Finding an oriented Hamiltonian path in a tournament.

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Abstract

We present here an $O(n^2)$ algorithm for finding a specicfied oriented path of order at most n in a tournament of order n. Using this algorithm, we present an $O(n^2)$ algorithm that finds a specified oriented path from a gien vertex if one exists.

One of the earliest results on tournaments is Rédei's theorem, which asserts that every tournament has a directed hamiltonian path. His proof yields an algorithm in $O(n^2)$ to find such a path in a tournament. Hell and Rosenfeld [4] gave a faster algorithm, in time $O(n \log n)$, for the same problem. This algorithm is an analog of binary tree insertion sort. In the particular case of a transitive tournament, it is optimal : finding a directed hamiltonian path is equivalent to sorting its vertices.

Hell and Rosenfeld [4] also showed that, for all $0 \leq \alpha \leq 1$, there exist paths that require $O(n \log^{\alpha} n)$ comparisons to be found. In particular, they proved that finding a alternating hamiltonian path (in which adjacent arcs have opposite sense) necessitates $O(n)$ comparisons. Bampis, Hell, Manoussakis and Rosenfeld [1] described an algorithm in time $O(n)$ which finds an alternating hamiltonian path in a tournament.

In 1986, Thomason [6] proved that every sufficiently large tourmanent (of order at least 2^{128}) contains every oriented hamiltonian path. However no efficient algorithm was known to find an arbitrary oriented hamiltonian path in a tournament, even sufficiently large, Thomason's proof was not constructive. Recently, Havet and Thomassé [3] proved that with three exceptions (of orders 3, 5 and 7), every tournament of order n contains every oriented hamiltonian path.

Using results of [3], we present here an algorithm in time $O(n^2)$ that, given a tournament T of order n and an oriented path P of order at most n, finds an occurrence of P in T. This algorithm is based on Theorem 1 below. Because of the exceptional cases in the theorem, requires some special subroutines to examine these exceptions. In order to give a clearer view of this algorithm, we first present it when we restrict ourselves to paths of order at most $n-1$. In this case, there are no exceptions to Theorem 1 and thus no special subroutines are needed.

To cnclude, we apply our algorithm to show how to find a specified oriented path of order at most n starting from a given vertex in a tournament of order n, if such a path exists.

1 Preliminaries

Definition 1 A *tournament* is an orientation of the edges of a complete graph. Let T be a tournament. If X is a subset of $V(T)$, the subtournament of T induced by X is denoted by $T(X)$; for convenience we denote $T(V \setminus X)$ by $T-X$ and $T-\{x\}$ by $T-x$. Let D be a digraph. We say that T *contains* D if D is a subgraph of T. The *order* of a digraph D, denoted by $|D|$, is its number of vertices.

An *oriented path of order n* is an orientation of the path $P=(x_1,\cdots,x_n)$. Occasionally, we abbreviate 'oriented path' to 'path'. Let $P=(x_1,\cdots,x_n)$ be a path. We say that x_1 is the *origin* of P and x_n is the *terminus* of P. If $x_1 \to x_2$, P is an *outpath*, otherwise P is an *inpath*. The *directed outpath* of order n is the path $P=(x_1,\cdots,x_n)$ in which $x_i \to x_{i+1}$ for all i, $1 \leq i < n$; the dual notion is *directed inpath*. The *length* of a path is its number of arcs. We denote the path $(x_2,\cdots,x_n)$ by ${}^*P$. The *blocks* of P are the maximal directed subpaths of P. We enumerate the blocks of P from the origin to the terminus. The first block of P is denoted by $B_1(P)$ and its length by $b_1(P)$. Likewise, the i^{th} block of P is denoted by $B_i(P)$ and its length by $b_i(P)$. The path P is totally described by the signed sequence $sgn(P)(b_1(P),b_2(P),\cdots,b_k(P))$ where k is the number of blocks of P and $sgn(P)=+$ if P is an outpath and $sgn(P)=-$ if P is an inpath.

Let X be a subset of vertices of T. We denote by I_X (resp. O_X) an arbitrary directed hamiltonian inpath (resp. outpath) of $T(X)$.

The *dual* (sometimes called the converse) of a tournament T is the tournament $-T$ on the same set of vertices such that $x \to y$ is an arc of $-T$ if and only if $y \to x$ is an arc of T. Likewise, the *dual* of a path P is the path, denoted $-P$, on the same set of vertices such that $x \to y$ is an arc of $-P$ if and only if $y \to x$ is an arc of P.

A tournament is *strong* if for any two vertices x and y there exists a directed outpath with origin x and terminus y. A (strong) *component* of T is a strong subtournament of T which is maximal with respect to inclusion. A tournament is $k-strong$, if $|V(T)| \geq k+1$ and $T-X$ is strong for any set X of $k-1$ vertices.

Let X be a set of vertices of T. The *outsection* generated by X in T is the set of vertices y to which there exists a directed outpath from $x \in X$; we denote this set by $S^+_T(X)$ (note that $X \subseteq S^+_T(X)$ since we allow paths of length zero). We abbreviate $S^+_T(\{x\})$ to $S^+_T(x)$ and $S^+_T(\{x,y\})$ to $S^+_T(x,y)$. The dual notion, the *insection*, is denoted by $S^-(X)$. We also write $s^+_T(X)$ (resp. $s^-_T(X)$) for the number of vertices of $S^+_T(X)$ (resp. $S^-_T(X)$). If $X \subseteq Y \subseteq V$, we write $S^+_Y(X)$ instead of $S^+_{T(Y)}(X)$. An *outgenerator* of T is a vertex $x \in T$ such that $S^+(x)=T$; the dual notion is an *ingenerator*.

Our algorithm is based on the following theorem and its dual:

Theorem 1 (Havet and Thomassé [3]) *Let T be a tournament of order n and P an outpath and x and y two vertices of T such that $s^+_T(x,y) \geq b_1(P)+1$.*
1) If $|P| \leq n-1$ then x or y is an origin of P in T.
2) If $|P| = n \geq 8$ and $(T;P)$ is not an exception $\mathcal{F}_i$, then x or y is an origin of P in T.

Definition 2 - We now introduce the *exceptions* $\mathcal{F}_i(n)=(F_i(n),P)$, where $F_i(n)$ is the tournament on n vertices depicted as F_i in Figure 1. We denote by S the set of vertices of $F_i(n)$ which are not origins of P. Conditions are given on the tournament. Finally, we give the paths P with origin $x \notin S$.

$\mathcal{F}_1(n)=(F_1(n),(1,n-2))$; $S=\{1,2,3\}$. Conditions: $|A| \geq 1$. Paths: For any $u \in A$, $P=u132I_{A\setminus\{u\}}$.

$\mathcal{F}_2(n)=(F_2(n),(2,n-3))$; $S=\{3,4\}$. Conditions: $|A| \geq 1$. Paths: $P=1234I_A$, $P=2314I_A$ and for any $u \in A$, $P=u4321I_{A\setminus\{u\}}$.

$\mathcal{F}_3(n)=(F_3(n),(1,n-2))$; $S=\{1,3\}$. Conditions: $N^+(3) \neq \{2\}$ and 3 is an ingenerator of $T(A)$ (in particular there exists a directed hamiltonian inpath $u3R_1$ of $T(A)$ and another $3vR_2$ of $T(A)$). Paths: $P=21u3R_1$, and for any $y \in A\setminus\{3\}$, $P=y12I_{A\setminus\{y\}}$.

$\mathcal{F}_4(n)=(F_4(n),(2,n-3))$; $S=\{1,4\}$. Conditions: $N^+(3) \neq \{2\}$ and 3 is an ingenerator of $T(A)$ (in particular there exists a directed hamiltonian inpath $u3R_1$ of $T(A)$ and another $3vR_2$ of $T(A)$). Paths: $P=214u3R_1$, $P=3241vR_2$ and for any $y \in A\setminus\{3\}$, $P=y142I_{A\setminus\{y\}}$.

$\mathcal{F}_5(n)=(F_5(n),(1,n-2))$; $S=\{1,2\}$. Conditions: $n \geq 5$, $|B| \geq 2$ and 2 is an ingenerator of $T(A)$. Paths: for any $x \in A\setminus\{2\}$, $P=x1I_BI_{A\setminus\{x\}}$ and for any $y \in B$, $P=y1I_{B\setminus\{y\}}I_A$.

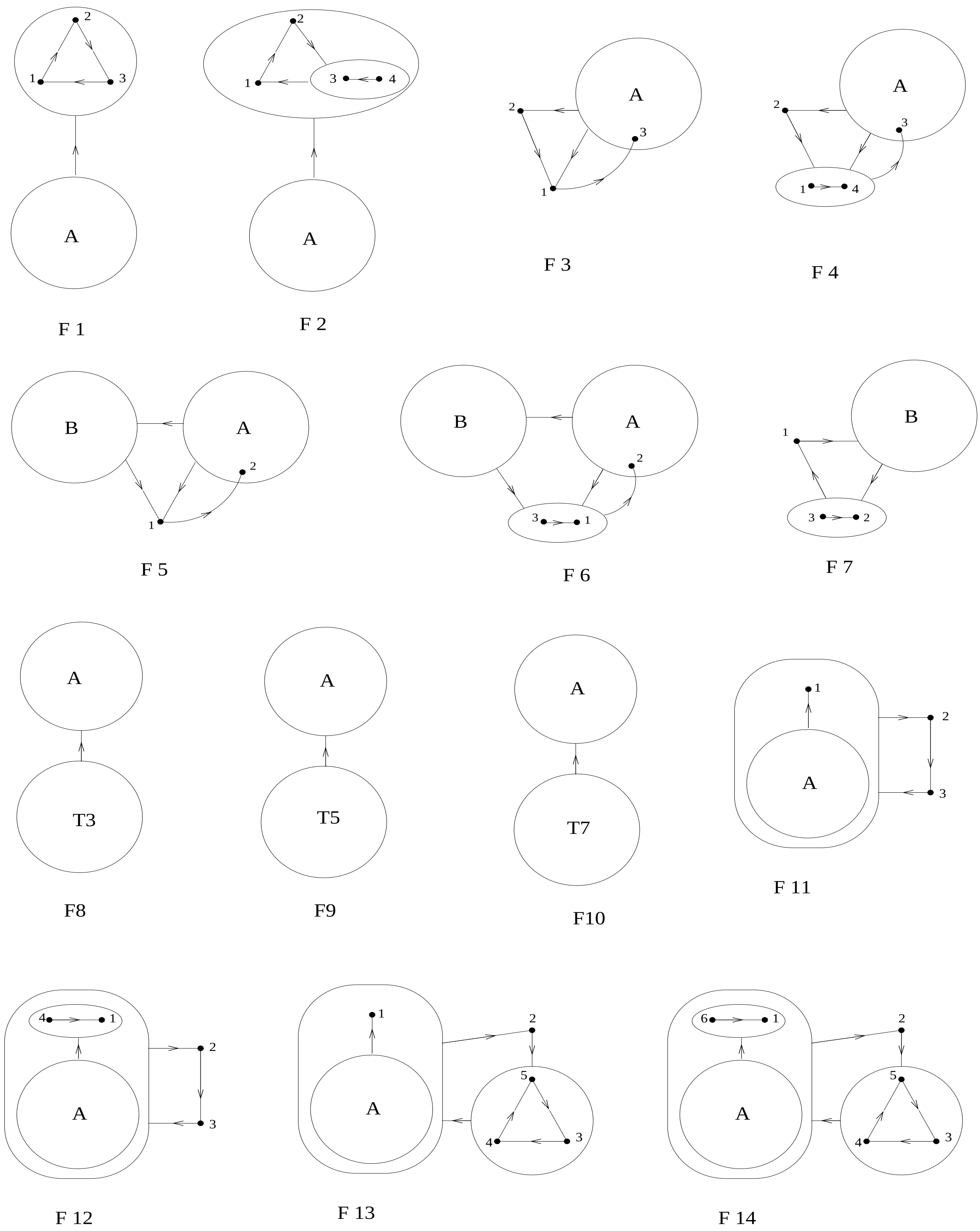

Figure 1: Tournaments F_i.

$\mathcal{F}_6(n) = (F_6(n), (2, n-3))$; $S = \{1,3\}$. Conditions: $|B| \geq 2$ and 2 is an ingenerator of $T(A)$. Paths: for any $x \in A$, $P = xz13I_{B\setminus\{z\}}I_{A\setminus\{x\}}$ (for a given $z \in B$) and for any $y \in B$, $P = y31I_{B\setminus\{y\}}I_A$.

$\mathcal{F}_7(n) = (F_7(n), (1,1,n-3))$, $S = \{2,3\}$. Conditions: $T(B)$ is not a 3-cycle (i.e. $T(B)$ is not isomorphic to T_3) and $|B| \geq 3$. Paths: since $T(B)$ is not the 3-cycle, there is a path $Q = -(1, n-5)$ in $T(B)$ (this is clear if $T(B)$ is not strongly connected; if $T(B)$ is strongly connected with at least 4 vertices, one can find a vertex $w \in B$ such that $T(B)\setminus\{w\}$ is strongly connected: such a vertex w is certainly an origin of Q), we then have $P = 1Q32$. For any $y \in B$, $P = y231O_{B\setminus\{y\}}$.

$\mathcal{F}_8 = (F_8(n), (n-4,1,1,1))$; $S = A$. Conditions: T_3 is the 3-cycle, its set of vertices is $\{1,2,3\}$, we furthermore need $|A| \geq 2$. Paths: $P = 1O_{A\setminus\{u\}}2u3$, $P = 2O_{A\setminus\{u\}}1u3$ and $P = 3O_{A\setminus\{u\}}2u1$ for a given $u \in A$.

$\mathcal{F}'_8(n) = (F_8(n), (n-4,2,1))$; $S = A$. Conditions: T_3 is the 3-cycle, its set of vertices is $\{1,2,3\}$, we furthermore need $|A| \geq 2$. Paths: $P = 1O_{A\setminus\{u\}}32u$, $P = 2O_{A\setminus\{u\}}13u$ and $P = 3O_{A\setminus\{u\}}21u$ for a given $u \in A$.

$\mathcal{F}_9(n) = (F_9(n), (n-6,1,1,1,1,1))$; $S = A$. Conditions: T_5 is the 2-regular tournament, its set of vertices is $\{1,2,3,4,5\}$, we furthermore need $|A| \geq 2$. Paths: $P = 1O_{A\setminus\{u\}}2u453$, $P = 2O_{A\setminus\{u\}}3u514$, $P = 3O_{A\setminus\{u\}}4u125$, $P = 4O_{A\setminus\{u\}}5u231$ and $P = 5O_{A\setminus\{u\}}1u342$ for a given $u \in A$.

$\mathcal{F}'_9(n) = (F_9(n), (n-6,2,1,1,1))$; $S = A$. Conditions: T_5 is the 2-regular tournament, its set of vertices is $\{1,2,3,4,5\}$, we furthermore need $|A| \geq 2$. Paths: $P = 1O_{A\setminus\{u\}}32u45$, $P = 2O_{A\setminus\{u\}}43u51$, $P = 3O_{A\setminus\{u\}}54u12$, $P = 4O_{A\setminus\{u\}}15u23$ and $P = 5O_{A\setminus\{u\}}21u34$ for a given $u \in A$.

$\mathcal{F}_{10}(n) = (F_{10}(n), (n-8,1,1,1,1,1,1,1))$; $S = A$. Conditions: T_7 is the Paley tournament on 7 vertices, its set of vertices is $\{1,2,3,4,5,6,7\}$, we furthermore need $|A| \geq 2$. Paths: $P = 1O_{A\setminus\{u\}}2u45376$, $P = 2O_{A\setminus\{u\}}3u56417$, $P = 3O_{A\setminus\{u\}}4u67521$, $P = 4O_{A\setminus\{u\}}5u71632$, $P = 5O_{A\setminus\{u\}}6u12743$, $P = 6O_{A\setminus\{u\}}7u23154$ and $P = 7O_{A\setminus\{u\}}1u34265$ for a given $u \in A$.

$\mathcal{F}'_{10}(n) = ((F_{10}(n), (n-8,2,1,1,1,1,1))$; $S = A$. Conditions: T_7 is the Paley tournament on 7 vertices, its set of vertices is $\{1,2,3,4,5,6,7\}$, we furthermore need $|A| \geq 2$. Paths: $P = 1O_{A\setminus\{u\}}32u4657$, $P = 2O_{A\setminus\{u\}}43u5761$, $P = 3O_{A\setminus\{u\}}54u6172$, $P = 4O_{A\setminus\{u\}}65u7213$, $P = 5O_{A\setminus\{u\}}76u1324$, $P = 6O_{A\setminus\{u\}}17u2435$ and $P = 7O_{A\setminus\{u\}}21u3546$ for a given $u \in A$.

$\mathcal{F}_{11}(n) = (F_{11}(n), (1,1,n-3))$; $S = \{1,2\}$. Conditions: $|A| \geq 2$. Paths: $P = 31O_A2$ and for all $u \in A$, $P = u1O_{A\setminus\{u\}}23$.

$\mathcal{F}_{12}(n) = (F_{12}(n), (2,1,n-4))$; $S = \{1,4\}$. Conditions: $|A| \geq 2$. Paths: $P = 231O_A4$ and for all $u \in A$, $P = u41O_{A\setminus\{u\}}23$.

$\mathcal{F}_{13}(n) = (F_{13}(n), (1,1,n-3))$; $S = \{1,2\}$. Conditions: $|A| \geq 2$. Paths: $P = 3425O_A1$, $P = 4523O_A1$, $P = 5324O_A1$ and for all u, v in A, $P = u1v2345O_{A\setminus\{u,v\}}$.

$\mathcal{F}_{14}(n) = (F_{14}(n), (2,1,n-4))$; $S = \{1,6\}$. Conditions: $|A| \geq 2$. Paths: for every vertex u of A, $P = 24u53O_{A\setminus\{u\}}61$, $P = 3u16245O_{A\setminus\{u\}}$, $P = 4u16253O_{A\setminus\{u\}}$, $P = 5u16234O_{A\setminus\{u\}}$ and $P = u61345O_{A\setminus\{u\}}2$.

In view of Theorem 1, we say that two vertices x and y are *potential origins* of P in T if $s^+_T(x,y) \geq b_1(P)+1$ when P is an outpath and if $s^-_T(x,y) \geq b_1(P)+1$ when P is an inpath.

The basic idea of our algorithm is to construct step by step the path P in T. Suppose that P is an outpath and that x and y are potential origins of P in T. By Theorem 1, at least one of these two vertices is an origin of T. We determine next which of x and y is an origin of P by finding two outneighbours of this fixed vertex t which are potential origins of ${}^*P$ in $T-t$. We then reapply this process until the whole path is constructed.

This process remains valid whenever Theorem 1 is valid. Thus whenever our process leads to an exception, we must incorporate a special routine. We first describe an algorithm to find a path of order at most $n-1$ in a tournament of order n. As there is no exception to Theorem 1 in this case, we do not require any special subroutine.

We will often use this proposition (without specifically refering to it) for finding two potential origins of a path in a tournament. The proposition guarantees that whenever we execute **n-1path**(P', T', u, v)

or $\mathbf{path}(P', T', u, v)$ described below, the vertices x and y are potential origins of P' in T'.

Proposition 1 *Let X be a set of vertices of a tournament T and t an outgenerator of X. Then $s^+_T(X) = s^+_T(t)$. In particular, if t is an outgenerator of $N^+_T(x)$ then $s^+_T(x) = s^+_{T-x}(t) + 1$.*

2 Finding a path of order at most $n-1$ in a tournament of order n.

2.1 The algorithm.

Routine **n-1path**(T, P, x, y) belw corresponds to Theorem 1-1) and its dual : with input a tournament of order n, a path of order at most $n-1$ and two potential origins x and y of P in T, it returns an occurrence of P in T with origin x or y. It is divided into two dual routines **n-1path+**and **n-1path-** depending on whether P is an outpath or an inpath.

Because these two routines are duals we present just one of them here, **n-1path+**.

n-1path+(T, P, x, y)
Begin
If $x \leftarrow y$, do **n-1path+**(T, P, y, x).
Now $x \rightarrow y$.
If $|P| = 1$, return (x, y).

If $b_1(P) \geq 2$,
  If $d^+_T(x) \geq 2$.
    Find an outgenerator z of $N^+_T(x)$ and $t \in N^+_T(x)\backslash\{z\}$.
    Return (x, **n-1path+**$(T - x, ^* P, z, t)$).
  Else $d^+_T(x) = 1$.
    Find an outgenerator z $N^+_T(y)$,
    Return (y, **n-1path+**$(T - y, ^* P, z, x)$).
Note that **n-1path+**$(T - y, ^* P, z, x)$ *certainly returns a path with origin z because $x \leftarrow u$ for every $u \in V(T - y)$ and $^*P$ is an outpath.*

Else $b_1(P) = 1$.
  If $d^+_T(x) \geq 2$,
    Find an ingenerator z of $N^+_T(x)$ and $t \in N^+_T(x)\backslash\{z\}$.
    If $s^-_{T-x}(z) \geq b_2(P) + 1$,
      Return (x, **n-1path-**$(T - x, ^* P, z, t)$).
    Else, set $Y = T - (x, S^-_{T-x}(z))$.
      Find an ingenerator u of Y and $v \in Y - u$.
      Return (y, **n-1path-**$(T - y, ^* P, u, v)$).
  Else $d^+_T(x) = 1$.
    If $d^+_T(y) < 2$,
      Find an outgenerator x' and an ingenerator y' of $N^-_{T-x}(y)$.
      Return (x, y, **n-1path**$(T - (x, y), ^{**} P, x', y')$).
    Else $d^+_T(y) \geq 2$,
      Find an ingenerator z of $N^+_T(y)$ and $t \in N^+_T(y)\backslash\{z\}$.
      If $s^-_{T-y}(z) \geq b_2(P) + 1$,
        Return (y, **n-1path-**$(T - y, ^* P, z, t)$).
      Else $s^-_{T-y}(z) \leq b_2(P)$.

Set $X = T - (S^-_{T-y}(z), x, y)$.
Find an ingenerator v of X.
If $|X| \geq 2$, let $u \in X - v$.
Return $(x,\, y,\, \mathbf{n\text{-}1path}(T - (x,y),^{**} P, v, u))$.
Else find a directed hamiltonian inpath H of $T - (x, y, v)$.
Return $(x, y, v,^{*} H)$.
End

This algorithm readily yields an algorithm **smallpath** to find an occurrence of a path P of order at most $n-1$ in a tournament of order n. It suffices to find an outgenerator x of T and an ingenerator y of $T - x$ and to run $\mathbf{n\text{-}1path}(T, P, x, y)$.

2.2 Complexity.

The above algorithm necessitates subroutines that
1) find the out- and inneighborhood of a vertex;
2) find an ingenerator (or outgenerator) of a tournament;
3) test whether the insection (or outsection) is greater than a fixed integer k;
4) remove a vertex of a tournament.

1) Finding the outneighbourhood and inneighbourhood of a vertex x in a tournament of order n requires $n-1$ comparisons (we must compare x to all other vertices).

2) One can find an ingenerator of a tournament of order n by doing $n-1$ comparisons. Indeed, assume that we have an ingenerator y of the tournament $T - x$. We just need to compare x and y to know which of them is an ingenerator of T. If $x \leftarrow y$, it is x, otherwise it is y.

3) Testing if the insection induced by a set S in a tournament of order n has size greater than k requires $O(k.n)$ operations. Indeed, one can find an ingenerator x of S or of any k-subset of S if $|S| \geq k$ with k comparisons. And if $|S| < k$, it suffices to construct an in-arborescence with root x (that is a tree such that all induced path with origin x are inpaths) as big as possible and stop whenever the tree has order k. This can be achieved with a breadth-first or depth-first search algorithm in $O(kn)$ operations.

4) Removing a vertex from a tournament can be effected in $O(n)$ operations. It is for example equivalent to removing a row and a column from the adjacency matrix of the tournament.

In the above algorithm, one computes the outneighbourhood and inneighbourhood of at most two vertices (x and y) in each loop of **n-1path**. So we determine for out- and inneighborhoods at most $2n-2$ times. To do this, we need fewer than n^2 comparisons.
In each loop of **n-1path**, one have to find at most two in- or outgenerators. This necessitates less than $2n^2$ comparisons.
We test if an insection (or outsection) is bigger than the length of the second block of the path only when the first block of teh path is of length one. Thus the algorithm does this whenever it has to fix the last edge of a block. So, if $P = (b_1, b_2, \ldots, b_l)$, computing the size of the in- or outsection requires $O(b_2.n) + O(b_3.n) + \cdots + O(b_l.n) = O(n^2)$ operations.
Finally, the algorithm removes a vertex from a tournament $n-2$ times, and this takes $O(n^2)$ operations.

Thus, the above algorithm runs in $O(n^2)$ time.

Remark 1 *This algorithm is easy to implement. It has been implemented in Scheme. (See [2] Annexe A.)*

3 Finding a hamiltonian path in a tournament.

3.1 The algorithm.

The general algorithm is based on a routine **path**(T, P, x, y) that corresponds to Theorem 1 and its dual. This routine is based on the same principles as **n-1path**. It is also divided into two dual routines **path+** and **path-**. Because of this duality, we present only **path+** here. With input a tournament T of order n, an outpath P of order at most n such that $(T; P)$ is not an exception, and two potential origins x and y of P in T, **path+** returns an occurrence of P in T with origin x or y.

The idea is to find the vertex $t \in \{x, y\}$ and two outneighbours u, v of t which are potential origins of $^*P$ in $T - t$. The problem is that Theorem 1 guarantees the existence of an origin of $^*P$ of $\{u, v\}$, only if $(T - t, ^* P)$ is not an exception. So we must check whether $(T - t, ^* P)$ is an exception or not. If it is not, we execute a recursive call to **path**$(T - t, ^* P, u, v)$. Otherwise, we have to run certian subroutines : **extension1**, **extension2** and **tombexcepi**, $i \in \{1, 2, 3, 4\}$.

Moreover, it is sometimes difficult to find the vertex t and two of its outneighbours which are potential origins of $^*P$ in $T - t$, but easy to find P in T with origin x or y. In such cases, we apply one of the routines **special** which returns such an occurrence of P in T.

Since Theorem 1 2) is valid when $n \geq 9$, we need a routine **path9-10-11**(T, P, x, y), that gives the result whenever T and P have the same order $n \in \{9, 10, 11\}$. Indeed, this routine for 8, 9 and 10 is enough since the following routine **path** calls itself with tournaments having at most three fewer vertices.

path+(T, P, x, y)
Begin
If $x \leftarrow y$, do **path+**(T, P, y, x).
Now $x \rightarrow y$.
If T has order 9, 10 or 11 , return **path9-10-11**(T, P, y, x).

If $b_1(P) \geq 2$,
  If $d^+_T(x) \geq 2$,
    If $(T - x;^* P)$ is an exception, do **extension1**(T, P, x, y).
    Else find an outgenerator z of $N^+(x)$ and $t \in N^+(x) - z$.
      Return $(x$, **path+**$(T - x,^* P, z, t))$.
  Else $d^+_T(x) = 1$.
    If $(T - y;^* P)$ is an exception, do **tombexcep1**(T, P, x, y).
    Else find an outgenerator z of $N^+(y)$.
      Return $(y$, **path+**$(T - y,^* P, z, x))$.

Else $b_1(P) = 1$.
  Set $X = S^-_{T-x}(N^+(x))$.

  If $d^+_T(x) \geq 2$ and $|X| \geq b_2(P) + 1$,
    If $(T - x;^* P)$ is an exception, do **extension2**(T, P, x, y).
    Else find an ingenerator z of $N^+(x)$ and $t \in N^+(x) - z$.
      Return $(x$, **path-**$(T - x,^* P, z, t))$.

  If $d^+_T(x) \geq 2$ et $|X| \leq b_2(P)$,
    If $P = +(1, n - 2)$, $P = +(1, n - 3, 1)$, $P = +(1, n - 4, 2)$ or $P = +(1, n - 4, 1, 1)$, do **special1**.
    Else we have $|X| \leq b_2(P) \leq n - 5$.
      Set $Y = T - (x, X)$.
      Find an ingenerator z of Y and $t \in Y - z$.

Return $(y,\, \mathbf{path}(T-y,^{*} P, z, t))$.

If $d^+_T(x) = 1$ et $|X| \geq b_2(P)+1$,
If $d^+_T(y) = 0$,
Find an outgenerator z of $T-(x,y)$ and an ingenerator t of $T-(x,y)$.
Return $(x,\, y,\, \mathbf{path}(T-(x,y),^{**} P, z, t))$.
If $d^+_T(y) = 1$,
If $(T-(x,y);^{**} P)$ is an exception, do **tombexcep2**(T, P, x, y).
Else $(T-(x,y);^{**} P)$ is not an exception.
If $P=(1,n-2)$ or $P=(1,1,n-3)$ do **special2**.
Else $b_1(^{**}P)+1 < n-2 = |T-(x,y)|$.
Find an outgenerator u and an ingenerator v of $(T-(x,y)$.
If $u=z$, replace u by an outgenerator of $N^+_{T-(x,y)}(u)$.
If $v=z$, replace v par by an ingenerator of $N^-_{T-(x,y)}(v)$.
Return $(x,\, y,\, \mathbf{path}(T-(x,y),^{**} P, u, v))$.
Else $d^+_T(y) \geq 2$.
Set $Z = S^-_{T-y}(N^+(y))$ and $W = T-(Z,x,y)$.
If $|Z| \geq b_2(P)+1$,
If $(T-y;^{*} P)$ is an exception, do **tombexcep3**(T, P, x, y).
Else find an ingenerator z of $N^+(y)$ and $t \in N^+(y)-z$.
Return $(y,\, $**path-**$(T-y,^{*} P, z, t))$.
Else $|Z| \leq b_2(P)$.
If $(T-(x,y);^{**} P)$ is an exception, do **tombexcep4**(T, P, x, y).
Else,
If $|W| \geq 2$,
Find an ingenerator u of W and $v \in W-u$.
Return $(x,\, y,\, \mathbf{path}(T-(x,y),^{**} P, u, v))$.
If $|W|=1$, say $W=\{w\}$,
If y is dominated by a vertex v of Z,
Return $x,\, y,\, \mathbf{path}(T-(x,y),^{**} P, w, v))$.
Else $Z = N^+(y)$.
Find an ingenerator u and an outgenerator v of $T-(x,y,w)$.
Return $(x,\, y,\, z,\, \mathbf{path}(T-(x,y,w),^{***} P, u, v))$.
Else $W=\emptyset$, thus $P=+(1,n-2)$ and y is an ingenerator of $T-x$.
Return **special3**(T, x, y).

If $d^+_T(x) = 1$ and $|X| \leq b_2(P)$,
Set $Y = T-(x,X)$.
We have $|Y| \geq 2$ and $b_2(P) \geq n-2$ because $(T;P)$ is not an exception.
If $(T-y;^{*} P)$ is an exception, do **tombexcep5**(T, P, x, y).
Else find an ingenerator u of Y and $v \in Y-u$.
Return $(y,\, \mathbf{path}(T-y,^{*} P, u, v))$.

End

This routine **path**, together with a routine **pathexcep** that for every exception $(T;P)$ returns an occurrence of P in T, yields an algorithm to find a path P of order at most n in a tournament T of order n; in particular, it finds hamiltonian paths in tournaments. If $(T;P)$ is an exception, we run **pathexcep**, otherwise, we find an ingenerator x and an outgenerator y of T and run $\mathbf{path}(T, P, x, y)$.

3.2 Subroutines.

The above algorithms require the use of subroutines called by **n-1path**. Moreover, we have to recognize exceptions and to execute the subroutines **extension**, **tombexcep** and **special**. We present in this subsection all these routines. Note that we only present the routines required by **path+**. So we also need the dual of each of these in order to perform **path-**.

3.2.1 Recognizing exceptions.

To check if a pair $(T;P)$ is an exception, we first check if P is a path occuring in one of the exceptions. Since every such path has at most eight blocks, this can be checked in constant time. Next, if P corresponds to an exception $\mathcal{F}_i$, we have to check if T is a tournament F_i. For this task, it is sufficient to look at vertices with small (≤ 3) out- or indegree and then to look at the subtournament induced by these vertices. For example, to check if a tournament belongs to F_1, it suffices to check if it has three vertices of outdegree 1. To check if a tournament belongs to F_2, it suffices to check that it has exactly two vertices of outdegree 1, say x_1 and x_3, and exactly two vertices of outdegree 2, say x_2 and x_4, such that $x_1 \rightarrow x_2$ and $x_2 \rightarrow \{x_3, x_4\}$.

Now one can find all the vertices of outdegree at most k in a tournament of order n with at most $(2k+1)n$ comparisons using the following algorithm. This algorithm examines each vertex of $\{x_1, x_2, \ldots, x_n\}$ in turn and memorizes the list L of vertices which might have a outdegree at most k, that is the vertices whose outdegree in $T(x_1, \ldots, x_i)$ is at most k.

Begin
Initialize L to empty list.
For each vertex x do:
    Initialize the temporary outdegree of x to 0.
    For each vertex y of L do:
        If x dominates y then increment the temporary outdegree of x .
            Else increment the temporary outdegree of y.
    Put x in L.
    Remove all vertices of L with temporary outdegree greater than k.
End

In this algorithm, each vertex in turn is compared with all vertices of L. Since the vertices of L have outdegree at most k in the tournament induced by the vertices already visited, L contains at most $2k+1$ vertices. So this algorithm requires at most $(2k+1)n$ comparisons.

Hence, we can check whether $(T;P)$ is an exception in time $O(n)$.

3.2.2 Subroutines pathexcep.

The subroutine **pathexcep**(T, P, x) returns an occurrence of P in T with origin x whenever $(T;P)$ is an exception $\mathcal{F}_i$ and x is an origin of P in T. Such a path of origin x is given in definition 2. Therefore, it only requires to find directed hamiltonian paths of subtournaments. So it can be executed in time $O(n \log n)$ with the algorithm of Hell and Rosenfeld [4].

3.2.3 Subroutines extension.

For all the following procedures, we will use the notation of figure 1. So the tournaments A and B and the vertices 1, 2, ... are those depicted in figure 1.

The subroutine **extension1** finds the desired path P whenever $b_1(P) \geq 2$ and the vertex x has outdegree 2 but **path** cannot be used on $(T - x;^* P)$ because it is an exception.

extension1(T, P, x, y)
Begin
Set S the set of vertices of T which are not origins of ${}^*P$ in $T - x$.
If $N_T^+(x) \not\subset S$,
  Pick u a vertex of $N_T^+(x) \setminus S$ and v an outneighbour of x distinct from u.
  Return $(x,$ **path**$(T - x,^* P, u, v))$.
Else, since $(T; P)$ is not an exception,
$(T - x,^* P)$ is an exception $\mathcal{F}_i$ for $i \in \{1, 2, 4, 6, 7, 12, 14\}$.
  If $(T - x;^* P)$ is exception $\mathcal{F}_1$,
    Find a directed hamiltonian inpath I_A of $A = N_T^-(x)$.
    Free to rename vertices, one can assume $y = 1$, return $(1, 2, 3, x, I_A)$.
  If $(T - x;^* P)$ is exception $\mathcal{F}_2$,
    Find a directed hamiltonian inpath I_A of A.
    If $y = 3$, return $(3, 1, 2, 4, x, I_A)$; if $y = 4$ return $(4, 1, 2, 3, x, I_A)$.
  If $(T - x;^* P)$ is exception $\mathcal{F}_4$,
    Find a directed hamiltonian inpath I_{A-3} of $A - 3$.
    If $y = 1$, return $(1, 3, x, 4, 2, I_{A-3})$; if $y = 4$, return $(4, 3, x, 1, 2, I_{A-3})$.
  If $(T - x;^* P)$ is exception $\mathcal{F}_6$,
    Find a directed hamiltonian inpath I_B of B
    Find a directed hamiltonian inpath I_{A-2} of $A - 2$.
    If $y = 1$, return $(1, 2, x, 3, I_B, I_{A-2})$; if $y = 3$, return $(3, 2, x, 1, I_B, I_{A-2})$.
  If $(T - x;^* P)$ is exception $\mathcal{F}_7$,
    Find O_B of B.
    If $y = 2$, return $(2, 1, x, O_B, 3)$; if $y = 3$ return $(3, 1, x, O_B, 2)$.
  If $(T - x;^* P)$ is exception $\mathcal{F}_{12}$,
    Find a directed hamiltonian outpath O_A of A.
    If $y = 1$, return $(1, 2, x, 4, 3, O_A)$; if $y = 4$, return $(4, 2, x, 1, 3, O_A)$.
  If $(T - x;^* P)$ is exception $\mathcal{F}_{14}$,
    Find a directed hamiltonian outpath O_A of A.
    If $y = 1$, return $(1, 2, x, 6, 3, 4, 5, O_A)$; if $y = 6$, return $(6, 2, x, 1, 3, 4, 5, O_A)$.
End

extension2(T, P, x, y)
Begin
Let S be the set of vertices of T which are not origins of ${}^*P$ in $T - x$.
If $N_T^+(x) \not\subset S$,
  Pick a vertex u of $N_T^+(x) \setminus S$ and v an outneighbour of x distinct from u.
  Return $(x,$ **path**$(T - x,^* P, u, v))$.
Else,
  If $(T - x;^* P)$ is exception $\mathcal{F}_1$ dual,
    If $N^+(x) = \{1, 2, 3\}$, then, by renaming if necessary, $y = 1$.
      Find a directed hamiltonian outpath O_A of A.
      Return $(1, 3, x, 2, O_A)$.
    Else $N^+(x) = \{1, 2\}$ (by renaming if necessary).
      Pick a vertex u of A.
      Find a directed hamiltonian outpath O_{A-u} of $A - u$.
      If $y = 1$, return $(1, u, 3, x, 2, O_{A-u})$; if $y = 2$, return $(2, u, 3, x, 1, O_{A-u})$.
  If $(T - x;^* P)$ is exception $\mathcal{F}_2$ dual,
    Find a directed hamiltonian outpath O_A of A.
    If $y = 3$, return $(3, 2, 4, 1, O_A, x)$; if $y = 4$, return $(4, 2, 3, 1, O_A, x)$.

If $(T-x;^* P)$ is exception $\mathcal{F}_3$ dual,
  If $y=1$, find a directed hamiltonian outpath $(u,3,R_1)$ of A.
    Return $(1,u,2,x,3,R_1)$.
  If $y=3$, find a directed hamiltonian outpath $(3,v,R_2)$ of A.
    Return $(3,v,1,2,R_2,x)$.
If $(T-x;^* P)$ is exception $\mathcal{F}_4$ dual,
  Find a directed hamiltonian outpath $(3,v,R_2)$ of A.
  If $y=1$, return $(1,v,3,2,x,4,R_2)$; if $y=4$, return $(4,v,3,2,x,1,R_2)$.
If $(T-x;^* P)$ is exception $\mathcal{F}_5$ dual,
  If $y=1$, find a directed hamiltonian outpath of A, $(2,Q,v)$.
    Find a directed hamiltonian outpath O_B of B.
    Return $(1,v,O_B,x,2,Q)$.
  If $y=2$, find a directed hamiltonian outpath O_B of B.
    Find a vertex u of A that is dominated by 2.
    Find a directed hamiltonian outpath O of $A-(u,2)$.
    Return $(2,u,O_B,x,1,O)$.
If $(T-x;^* P)$ is exception $\mathcal{F}_6$ dual,
  If $|A|\geq 2$, find a vertex u of A that is dominated by 2.
    Find a directed hamiltonian outpath O_B of B.
    Find a directed hamiltonian outpath O of $A-(u,2)$.
    If $y=1$, return $(1,u,2,O_B,x,3,O)$; if $y=3$, return $(3,u,2,O_B,x,1,O)$.
  If $|A|=2$, alors $|Y|\geq 3$.
    Pick three vertices u, v and w of B such that $u\leftarrow v\leftarrow w$.
    Find a directed hamiltonian outpath O of $B-(u,v,w)$.
    If $y=1$, return $(1,u,v,w,2,x,3,O)$; if $y=3$, return $(3,u,v,w2,x,1,O)$.
If $(T-x;^* P)$ is exception $\mathcal{F}_7$ dual,
  Pick two vertices u and v of B such that $u\leftarrow v$.
  Find a directed hamiltonian inpath $I_{B-(u,v)}$ of $B-(u,v)$.
  If $y=2$, return $(2,u,v,x,I_{B-(u,v)},3,1)$; if $y=3$, return $(3,u,v,x,I_{B-(u,v)},2,1)$.
If $(T-x;^* P)$ is exception $\mathcal{F}_8$ dual,
  Find a directed hamiltonian inpath I_{A-y} of $A-y$.
  Return $(y,1,I_{A-y},2,3,x)$.
If $(T-x;^* P)$ is exception $\mathcal{F}'_8$ dual,
  Find a directed hamiltonian inpath I_{A-y} of $A-y$.
  Return $(y,1,I_{A-y},2,x,3)$.
If $(T-x;^* P)$ is exception $\mathcal{F}_9$ dual,
  Find a directed hamiltonian inpath I_{A-y} of $A-y$.
  Return $(y,1,I_{A-y},2,4,3,5,x)$.
If $(T-x;^* P)$ is exception $\mathcal{F}'_9$ dual,
  Find a directed hamiltonian inpath I_{A-y} of $A-y$.
  Return $(y,1,I_{A-y},2,x,4,3,5)$.
If $(T-x;^* P)$ is exception $\mathcal{F}_{10}$ dual,
  Find a directed hamiltonian inpath I_{A-y} of $A-y$.
  Return $(y,1,I_{A-y},2,4,3,7,5,6,x)$.
If $(T-x;^* P)$ is exception $\mathcal{F}'_{10}$ dual,
  Find a directed hamiltonian inpath I_{A-y} of $A-y$.
  Return $(y,1,I_{A-y},2,x,4,3,7,5,6)$.
If $(T-x;^* P)$ is exception $\mathcal{F}_{11}$ dual,
  Pick two vertices u and v of A such that $u\leftarrow v$.

Find a directed hamiltonian inpath $I_{A-(u,v)}$ of $A-(u,v)$.
If $y=1$, return $(1,3,v,u,2,x,I_{A-(u,v)})$; if $y=2$, return $(2,u,v,x,3,I_{A-(u,v)},1)$.
If $(T-x;^* P)$ is exception $\mathcal{F}_{12}$ dual,
Pick a vertex u of A.
Find a directed hamiltonian inpath I_{A-u} of $A-u$.
If $y=1$, return $(1,u,2,3,x,I_{A-u},4)$; if $y=4$, return $(4,u,2,3,x,I_{A-u},1)$.
If $(T-x;^* P)$ is exception $\mathcal{F}_{13}$ dual,
If $y=1$, find a directed hamiltonian inpath I_A of A.
Return $(1,3,4,2,x,5,I_A)$.
If $y=2$, pick two vertices u and v of A such that $u \leftarrow v$.
Find a directed hamiltonian inpath $I_{A-(u,v)}$ of $A-(u,v)$.
Return $(2,u,v,x,3,4,5,I_{A-(u,v)},1)$.
If $(T-x;^* P)$ is exception $\mathcal{F}_{14}$ dual,
Find a directed hamiltonian inpath I_A of A.
If $y=1$, return $(1,3,4,5,x,I_X,6,2)$; if $y=6$, return $(6,3,4,5,x,I_A,1,2)$.
End

3.2.4 Subroutines tombexcep.

We run **tombexcep1**(T,P,x,y) when $b_1(P) \geq 2$, $N^+(x)=y$, $(T-y;^* P)$ is an exception but $(T;P)$ is not any. Thus there exists a vertex z of $N_T^+(y) \setminus S$ with S the set of vertices which are not origins of $^*P$ in $T-y$. So we find such a z and return $(y, \textbf{path}(T-y,^* P,z,x))$. Note that $\textbf{path}(T-y,^* P,z,x)$ will certainly return a path with origin z since x is dominated by every vertex of $T-(x,y)$ and therefore cannot be an origin of $^*P$ in $T-y$.

We run **tombexcep2**(T,P,x,y) when $b_1(P)=1$, $N^+(x)=y$, $d^+(y)=1$, $|X| \geq b_2(P)+1$ and $(T-(x,y);^{**} P)$ is an exception. Since there at least two origins of $^{**}P$ in $T-(x,y)$, there exists a vertex u of $T-(x,y)$ which dominates y and which is an origin of $^{**}P$. We just have to find this vertex u and to return $(x,\ y,\ \textbf{path}(T-(x,y),^{**} P,u,v))$, with v any vertex of $T-(x,y,z)$ distinct from u.

We execute **tombexcep3** when $b_1(P)=1$, $N^+(x)=y$, $d^+(y) \geq 2$, $|X| \geq b_2(P)+1$, $|Z| \geq b_2(P)+1$ and $(T-y;^* P)$ is an exception. Therefore, it is easy to check that $(T-y;^* P)$ is the dual of one of the exceptions $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_4$, $\mathcal{F}_5$ or $\mathcal{F}_6$. In the following routine, the vertices denoted 1, 2, ..., and the sets A and B, represent the vertices and sets depicted as such in figure 1.

tombexcep3(T,P,x,y)
Begin
If $(T-y;^* P)$ is exception $\mathcal{F}_1$ dual,
If y has an inneighbour $u \in A$,
Find O_{A-u} a directed hamiltonian outpath of $A-u$.
Return $(y,u,1,3,2,O_{A-u})$.
If y has an inneighbour $u \notin A$, say 1 (by renaming if necessary),
Find O_{A-x} a directed hamiltonian outpath of $A-x$.
Return $(x,y,1,3,2,O_{A-x})$.
If $(T-y;^* P)$ is exception $\mathcal{F}_2$ dual,
If y has an outneighbour $u \notin \{3,4\}$,
Return $(y,\ \textbf{pathexcep}(T-y,^* P,u))$.
Else pick $u \in A-x$. Find a directed hamiltonian outpath $O_{A-(x,u)}$of $A-(x,u)$.
Return $(x,y,u,2,1,3,4,O_{A-(x,u)})$.
If $(T-y;^* P)$ is exception $\mathcal{F}_3$ dual,
If y has an outneighbour $u \notin \{1,3\}$,
Return $(y,\ \textbf{pathexcep}(T-y,^* P,u))$.

Else find a directed hamiltonian outpath $O_{A-(x,3)}$ of $A-(x,3)$.
Return $(x, y, 2, 3, 1, O_{A-(x,3)})$.
If $(T-y;^* P)$ is exception $\mathcal{F}_4$ dual,
If y has an outneighbour $u \notin \{1,4\}$,
Return $(y$, **pathexcep**$(T-y,^* P, u))$.
Else pick $u \in A-(x,3)$.
Find $O_{A-(x,u,3)}$ a directed hamiltonian outpath of $A-(x,u,3)$.
Return $(x, y, u, 2, 3, 4, 1, O_{A-(x,u,3)})$.
If $(T-y;^* P)$ is exception $\mathcal{F}_5$ dual,
If y has an outneighbour $u \notin \{1,2\}$,
Return $(y$, **pathexcep**$(T-y,^* P, u))$.
Else find $O_{A-(x,2)}$ and O_B directed hamiltonian outpath of $A-(x,2)$ and B.
Return $(x, y, O_B, 2, 1, O_{A-(x,2)})$.
If $(T-y;^* P)$ is exception $\mathcal{F}_6$ dual,
If y has an outneighbour $u \notin \{1,3\}$,
Return $(y$, **pathexcep**$(T-y,^* P, u))$.
Else pick an arc $u \to v$ of B.
Find $O_{A-(x,2)}$ and $O_{B-(u,v)}$ directed hamiltonian outpaths of $A-(x,2)$ and $B-(u,v)$.
Return $(x, y, u, v, 2, 1, 3, O_{B-(u,v)}, O_{A-(x,2)})$.
End

We run **tombexcep4**(T, P, x, y) when $b_1(P)=1$, $N^+(x)=y$, $d^+(y)\geq 2$, $|X|\geq b_2(P)+1$, $|Z|\leq b_2(P)$, $(T-(x,y);^{**} P)$ is an exception, $T-(x,y)$ is reducible with decomposition $W \leftarrow Z$ and $^{**}P$ is an inpath. We find a vertex $u \in W$ which is an origin of $^{**}P$ in $T-(x,y)$ (it is easy to check that such a vertex exists) and we return $(x$, y, **pathexcep**$(T-(x,y),^{**} P, u))$.

We run **tombexcep5**(T, P, x, y) when $b_1(P)=1$, $N^+(x)=y$, $|X|\leq b_2(P)$, $(T-y;^* P)$ and $y \to (T-(x,y)$. Hence, it suffices to pick a vertex $u \in T-(x,y)$, an origin of $^*P$ in $T-y$ and return $(y$, **pathexcep**$(T-y,^* P, u))$.

3.2.5 Subroutines special.

We execute **special1**, when $d^+(x)\geq 2$, $|X|\leq b_2(P)$ and $(T;P)$ is not an exception and P is one of the following paths, $+(1,n-2)$, $+(1,n-3,1)$, $+(1,n-4,2)$ and $+(1,n-4,1,1)$.

special1(T, P, x, y)
Begin
If $P=+(1,n-2)$,
Pick a vertex z of $T-x$.
Find $I(z)$ a directed hamiltonian inpath of $T-y$ with origin z.
Return $(y, I(z))$.
If $P=(1,n-3,1)$,
Set $A^+ = X \cap (N^+(x)\setminus\{y\})$ and $A^- = X \cap N^-(x)$.
If A^- is not empty, pick $v \in Y$.
Find I_{Y-y}, I_{A^+} and I_{A^-} hamiltonian directed inpaths of $Y-y$, A^+ and A^-.
Return $(y, I_{Y-v}, I_{A^+}, x, I_{A^-}, v)$.
Else A^- is empty.
If $|A^+|\geq 2$, pick $z \in A^+$.
Find I_Y and $I_{X-(y,z)}$ hamiltonian directed inpaths of Y and $X-(y,z)$.
Return $(y, I_Y, I_{X-(y,z)}, x, z)$.
Else A^+ is the arc $z \to y$.

Let P'= **smallpath**$(Y,^{****} P)$ and v the vertex of $Y - P'$.
Return (y, v, z, x, P').
If $P = +(1, n-4, 2)$
Pick a vertex v of Y.
Find I_{Y-v} and I_{X-y} directed hamiltonian inpath of $Y - v$ and $X - y$.
Return $(y, I_{Y-v}, I_{X-y}, v, x)$.
If $P = +(1, n-4, 1, 1)$,
Set $A^+ = X \cap (N^+(x) \setminus \{y\})$ and $A^- = X \cap N^-(x)$.
If A^- is not empty,
Pick two vertices v and w of Y such that $v \to w$.
Find $I_{Y-(v,w)}$, I_{A^+} and I_{A^-} hamiltonian directed inpaths of $Y - (v, w)$, A^+ and A^-.
Return $(y, I_{Y-(v,w)}, I_{A^+}, x, I_{A^-}, w, v)$.
Else A^- is empty.
If $|Y| \geq 4$,
Set $P' = -(|Y| - 4, 1, 1)$ (or $+(1, 1)$ if $|Y| = 4$).
Let Q:=**smallpath**(P', Y).
Pick a vertex v of $Y - Q$ and find I_{X-y} a directed hamiltonian inpath of $X - y$.
Return (y, v, I_{X-y}, x, Q).
Else $V(Y) = \{u, v, w\}$ with $u \to v$.
Pick $z \in X - y$ and find $I_{X-(y,z)}$ a directed hamiltonian inpath of $X - (y, z)$.
Return $(y, w, I_{X-(y,z)}, x, u, v, z)$.
End

We run the routine **special2**, when $b_1(P) = 1$, $d^+(x) = d^+(y) = 1$, $|X| \geq b_2(P) + 1$, $(T; P)$ is not an exception and P is $+(1, n-2)$ or $+(1, 1, n-3)$.

special2(T, x, y)
Begin
Let z be the outneighbour of y.
If $P = +(1, n-2)$,
Pick an outneighbour w of z distinct from x.
Grow (w, z) into a directed hamiltonian inpath I of $T - (x, Y)$
This is possible applying the algorithm of Hell and Rosenfeld and guarantee that z is not the origin of I.
Return (x, y, I).
If $P = +(1, 1, n-3)$,
Pick an inneighbour w of z distinct from y.
Grow (w, z) into a directed hamiltonian outpath O of $T - (x, Y)$
Return (x, y, O).
End

We execute routine **special3**, when $P = +(1, n-2)$ and y is an ingenerator of $T - x$. We just have to find an directed hamiltonian inpath I of $T - x$ with origin y and return (x, I).

3.3 Complexity of the algorithm.

The routine **hamilt** run the same elementary subroutines as **smallpath**; these take time $O(n^2)$. Moreover,for each recursive call of **path**, it must check whether (T, P) is an exception. Since there are at most n calls, (because the size of the path decreases at each call) and one can recognize exceptions in time $O(n)$, the algorithm spends in all $O(n^2)$ time on the recognition of exceptions.

The subroutines **extension** and **tombexcep** either perform $O(n)$ operations before a recursive call to **path** (with a smaller path), or else find a directed hamiltonian path of a prescribed subtournament and return the path P in T. Using the algorithm of Hell and Rosenfeld [4], this can be achieved in $O(n \log n)$.

The subroutines **special1** and **special2** for $1 \leq i \leq 6$, find a directed hamiltonian path of prescribed subtournaments and return P. This can be achieved in $O(n \log n)$.

In order to achieve subroutines **special3** and **extension2** (only when $(T - x;^* P) = \mathcal{F}_4$), we must find a directed hamiltonian inpath of a tournament S with prescribed origin v. This can be done in time $O(n^2)$ in the following way : First find the strong component C of v in S. This can be done in time $O(n^2)$ by a modified depth-first search algorithm. Then find a directed hamiltonian cycle of C and pick the directed hamiltonian inpath I with origin v contained in C. One can find a directed hamiltonian cycle in $O(n^2)$, as shown by Manoussakis [5]. Finally find directed a hamiltonian inpath I' of $T - C$ and return (I, I').

Thus, **path** and **hamilt** run in time $O(n^2)$.

Remark 2 This algorithm to find a hamiltonian path in a tournament is very difficult to implement because of the exceptions and the special subroutines we need accordingly. But the algorithm can be much simplified and shortened if we restrict ourselves to paths of a given class. For example, if we wish to find paths with blocks of length less than k. Since there are no exceptions of order $k + 8$ in wich all blocks are of length less than than k, we do not need to check whether $(T; P)$ is an exception. Moreover, since we run **special** routines for paths with long blocks, we do not need these either.

Remark 3 In the same way as described above, one can construct an algorithm for finding hamiltonian oriented paths (if it exists) in tournaments of every order. Indeed Havet and Thomassé [3] described all the 52 exceptions of order at most 8 of Theorem 1-2). So taking them into account the subroutines **pathexcep**, **tombexcep** and **special** yields an algorithm valid even for small tournaments. Moreover, we do need the subroutine **path9-10-11** anymore.

4 Application : path with given origin.

Using **path**, it is easy, following the proofs of Corollary 2.2 and 2.3 of [3] to construct algorithms for finding a path P with origin a prescribed vertex x in a 3-strong tournament and for finding a path P or its dual $-P$ with origin a prescribed vertex x in a 2-strong tournament.

We will now present a more general algorithm that finds a path of order at most n starting from a given vertex in a tournament of order n, provided that w such a path exists (and returns an error message otherwise). This algorithm is divided in two dual routines according to whether if P is an outpath or not. We present here the routine need when P is an outpath.

origin+(T, P, x)
Begin
If $s^+(x) \leq b_1(P)$, write "x is not an origin of P in T".
If $(T - x;^* P)$ is an exception,
  If there is an outneighbour y of x that is an origin of $^*P$ in T, return (x,**pathexcep**$(T - x,^* P, y)$).
  Else write "x is not an origin of P in T".
Else $(T - x;^* P)$ is not an exception,
  If x has a unique neighbour y, return (x,**origin**$(T - x,^* P, y)$).
  Else $d^+(x) \geq 2$.
    If $b_1(P) \geq 2$, find a outgenerator u of $N^+(x)$, an outneighbour v of x distinct from u.
      Return (x,**path+**$(T - x,^* P, u, v)$.
    Else $b_1(P) = 1$.
      If $s^-_{T-x}(N^+(x)) \geq b_2(P) + 1$, find a ingenerator u of $N^+(x)$, an outneighbour v of x distinct from u.

Return $(x,$**path-**$(T-x,^* P,u,v)$.
Else write "x is not an origin of P in T".
End

This algorithm clearly runs in time $O(n^2)$.

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