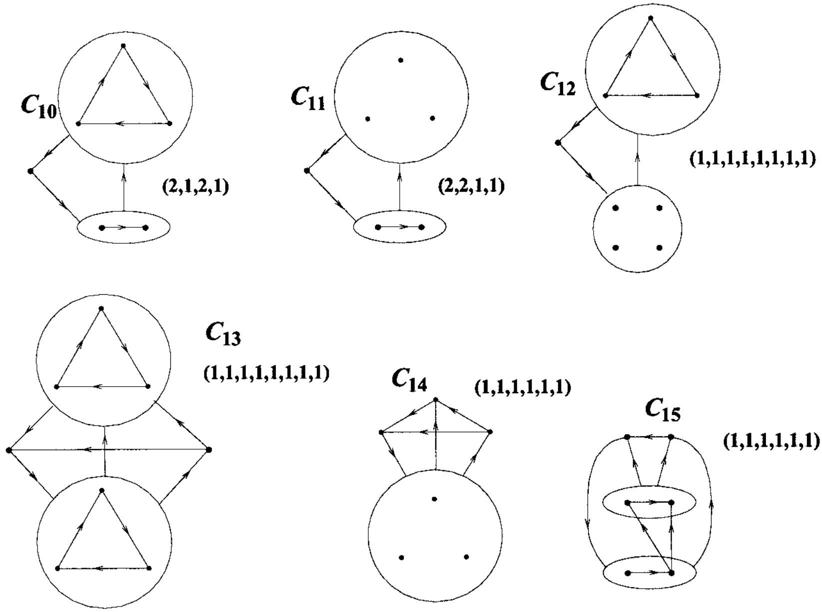

FIG. 7. 1-strong exceptions.

The bound 68 of Corollary 8.2 is of course not sharp. The right value is very likely to be 9, as remarked by Thomason [11]. This bound could be proved by our methods with the aid of a computer, there being many cases to check. Indeed, Theorem 2.2 has 50 exceptions of order less than 8. Thus, one can prove lemmas analogous to Lemmas 6.1 and 6.2 for tournaments of order less than 8, but with numerous exceptions. By studying the many cases generated by these exceptions, one could prove that every 5-strong tournament contains every Hamiltonian cycle as in the proof of Theorem 7.1. On the other hand, using Lemmas 2.1, 2.2, and 2.3, and proofs in the same vein as those of Theorems 4.1, 4.2, and 5.1 it would be possible to prove the result for tournaments which are not 5-strong.

In [6], we prove it for $(=1)$-strong tournaments:

THEOREM 8.1. *Let T be a* 1-*strong tournament of order n and C a cycle of order n. Then T contains C if and only if $(T; C)$ is neither one of the exceptions C_i for $i \in \{1, 4, 5, 6, 9, 10, 11, 12, 13, 14, 15\}$ (Figs.* 2 *and* 7) *nor one of their duals.*

ACKNOWLEDGMENTS

We gratefully thank J. A. Bondy for his help during the preparation of this paper.

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