Trees with three leaves are $(n+1)$-unavoidable.

S. Ceroi and F. Havet

April 19, 2001

Abstract

We prove that every tree of order $n \geq 5$ with three leaves is $(n+1)$-unavoidable. More precisely, we prove that every tree A with three leaves of order n is contained in every tournament T of order $n+1$ except if $(T;A)$ is $(R_5;S_3^+)$ or its dual, where R_5 is the regular tournament on five vertices and S_3^+ is the *outstar* of degree three, that is, the tree consisting of a root dominating three leaves. We then deduce that Sumner's conjecture is true for trees with four leaves, i.e. every tree of order n with four leaves is $(2n-2)$-unavoidable.

1 Introduction.

A *tournament* is an orientation of a complete graph. An *oriented tree* is an orientation of a tree; in particular, an *oriented path* is an orientation of a path. Throughout this paper, since we only consider oriented trees and oriented paths, we abbreviate 'oriented tree' to 'tree' and 'oriented path' to 'path'. A digraph is said to be n-*unavoidable* if every tournament of order n contains it as a subgraph. Let $f(n)$ be the smallest integer such that every (oriented) tree of order n is $f(n)$-unavoidable. Sumner (see [8]) noted that $f(n) \geq 2n-2$. Indeed, consider the *outstar* (resp. the *instar*) of degree $n-1$, S_{n-1}^+ (resp. S_{n-1}^-), that is the oriented tree consisting of a root dominating (resp. dominated by) $n-1$ leaves. The regular tournaments of order $2n-3$ have no vertex with outdegree or indegree greater than $n-2$ and thus do not contain S_{n-1}^+ or S_{n-1}^-. Sumner also conjectured that equality holds:

Conjecture 1 (Sumner) *Every tree of order n is $(2n-2)$-unavoidable.*

Häggkvist and Thomason [1] proved that $f(n) \leq 12n$ and also obtained the asymptotic bound $f(n) \leq (4+o(1))n$. Havet and Thomassé [5] improved both of these bounds by showing that $f(n) \leq 3.5n$. They also proved Sumner's conjecture for arborescences.

Denote by $g(k)$ the smallest integer such that every tree of order n with k leaves is $(n+g(k))$-unavoidable. Häggkvist and Thomason [1] proved that $g(k) \leq 2^{512k^3}$. The outstar S_k^+ (and instar S_k^-) in regular tournaments of order $2k-3$ show that $g(k) \geq k-1$. Havet and Thomassé (see [2]) conjectured that equality holds :

Conjecture 2 (Havet and Thomassé) *Every tree of order n with k leaves is $(n+k-1)$-unavoidable, i.e. $g(k) = k-1$.*

This conjecture implies Sumner's because a tree of order n has at most $n-1$ leaves. Since trees with two leaves are paths, a result of Thomason [9] confirms this conjecture for $k=2$. Havet proved that $g(3) \leq 5$ [2] and also verified the conjecture for a large class of trees [3].

If true, the bound $k-1$ of Conjecture 2 is minimum because of the outstar S_k^+ (and instar S_k^-) in regular tournaments of order $2k-1$. However, we conjecture that for n sufficiently large compared to k, every tree of order n with k leaves is $(n+k-2)$-unavoidable. It has been established for paths (trees with two leaves) by Havet and Thomassé [4], who proved that every path of order $n \geq 8$ is n-unavoidable. For $k \geq 3$, we have the following conjecture:

Conjecture 3 *Let T be a tournament of order $n+k-2$ and A a tree of order n with $k \geq 3$. If T is not a regular tournament or A is not an outstar or an instar then T contains C.*

The bound $k-2$ of Conjecture 3 is tight even when n goes to infinity. Indeed, consider a tournament T consisting of $n-2k+4$ vertices dominating a regular tournament on $2k-3$ vertices, and a tree A of order n that is the merging of one directed inpath of length $n-k$ and $k-1$ outpaths of length one. Clearly, T does not contain A.

The main result of this paper is to show that this conjecture holds for $k=3$:

Theorem 1 *Let T be a tournament of order $n+1$ and A a tree of order n with three leaves. If $(T;A) \notin \{(R_5;S_3^+),(R_5;S_3^-)\}$ then T contains C.*

Therefore, Conjecture 2 is also true for $k=3$, that is $g(3)=2$. We then deduce that Conjecture 1 holds for trees with four leaves.

2 Definitions

Let T be a tournament. Let x and y be two vertices of T, we write $x \to y$ (or $y \leftarrow x$) if (x,y) is an arc of T. In the same way, let X and Y be two subdigraphs of T. We write $X \to Y$ if $x \to y$ for all pairs $(x,y) \in V(X) \times V(Y)$. A vertex y is an *outneighbour* (resp. *inneighbour*) of a vertex x if $x \to y$ (resp. $x \leftarrow y$). The *outneighbourhood* (resp. *inneighbourhood*) of a vertex x in T, denoted by $N_T^+(x)$ (resp. $N_T^-(x)$) is its set of outneighbours (resp. inneighbours). Its cardinality, called the *outdegree* (resp. *indegree*) of x in T is denoted by $d_T^+(x)$ (resp. $d_T^-(x)$). The minimum outdegree (resp. indegree) over all the vertices of a tournament T is denoted by $\delta^+(T)$ (resp. $\delta^-(T)$).

Let $A_1, A_2, \ldots, A_k$ be a family of sets of vertices of T. We denote by $T[A_1, A_2, \ldots, A_k]$ the subtournament induced by T on the set of vertices $\bigcup_{1 \leq i \leq k} A_i$. and by $T-[A_1, A_2, \ldots, A_k]$

the subtournament induced by T on the set of vertices $V(T) \setminus \bigcup_{1 \leq i \leq k} A_i$. We often abbreviate $T - [A_1]$ to $T - A_1$ and $T - [\{x\}]$ to $T - x$.

Let $P = (x_1, \cdots, x_n)$ be a path. We say that x_1 is the *origin* of P and x_n is the *terminus* of P. If $x_1 \to x_2$, P is an *outpath*, otherwise P is an *inpath*. The *directed outpath* of order n is the path $P = (x_1, \cdots, x_n)$ in which $x_i \to x_{i+1}$ for all i, $1 \leq i < n$; the dual notion is that of a *directed inpath*. The *length* of a path is its number of arcs. We denote the path $(x_2, \cdots, x_n)$ by $^*P$.

The *blocks* of P are the maximal directed subpaths of P. We number the blocks of P from the origin to the terminus. The first block of P is denoted $B_1(P)$ and its length $b_1(P)$. Likewise, the i^{th} block of P is denoted $B_i(P)$ and its length $b_i(P)$. The path P is totally described by the signed sequence $sgn(P)(b_1(P), b_2(P), \cdots, b_k(P))$ where k is the number of blocks of P and $sgn(P) = +$ if P is an outpath and $sgn(P) = -$ if P is an inpath.

A tournament is *strong* (or *strongly connected*) if for any two vertices x and y there exists a directed outpath with origin x and terminus y. A nonstrong tournament is said to be *reducible*. The *strong components* of a tournament are its maximal strong subtournaments.

A tournament T is *k-strong*, if $T - Y$ is strong for any set Y of $k - 1$ vertices. A tournament is *$(=k)$-strong* or *exactly k-strong*, if it is k-strong and not $(k+1)$-strong. A *k-cut* of a tournament is a set of k vertices X such that $T - X$ is reducible.

Let X be a set of vertices of T. The *outsection* generated by X in T is the set of vertices y to which there exists a directed outpath (possibly restricted to a single vertex) from $x \in X$; we denote this set by $S_T^+(X)$ (or by $S_T^+(x)$ if $X = \{x\}$ and $S_T^+(x, y)$ if $X = \{x, y\}$). Note that $X \subseteq S_T^+(X)$. The dual notion, the *insection*, is denoted by $S_T^-(X)$. We also write $s_T^+(X)$ (resp. $s_T^-(X)$) for the number of vertices of $S_T^+(X)$ (resp. $S_T^-(X)$). An *outgenerator* of T is a vertex x of T such that $S_T^+(x) = V(T)$; the dual notion is an *ingenerator*. Note that since every tournament contains a hamiltonian directed path it has an outgenerator (and also an ingenerator). The set of outgenerators (resp. ingenerators) in T is denoted by $Out(T)$ (resp. $In(T)$) and its cardinal by $out(T)$ (resp. $in(T)$). When the tournament T is clearly understood, we often omit the subscript T in the above notations.

Let $P_1, P_2, \ldots P_k$ be paths. The *disjoint union* of $P_1, P_2, \ldots, P_k$ is denoted $\bigsqcup_{i=1}^k P_i$. The *merging* of $P_1, P_2, \ldots, P_k$, denoted $\bigvee_{i=1}^k P_i$, is the digraph $\bigcup_{i=1}^k P_i'$ where P_i' is isomorphic to P_i with origin x and $V(P_i') \cap V(P_j') = \{x\}$, $1 \leq i < j \leq k$. We say that x is the *origin* of $\bigvee_{i=1}^k P_i$.

A tree with three leaves is thus the merging of three paths.

3 Paths in tournaments

This section introduces some results we will use later. The following proposition follows directly from the definition of an outgenerator:

Proposition 1 *Every outgenerator of a tournament T is an origin of a hamiltonian directed outpath in T.*

Theorem 2 (Havet and Thomassé [4]) *Let T be a tournament of order n, P an outpath of order $k < n$, and x and y two vertices of T such that $s_T^+(x, y) \geq b_1(P) + 1$. Then x or y is an origin of P in T.*

Corollary 1 *Let T be a tournament of order n, P an outpath of order $k < n$. There are at least two origins of P in T.*

Theorem 3 (Havet and Thomassé [4]) *Let T be a tournament of order n and P a path of order $k \leq n$. Then T contains P if and only if the pair (T, P) is not one of Grünbaum's exceptions.*

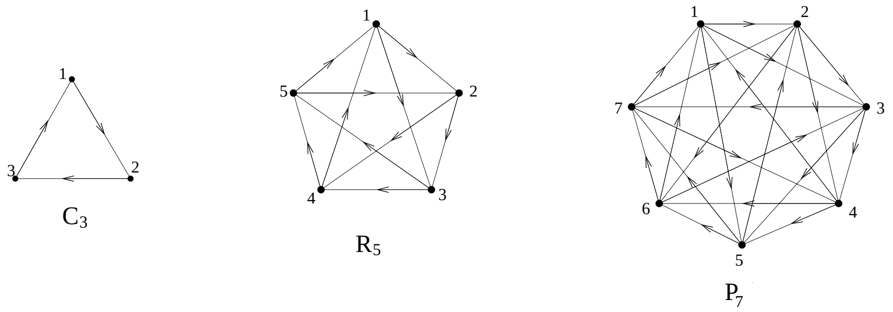

Figure 1: Grünbaum's tournaments.

Grünbaum's exceptions are the pairs $(C_3, \pm(1, 1))$, $(R_5, \pm(1, 1, 1, 1))$ and $(P_7, \pm(1, 1, 1, 1, 1, 1))$ where C_3 is the 3-circuit, R_5 the regular tournament on five vertices and P_7 the Paley tournament on seven vertices. These three tournaments are called *Grünbaum's tournaments*, and are depicted in Figure 1 and these six paths are called *Grünbaum's paths.*

Corollary 2 *Let T be a tournament of order n and P_i, $1 \leq i \leq k$, $k \geq 2$ paths of order n_i such that $n \geq \sum_{i=1}^{k} n_i$. Then T contains $\bigsqcup_{i=1}^{k} P_i$.*

4 Reduction of the problem

In [2], Havet proved the following :

Theorem 4 (Havet [2]) *The merging of order n of two outpaths and one inpath is $(n+1)$-unavoidable.*

In Remark 3.1 of the same article, the following is established :

Theorem 5 (Havet [2]) *A merging of order n of k outpaths whose first blocks are of length at least two and l of which are directed is $(\max(n+1, n+l-1))$-unavoidable.*

And Lu, Wang and Wong [7], showed :

Theorem 6 (Lu, Wang and Wong [7]) *The merging of order n of at most $\frac{19}{50}n$ directed outpaths is n-unavoidable.*

Theorems 5 and 6 yield the following corollary :

Corollary 3 *The merging of order n of three outpaths with first block of length at least two is $(n+1)$-unavoidable.*

Proof. If one of the outpaths is not directed, A is $(n+1)$-unavoidable by Theorem 5. If the 3 outpaths are directed, if $n \geq 8$ then $\frac{19}{50}n \geq 3$ so by Theorem 6 A is n-unavoidable and *a fortiori* $(n+1)$-unavoidable. Finally if $n = 7$, consider the merging A' of 3 directed outpaths of length 2,2 and 3 respectively. As previously, A' is 8-unavoidable hence A, which is a subtree of A', is also 8-unavoidable. ∎

Thus, by duality, in order to prove Theorem 1 it suffices to prove that a merging of order $n \geq 5$ of three outpaths, one of which has first block of length one, is $(n+1)$-unavoidable.

5 Mergings of three outpaths in non 2-strong tournaments

Lemma 1 *Let x be a vertex with maximum outdegree in a tournament T of order n. If $d^+(x) \geq 3$, then for any positive integers b_1 and b_2 such that $b_1 + b_2 \leq n-2$ there exists a partition (S_1, S_2, S_3) of $T - x$ and three vertices $y_i \in S_i$ for $i = 1, 2, 3$ such that $x \to \{y_1, y_2, y_3\}$ and for $i \in \{1, 2\}$, $s^+_{S_i}(y_i) = |S_i| = b_i$.*

Proof. Without loss of generality, we may suppose that $b_1 \geq b_2$. If $b_2 + 2 \leq n^+(x)$, let v be an outgenerator of $N^-(x)$ and u one of its inneighbours in $N^+(x)$. Let y_3 be a vertex of $N^+(x) - u$ and S_2 a subset of $N^+(x) - (u, y_3)$ with order b_2. Let A be the set of the $a = b_1 - N^+(x) + s_2 + 1$ first vertices of a hamiltonian directed path of $N^-(x)$ with origin x (or the empty set if a is non-positive). Let $S_1 = N^+(x) - (S_2, y_3) \cup A$ and $S_3 = T - (S_1, S_2, x)$. Let y_1 be an outgenerator of $S_1 \cap N^+(x)$ and y_2 an outgenerator of S_2. Clearly, (S_1, S_2, S_3), y_1, y_2 and y_3 satisfy the result.

So we may suppose that $\lfloor \frac{n-2}{2} \rfloor + 1 \geq b_2 + 1 \geq n^+(x) \geq \lceil \frac{n-1}{2} \rceil$. Thus $n = 2p$ or $n = 2p + 1$ and $n^+(x) = p = b_2 + 1$.

If $n = 2p$, then $b_1 = p - 1$ and $n^-(x) = p - 1$. Let v be a vertex of $N^-(x)$ with non zero outdegree in $N^-(x)$. Then v is dominated by at least two vertices in $N^+(x)$ otherwise $d^+(v) > d^+(x)$. Let v_1 be an outgenerator of $N^-(x) - v$, y_1 be one of its inneighbours in $N^+(x)$, u be an inneighbour of v in $N^+(x)$ distinct from u', y_3 be a vertex of $N^+(x) - (u, y_1)$ and y_2 be an outgenerator of $N^+(x) - (y_1, y_3)$. Let $S_1 = T[y_1, N^-(x) - v]$, $S_2 = N^+(x) - (u, y_1)$ and $S_3 = \{y_3\}$. Clearly, (S_1, S_2, S_3), y_1, y_2 and y_3 satisfy the result.

In the same way, we obtain the result if $n = 2p + 1$. ■

Definition 1 A tournament T is in $\mathcal{T}_8$ if:

(i) it has order 8;

(ii) it is $(= 1)$-strong;

(iii) it has a vertex x such that $T - x$ is reducible with decomposition $T_1 \rightarrow T_2$ such that T_1 is a 3-circuit.

In order to prove Theorem 7, we need the following proposition, whose easy proof is left to the reader.

Proposition 2 *Every tournament of $\mathcal{T}_8$ contains every merging of three paths, each having length 2.*

Note that one can also prove that every vertex in T_1 defined in (iii) is an origin of such a merging.

Theorem 7 *Let T be a non 2-strong tournament of order $n + 1$ and $C = P_1 \vee P_2 \vee P_3$ a merging of order n of three outpaths such that $b_1(P_3) = 1$. Then P contains C.*

Proof. Let us define $\tilde{T}$ in the following way : if T is reducible $\tilde{T} = T$ and if T is (=1)-strong then $\tilde{T}$ is a maximum reducible subtournament, so for some vertex d, $\tilde{T} = T - d$.

Suppose $out(\tilde{T}) = 1$, say x is the only outgenerator of $\tilde{T}$. By Corollary 2, in $\tilde{T} - x$, one can find ${}^*P_1 \sqcup {}^*P_2 \sqcup {}^*P_3$. Thus, x is an origin of C.

Suppose that $in(\tilde{T}) = 1$, say $In(\tilde{T}) = \{t\}$. By Corollary 2, in $\tilde{T} - t$, one can find $(P_1 \vee P_2) \sqcup {}^{**}P_3$. Let x be the origin of $P_1 \vee P_2$, then x is the origin of $P_3 = (x, t, {}^{**}P_3)$ in $\tilde{T} - (P_1 \vee P_2)$. Thus, x is an origin of C.

Thus we may now suppose that $in(\tilde{T})$ and $out(\tilde{T})$ are greater than 2. In particular, $n+1 \geq 6$, thus one of the three paths P_i has length at least two.

Let $T_1 \to T_2$ be a decomposition of $\tilde{T}$. Then $Out(\tilde{T}) \subseteq T_1$ and $In(\tilde{T}) \subseteq T_2$.

- Suppose first that $|T_2| \geq 4$. Let y_1, z_1, z_2 and z_3 be four vertices of T_2. If $T \in \mathcal{T}_8$ and C is a merging of three paths of length two, we have the result by Proposition 2. If $T \notin \mathcal{T}_8$, then there is a subtournament T' of $\tilde{T} - (z_1, z_2, z_3)$ such that $|T'| = |P_1| + |P_2| - 3$, $|T' \cap T_1| \geq 2$ and T' is not a 3-circuit.

  By Theorem 3, in T' there is an origin x of $Q_1 \vee Q_2$, where Q_i, $i = 1, 2$, is a directed outpath of order $|P_i| - 1$. Then $s^+_{T'}(x) \geq |T'| - 1$, so $x \in T_1$. For $i \in \{1, 2\}$, let $S_i = V(Q_i) + z_i - x$ and y_i the origin of ${}^*Q_i$. Moreover set $S_3 = T - T'$.

  Note that $x \notin S_i$, $x \to \{y_1, y_2, y_3, z_1, z_2, z_3\}$, $s^+_{S_i}(y_i, z_i) = s^-_{S_i}(y_i, z_i) = |P_i| - 1$ for $i \in \{1, 2\}$, $s^-_{S_3}(y_3, z_3) = |P_3| - 1$ and $S_3 = |P_3| + 1$. By Theorem 2, y_3 or z_3 is an origin of ${}^*P_3$ in S_3 thus x is an origin of P_3 in $T[S_3, x]$. Let u be the vertex of $T[S_3, x]$ which is not in this P_3. By Theorem 2, y_2 or z_2 is an origin of ${}^*P_2$ in $T[S_2, u]$ thus x is an origin of P_2 in $T[S_2, u, x]$. Let v be the vertex of $T[S_2, x]$ which is not in this P_2. By Theorem 2, y_1 or z_1 is an origin of ${}^*P_1$ in $T[S_1, v]$ thus x is an origin of P_1 in $T[S_1, v, x]$. Therefore, x is an origin of C in T.

- Suppose now that $|T_2| = 3$, say $V(T_2) = \{z_1, z_2, z_3\}$. Then $|T_1| \geq 6$. Thus there is a vertex x of T_1 with outdegree at least three. So, by Lemma 1, there exists a partition (S_1, S_2, S_3) of T_1 and three outneighbours y_1, y_2 and y_3 of v such that, for $i \in \{1, 2\}$, $s^+_{S_i}(y_i) \geq |P_i| - 1$. Let $S'_1 = T[S_1, y_1]$, $S'_2 = T[S_2, y_2]$ and $S_3 = T - [S_1, S_2, x]$. Once more, we have $s^+_{S_i}(y_i, z_i) = s^-_{S_i}(y_i, z_i) = |P_i| - 1$ for $i \in \{1, 2\}$, $s^-_{S_3}(y_3, z_3) = |P_3| - 1$ and $S_3 = |P_3| + 1$. Thus as previously, x is an origin of C in T. ■

6 Mergings of outpaths in large 2-strong tournaments

Lemma 2 *Let T be a 2-strong tournament of order $n+1$ and $C = \bigvee_{i=1}^k P_i$ a merging of order n of $k \geq 3$ outpaths. If $|P_1| - 2 \geq \delta^-(T)$, then T contains C.*

Proof. Let t be a vertex with indegree $\delta^-(T) = \delta$ in T. Set $X = N^-(t)$ and $Y = N^+(t)$.

Since T is 2-strong, there is an ingenerator x of X that dominates a vertex u of Y. Otherwise, t is a 1-cut of T. By Proposition 1, there is a hamiltonian directed outpath $(x_1, x_2, \ldots, x_\delta)$ of X with terminus $x_\delta = x$. Since $d^-(x_1) \geq \delta$, there is a vertex $v \in Y$ that dominates x_1. Now $|Y - [u, v]| \geq \sum_{i=2}^k |{}^*P_i|$. So, by Corollary 2, one can find $\bigsqcup_{i=2}^k {}^*P_i$ in $Y - [u, v]$. Set $S_1 = Y - [\bigsqcup_{i=2}^k {}^*P_i]$ and let y_1 and z_1 be respectively an outgenerator and an ingenerator of S_1 such that $y_1 \neq z_1$. Such vertices exist because $|S_1| \geq 2$. Since $v \in S_1$, $S^+_{T[X,S_1]}(y_1) = T[X, S_1]$, and because $u \in S_1$, $S^-_{T[X,S_1]}(z_1) = T[X, S_1]$. Therefore, by Lemma 2, y_1 or z_1 is an origin of ${}^*P_1$ in $T[X, S_1]$. Thus t is an origin of C in T. ■

Lemma 3 *Let T be a 2-strong tournament of order $n+1$ and $C = \bigvee_{i=1}^{k} P_i$ a merging of order n of $k \geq 3$ outpaths. If $|P_1| + |P_2| - 5 \geq \delta^-(T)$ and $b_1(P_1) \geq 2$ then T contains C.*

Proof. Let t be a vertex with indegree $\delta^-(T) = \delta$ in T. Set $X = N^-(t)$ and $Y = N^+(t)$.

By Lemma 2, we may suppose that $|P_1| - 2 < \delta$ and also $|P_2| - 2 < \delta$. So $|P_i| \geq 4$, for $i = 1, 2$. Set $l_1 = |P_1| - 3$.

Since T is 2-strong, there is an ingenerator x of X that dominates a vertex u of Y. Otherwise, t is a 1-cut of T. By Proposition 1, there is a hamiltonian directed outpath $(x_1, x_2, \ldots, x_\delta)$ of X with terminus $x_\delta = x$. Since $d_T^-(x_{l_1+1}) \geq \delta$, there is a vertex $v \in Y$ that dominates x_{l_1+1}. Set $y_2 = u$ if $b_1(P_2) = 1$, and $y_2 = v$ otherwise. Because $d_T^-(x_1) \geq \delta$, $|N_T^-(x_1) \cap Y| \geq 2$. Thus, there is a vertex y_1 of $Y - y_2$ such that $x_1 \leftarrow y_1$.

Since $|P_1| + |P_2| - 5 \geq \delta$, $|Y - [y_1, y_2]| \geq |\bigsqcup_{i=3}^{k} {}^*P_i| + 1$. Hence, by Theorem 3 and Corollary 2, in $Y - [y_1, y_2]$, one can find $\bigsqcup_{i=3}^{k} {}^*P_i$. Let u_1 be a vertex of $Y - [\bigsqcup_{i=3}^{k} {}^*P_i, y_1, y_2]$.

Set $Y_2 = Y - [\bigsqcup_{i=3}^{k} {}^*P_i, y_1, y_2, u_1]$. Let u_2 be an ingenerator of Y_2 if $b_1(P_2) = 1$, and an outgenerator of Y_2 if $b_1(P_2) \geq 2$. Note that Y_2 is not empty, because $|P_1| + |P_2| - 5 \geq \delta$. Set $T_2 = T[Y_2, y_2, x_{l_1+1}, x_{l_1+2}, \ldots, x_\delta]$. By Theorem 2, u_2 or y_2 is an origin of ${}^*P_2$ in T_2. Let z be the vertex of T_2 which is not in this ${}^*P_2$. Setting $T_1 = T[z, u_1, y_1, x_1, x_2, \ldots, x_{l_1}]$, we have $s_{T_1}^+(u_1, y_1) \geq |P_1| - 1$. Thus, by Theorem 2, u_1 or v_1 is an origin of ${}^*P_1$ in T_1. So t is an origin of C in T. ■

Lemma 4 *Let T be a 2-strong tournament of order $n+1$ and $C = P_1 \vee P_2 \vee P_3$ a merging of order n of three outpaths. If $|P_1| + |P_2| - 5 \geq \delta^-(T)$ and $b_1(P_1) = b_1(P_2) = 1$ then T contains C.*

Proof. Let t be a vertex with indegree $\delta^-(T) = \delta$ in T. Set $X = N^-(t)$ and $Y = N^+(t)$.

Since T is 2-strong, there is an ingenerator x of X that dominates a vertex u of Y. Otherwise, t is a 1-cut of T. Let $(x_1, x_2, \ldots, x_\delta = x)$ be a hamiltonian directed outpath of X.

By Lemma 2, we may suppose that $|P_1| - 2 < \delta$ and $|P_2| - 2 < \delta$. So $|P_i| \geq 4$, for $i = 1, 2$. Set $l_i = |P_i| - 2$.

Suppose that there are two distinct vertices y_1 and y_2 of Y such that $x_{l_1} \to y_1$ and $x_\delta \to y_2$. By Theorem 3, in $Y - [y_1, y_2]$, one can find ${}^*P_3$. Let (S_1, S_2) be a partition of $Y - {}^*P_3$ such that $y_i \in S_i$, for $i = 1, 2$, $|S_1| = |P_1| - l_1$. For $i = 1, 2$, let u_i be an ingenerator of $S_i \cap Y$ and v_i a vertex of $S_i \cap Y$ distinct from u_i. Set $T' = T[S_1, x_1, x_2, \ldots, x_{l_1}]$. $s_{T'}^-(u_1) \geq |P_1| - 1$ and $|T'| = |P_1|$. Thus, by Theorem 2, u_1 or v_1 is an origin of ${}^*P_1$ in T' and so t is an origin of P_1 in $T' + t$. Let z be the vertex of T' which is not in this P_1 and $T'' = T[S_2, x_{l_1+1}, x_{l_1+2}, \ldots, x_\delta, z]$. Again, by Theorem 2, u_2 or v_2 is an origin of ${}^*P_2$ in T'' and so t is an origin of P_2 in $T'' + t$.

In the same way, we have the result if there are two distinct vertices y_1 and y_2 of Y such that $x_{l_1-1} \to y_1$ and $x_\delta \to y_2$.

We also have the result, if there are two distinct vertices y_1 and y_2 of Y such that $x_{l_2} \to y_2$ or $x_{l_2-1} \to y_2$ and $x_\delta \to y_1$.

Therefore, we have $Y - u \to \{x_{l_2-1}, x_{l_2}, x_{l_1-1}, x_{l_1}\}$.

Without loss of generality, we may suppose that $l_1 \geq l_2$.

If $|P_2| \geq 5$, then taking $l'_1 = |P_1| - 3$, and by the usual reasoning with l'_1 instead of l_1, we have the result if there are two vertices y_1 and y_2 such that $x_{l_1-2} \to y_1$ and $x_\delta \to y_2$.

- Suppose that either $|P_2| \geq 5$ or $l_1 > l_2$. Then there is an integer $j < l_1 - 1$ such that $Y - u \to x_j$. We have $Y - u \to \{x_j, x_{l_1-1}, x_{l_1}\}$. Now $|Y - u| \geq |P_3| + 2$, so by Theorem 3, one can find $P_3 \vee Q$ in $Y - u$, where Q is the directed outpath of length 1. Let y be the origin of this $P_3 \vee Q$ and set $Q = (y, z)$. Let v be a vertex of $Y - [u, P_3 \vee Q]$ and set $S_1 = T[t, v, x_1, x_2, \ldots, x_{l_1-1}]$. Clearly, $|S_1| = |P_1| - 1$ and $S^-_{S_1}(x_{l_1-1}, x_j) = S_1$. And let $S_2 = T - [P_3, S_1]$. We have $|S_2| = |P_2|$ and $s^-_{S_2}(x_{l_1}, z) \geq |S_2| - 1$. (Indeed, only u may not be in $S^-_{S_2}(x_{l_1}, z)$.) Thus, by Theorem 2, x_{l_1} or z is an origin of ${}^*P_2$ in S_2. Let a be the vertex of S_2 which is not in this path ${}^*P_2$. By Theorem 2, x_{l_1-1} or x_j is an origin of ${}^*P_1$ in $T[S_1, a]$. Thus y is an origin of C in T.

- Suppose now that $|P_1| = |P_2| \leq 4$. Since $|P_1| + |P_2| - 5 \geq \delta^-(T) = k$, $|P_1| - 2 < \delta$ and T is 2-strong, $|P_1| = |P_2| = 4$ and $\delta = 3$.

  Firstly, suppose that $P_1 = +(1,2)$. Then, by Corollary 2, one can find ${}^*P_2 \sqcup {}^*P_3$ in $Y - u$. And t is an origin of $P_1 = (t, u, x_3, x_2)$. Thus, t is an origin of C in T.

  Suppose now that $P_1 = +(1,1,1)$. There is a vertex $v \in Y$, such that $v \to x_2$ since $d^-_T(x_2) \geq 3$. Now t is an origin of $P_1 = (t, v, x_2, x_3)$. And, by Corollary 2, there is ${}^*P_2 \sqcup {}^*P_3$ in $Y - v$. Thus, t is an origin of C in T. ■

Lemmas 3 and 4 together with Theorem 4 yield the following result:

Theorem 8 *Every tree with three leaves of order $n \geq 17$ or $n = 15$ is contained in every 2-strong tournament of order $n + 1$.*

Proof. Let T be a tournament of order $n + 1$ and $C = P_1 \vee P_2 \vee P_3$ a tree with three leaves. By duality, we may suppose that C is a merging of at least two outpaths. If C is the merging of two outpaths and one inpath we have the result by Theorem 4. So we may assume that C is the merging of three outpaths.

Without loss of generality, we may suppose that $|P_1| \geq |P_3|$ and $|P_2| \geq |P_3|$. Since $n \geq 17$ or $n = 15$,

$$|P_1| + |P_2| \geq \left\lceil \frac{2n-2}{3} \right\rceil + 2 \geq \left\lfloor \frac{n}{2} \right\rfloor + 5.$$

Therefore T contains C, by Lemma 4 if $b_1(P_1) = b_1(P_2) = 1$ or by Lemma 3 otherwise. ■

7 Mergings of three outpaths in small 2-strong tournaments

Lemma 5 *Let T be a 2-strong tournament of order $n+1$ and $C = \bigvee_{i=1}^{k} P_i$ the merging of order n of $k \geq 3$ outpaths. If $\delta^-(T) = 2$ and $|P_1| \geq 3$, then T contains C.*

Proof. By Lemma 2, we may assume that $|P_1| = 3$. Let t be vertex with only two inneighbours x_1 and x_2 with $x_1 \to x_2$. Set $Y = N_T^+(t)$. Let y_1 be a vertex of Y such that $y_1 \to x_1$ if $P_1 = +(2)$ and $y_1 \leftarrow x_2$ if $P_1 = +(1,1)$. Such a vertex exists since T is 2-strong.

Now, in $Y - y_1$, we can find $\bigsqcup_{i=2}^{k} {}^*P_i$ in $Y - y_1$. And t is clearly an origin of P_1 in $T[t, x_1, x_2, y_1]$. Then t is an origin of C in T. ■

Lemma 6 *Let T be a 2-strong tournament of order $n+1$ and $C = P_1 \vee P_2 \vee P_3$ the merging of order n of three outpaths. If $\delta^-(T) = 3$, then T contains C.*

Proof. Since $\delta^-(T) = 3$, then $|T| \geq 7$ and so $|P_1| + |P_2| + |P_3| \geq 8$. Without loss of generality we may suppose that $|P_1| \geq |P_2| \geq |P_3|$.

By Lemma 2, we may assume that $|P_1| \leq 4$. Let t be vertex with three inneighbours. Set $X = N_T^-(t)$ and $Y = N_T^+(t)$. Let (x_1, x_2, x_3) be a hamiltonian directed outpath of X such that x_3 dominates a vertex in Y.

- Suppose first that $|P_1| = 4$. Then, by Lemma 3 and 4, $|P_2| \leq 3$. If $P_1 \neq +(1,1,1)$, let y_1 be the vertex such that:
  - $y_1 \to x_1$ if $P_1 = +(3)$,
  - $y_1 \to x_2$ if $P_1 = +(2,1)$, and
  - $y_1 \leftarrow x_3$ if $P_1 = +(1,2)$.

  Then in $Y - y_1$, one can find ${}^*P_2 \sqcup {}^*P_3$. Therefore, t is an origin of C in T.

  So we may assume that $P_1 = +(1,1,1)$.

  Suppose that $|P_2| \geq 3$. Let y_2 be a vertex dominating (resp. dominated by) x_3 if ${}^*P_2$ is an outpath (resp. inpath). In $Y - y_2$, one can find ${}^*P_3$. Let y_1 and z_1 be two vertices of $Y - [y_2, {}^*P_3]$. and z_2 be an outgenerator of $Y_2 = Y - [y_1, z_1, y_2, {}^*P_3]$ if $b_1(P_2) = 1$ and an ingenerator of Y_2 otherwise. By Theorem 2, in $T_1 = T[y_1, z_1, x_1, x_2]$, y_1 or z_1 is an origin of ${}^*P_1$. Let a be the vertex of $T_1 - {}^*P_1$. Now, in $T[Y_2, y_2, x_3, a]$, y_2 or z_2 is an origin of ${}^*P_2$. (If Y_2 is empty then y_2 is trivially an origin of ${}^*P_2$). Thus t is an origin of C in T.

  If $|P_2| = 2$, then $|P_3| = 2$ and $d_T^-(v) = d_T^+(v) = 3$ for all $v \in T$. There are only two regular tournaments on 7 vertices. And one can easily check that they contain C.

- Suppose now that $|P_1| = 3$. Then $|P_2| = 3$. Moreover $|T| = 8$ or $|T| = 7$.

  If P is directed, there are two distinct vertices y_1 and y_2 of Y such that $y_1 \to x_2$ and $y_2 \to x_3$ if $b_1(P_2) = 2$ and $y_2 \leftarrow x_3$ otherwise.

  We have $P_1 = (t, y_1, x_2)$ and $P_2 = (t, y_2, x_3)$. Now by Corollary 1, there is a vertex of $Y' = Y - [y_1, y_2]$ which is an origin of ${}^*P_3$ in $T[Y', x_1]$. Hence t is an origin of C in T.

  In the same way we have the result if P_2 or P_3 is $+(2)$. So we may suppose that $P_1 = P_2 = +(1,1)$ and $P_3 \in \{+(1,1), +(1)\}$.

  And by the same argument, we have the result if there are two distinct vertices y_1 and y_2 which are dominated by two different vertices in X. This is verified if T is a regular tournament of order 7 because $\delta^+(T) = 3$.

  So T has at least 8 vertices x_3 dominates a vertex $u \in Y$ and $Y - u \to x_2$. By Theorem 3, there is a vertex v of $Y - u$ which is origin of $+(1) \vee +(1,1)$. And v is origin of $+(1,1) = (v, x_2, x_1)$. Thus v is an origin of C in T.

■

Corollary 4 *Let T be a 2-strong tournament of order $n+1 \leq 8$ and C a tree of order n with three leaves. Then T contains C unless $(T; C) \in \{(R_5; S_3^+), (R_5; S_3^-)\}$.*

Proof. Since R_5 is the unique 2-strong tournament with $\delta^- = 2$, Lemma 6 yields the result. ■

Proposition 3 *Every 2-strong tournament of order* 17 *contains every merging of order* 16 *of three outpaths.*

Proof. Let T be a 2-strong tournament of order 17 and C a merging of order 16 of three outpaths P_1, P_2 and P_3. By Lemmas 3 and 4 and Corollary 3, we may assume that $|P_1| = |P_2| = |P_3| = 6$, $b_1(P_3) = 1$ and $\delta^-(T) = 8$. This implies that for every vertex $v \in T$, $d_T^-(v) = d_T^+(v) = 8$. Let t be a vertex. Let $X = N_T^-(t)$ and $Y = N_T^+(t)$. Let x_8 be a vertex of X with minimum outdegree in X. Then $|N_T^+(x_8) \cap Y| \geq 3$ and there is a hamiltonian directed outpath $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$ of X.

Because both in- and outdegree of a vertex are 8, one can find two vertices y_1 and y_2 in Y such that :

- $y_1 \to x_1$ if ${}^*P_1$ is an outpath, and $y_1 \leftarrow x_3$ otherwise;
- $y_2 \to x_4$ if ${}^*P_2$ is an outpath, and $y_2 \leftarrow x_6$ otherwise.

Let y_3 be a vertex of $N_T^+(x_8) \cap (Y \setminus \{y_1, y_2\}$ and set $\{z_1, z_2, z_3, z_4, z_5\} = Y \setminus \{y_1, y_2, y_3\}$. Now, in $T_1 = T[y_1, z_1, x_1, x_2, x_3, z_5]$, y_1 or z_1 is an origin of ${}^*P_1$ according to Theorem 2. Let a_1 be the vertex of $T_1 - [{}^*P_1]$. By Theorem 2, in $T_2 = T[y_2, z_2, x_4, x_5, x_6, a_1]$, y_2 or z_2 is an origin of ${}^*P_2$. Let a_2 be the vertex of $T_2 - [{}^*P_2]$. Free to rename the vertices, we have $z_3 \leftarrow z_4$. Again, by Theorem 2, in $T[y_3, z_3, z_4, x_7, x_8, a_2]$, y_3 or z_3 is an origin of ${}^*P_3$. Thus t is an origin of C in T. ■

Proposition 4 *Every 2-strong tournament of order* 15 *contains every merging of order* 14 *of three outpaths.*

Proof. Identical to Proposition 3. ∎

Proposition 5 *Every 2-strong tournament of order* 14 *contains every merging of order* 13 *of three outpaths.*

Proof. Let T be a tournament of order 14 and C a merging of order 13 of three outpaths P_1, P_2 and P_3. By Lemmas 3 and 4, Corollary 3, we may assume that $|P_1| = |P_2| = |P_3| = 5$, $b_1(P_1) = 1$ and $\delta^-(T) = 6$. Let t be a vertex with indegree 6. Let $X = N_T^-(t)$ and $Y = N_T^+(t)$.

For degree reason, there are at least 14 arcs oriented from a vertex in X to a vertex in Y, there are at least 4 vertices in Y having an inneighbour in X and at least 3 vertices in X having an outneighbour in Y.

- Suppose that $b_1(P_2) \geq 2$. Let x_1, x_2, x_3 be three vertices in X having an outneighbour in Y and $x_4 \in X - \{x_1, x_2, x_3\}$. By Corollary 1, at least two vertices in $\{x_1, x_2, x_3, x_4\}$ are an origin of $^{**}P_1$ in $T[x_1, x_2, x_3, x_4]$. Hence, a vertex of $\{x_1, x_2, x_3\}$, say x_1, is an origin of $^{**}P_1$ in $\{x_1, x_2, x_3, x_4\}$. Let y_1 be an outneighbour of x_1 in Y. Let $\{x'_1, x'_2, x'_3\} = X' = X - [\ ^{**}P_1]$. At least two vertices in X' have indegree at most four in X and thus have at least one inneighbour in $Y - y_1$. Therefore, there exist two vertices y_2, y_3 in Y such that $s^+_{T'}(y_2, y_3) \geq 4 \geq b_1(^*P_2)+1$, where $T' = T[y_2, y_3, x'_1, x'_2, x'_3]$. By Theorem 2, y_2 or y_3 is an origin of $^*P_2$ in T'. Then $Y - [y_1, y_2, y_3]$ has four vertices and then contains $^*P_3$ by Theorem 3. Hence t is an origin of C in T.

- Suppose $b_1(P_1) = b_1(P_2) = b_1(P_3) = 1$.

  - Suppose $b_2(P_1) = 3$ and exactly three vertices x_1, x_2, x_3 in X have outneighbours in Y. One of these three vertices, say x_1, has at least five outneighbours in Y. The vertex x_1 has at most one outneighbour in X, so it has at least two inneighbours x_4 and x_5 in $X - \{x_1, x_2, x_3\}$. Thus, in $T[x_1, x_4, x_5]$, x_1 is an origin of $^{**}P_1$. Set $\{x_6\} = X - \{x_1, x_2, x_3, x_4, x_5\}$. Let y_1, y_2 and y_3 be three distinct vertices in Y so that y_i is dominated by x_i, $i = 1, 2, 3$, and $T' = T[y_2, y_3, x_2, x_3, x_6]$. By Theorem 2, y_2 or y_3 is an origin of $^*P_2$ in T'. Now, by Theorem 3, $Y - [y_1, y_2, y_3]$ contains $^*P_3$. Hence t is an origin of C in T.
  - If $b_2(P_1) = 3$ and at least four vertices in X dominate at least one vertex in Y. Let x_1, x_2, x_3 and x_4 be four vertices having the maximum number of outneighbours in Y. Let x_5 and x_6 be the two remaining vertices of X, with $x_5 \to x_6$. As T is 2-strong, x_6 has an outneighbour distinct from t.

- $*$ If x_6 has no outneighbour in Y, it has an outneighbour in $\{x_1, x_2, x_3, x_4\}$, say $x_6 \to x_1$. As 14 edges go from X to Y, we may find in Y an outneighbour y_1 of x_1 and two vertices y_2 and y_3 in $Y - y_1$ such that $s^-_{T[y_2,y_3,x_2,x_3,x_4]}(y_2, y_3) \geq 4$. Then by Theorem 2, y_2 or y_3 is an origin of $^*P_2$ in $T[y_2, y_3, x_2, x_3, x_4]$. Set $P_1 = (t, y_1, x_1, x_6, x_5)$ and $*P_3$ in contained in $T[Y] - [y_1, y_2, y_3]$ according to Theorem 3.
- $*$ If x_6 has an outneighbour y_6 in Y. By Corollary 1, in $\{x_1, x_2, x_3, x_4\}$, two vertices, say x_1 and x_2 are an origin of $^{**}P_2$. If one of these two vertices, say x_1, has an outneighbour y_1 in $Y - y_6$. Then y_1 is the origin of $^*P_2$ in $T[y_1, x_1, x_2, x_3, x_4]$, and as x_5 has less than 5 outneighbours in Y, it has an inneighbour y_5 in $Y - \{y_1, y_6\}$. Then set $P_1 = (t, y_6, x_6, x_5, y_5)$. If both x_1 and x_2 are dominated by $Y - y_6$, then by Corollary 1, y_6 is an origin of $^*P_1$ in $T[y_6, x_6, x_5, x_1, x_2]$. Since x_3 and x_4 have at least 5 outneighbours in Y, two of them, say y_3 and y_4, satisfy $s^-_{T'}(y_3, y_4) \geq 4$, with $T' = T[\{y_3, y_4\} \cup X - P_1]$. Then by Theorem 2, y_3 or y_4 is an origin of $^*P_2$ in T', and we may find $^*P_3$ in the remaining vertices thanks to Theorem 3.

– Suppose now that $b_2(P_i) \leq 2$, $i = 1, 2, 3$. Let $X = \{x_i, i \in [1, 6]\}$ and $Y = \{y_i, i \in [1, 7]\}$. There exist two disjoint edges from X to Y, say $x_1 \to y_1$ and $x_4 \to y_3$. Let $T' = \{x_1, x_2, x_3, y_1, y_2\}$, $T'' = \{x_4, x_5, x_6, y_3, y_4\}$. By Theorem 2, y_1 or y_2 is an origin of $^*P_1$ in T' and y_3 or y_4 is an origin of $^*P_2$ in T''. Without loss of generality, y_5 is an ingenerator of $T[y_5, y_6, y_7]$. In $T - [^*P_1, ^*P_2, t]$, by Theorem 2, y_5 or y_6 is an origin of $^*P_3$. Thus t is an origin of C in T.

■

Proposition 6 *Every 2-strong tournament of order* 13 *contains every merging of order* 12 *of three outpaths.*

Proof. Let T be a tournament of order 13 and $C = P_1 \vee P_2 \vee P_3$ a merging of order 12 of three outpaths. By Lemmas 3 and 4, we may assume that $\delta^-(T) = 6$ and :

1. either $|P_1| = |P_2| = 5$, $|P_3| = 4$,
2. or $|P_1| = 6$ and $|P_2| = |P_3| = 4$.

Let t be a vertex. Let $X = N^-_T(t)$ and $Y = N^+_T(t)$.

1. Suppose that $|P_1| = |P_2| = 5$, $|P_3| = 4$. Let x_6 be a vertex of X with minimum outdegree in X. Then $|N^+_T(x_6) \cap Y| \geq 3$ and there is a hamiltonian directed outpath $(x_1, x_2, x_3, x_4, x_5, x_6)$ of X. According to Corollary 3, we may assume that $b_1(P_3) = 1$ or $b_1(P_2) = 1$.

Supoose first that $b_1(P_3) = 1$. Because both in- and outdegree of each vertex are equal to 6, one can find two vertices y_1 and y_2 in Y such that :

- $y_1 \to x_1$ if ${}^*P_1$ is an outpath, and $y_1 \leftarrow x_3$ otherwise;

- $y_2 \to x_4$ if ${}^*P_2$ is an outpath, and $y_2 \leftarrow x_5$ otherwise.

Let y_3 be a vertex of $N_T^+(x_6) \cap Y \setminus \{y_1, y_2\}$, and set $\{z_1, z_2, z_3\} = Y \setminus \{y_1, y_2, y_3\}$. Now, in $T_1 = T[y_1, z_1, x_1, x_2, x_3]$, y_1 or z_1 is an origin of ${}^*P_1$ according to Theorem 2. Let a_1 be the vertex of $T_1 - [{}^*P_1]$. By Theorem 2, in $T_2 = T[y_2, z_2, x_4, x_5, a_1]$, y_2 or z_2 is an origin of ${}^*P_2$. Let a_2 be the vertex of $T_2 - [{}^*P_2]$. Again, by Theorem 2, in $T[y_3, z_3, x_6, a_2]$, y_3 or z_3 is an origin of ${}^*P_3$. Thus t is an origin of C in T.

If $b_1(P_2) = 1$: we proceed analogously, by finding first P_1 and P_3, and then P_2 in the five remaining vertices.

2. Suppose that $|P_1| = 6$ and $|P_2| = |P_3| = 4$. If $b_1(P_3) \leq 2$. We proceed as in first case by first finding P_1 and P_2, and then P_3 in the 4 remaining vertices. If $b_1(P_2) = b_1(P_3) = 3$. Let $x_1, x_2, x_3, x_4, x_5, x_6$ be the vertices of X ordered by increasing outdegree in X. By Theorem 3, $T[x_2, x_3, x_4, x_5]$ contains ${}^{**}P_1$. Because every vertex of $\{x_2, x_3, x_4, x_5\}$ has at least two inneighbours and one outneighbour in Y, there is a vertex $y_1 \in Y$ that is an origin of ${}^*P_1$ in $T[y_1, x_2, x_3, x_4, x_5]$. Let y_2 be an inneighbour of x_6 distinct from y_1, and $y_3 \in Y - \{y_1, y_2\}$. By Theorem 2 in $T_1 = T[y_2, y_3, x_1, x_6]$, y_2 or y_3 is an origin of ${}^*P_2$. Finally, $T - [t, {}^*P_1, {}^*P_2]$ has four vertices and at least three of them are in Y. So by Corollary 1 a vertex $y_4 \in Y$ is an origin of ${}^*P_3$ in $T - [t, {}^*P_1, {}^*P_2]$. Hence, t is an origin of C in T.

■

Proposition 7 *Every tournament of order* 12 *contains every merging of order* 11 *of three outpaths.*

Proof.
Let T be a tournament of order 12 and $C = P_1 \vee P_2 \vee P_3$ a merging of order 11 of three outpaths. By Lemmas 3 and 4, we may assume that $|P_1| = 5$, $|P_2| = |P_3| = 4$, and that $\delta^-(T) = 5$. Let t be a vertex having indegree 5. Let $X = N_T^-(t)$ and $Y = N_T^+(t)$.

For degree reason, there are at least 9 arcs from X to Y, and at least 15 arcs from Y to X.

- Suppose that $b_1(P_2) \geq 2$ and $b_1(P_3) = 1$. Let $X = \{x_1, x_2, x_3, x_4, x_5\}$, with x_1 and x_2 having the maximum number of outneighbours in Y. In $\{x_1, x_2\}$, we may find ${}^{**}P_3$, say ${}^{**}P_3 = (x_1, x_2)$. Let y_1 be an outneighbour of x_1 in Y. Then ${}^*P_3 = (y_1, x_1, x_2)$. Among $\{x_3, x_4, x_5\}$, at least two vertices, say x_3 and x_4 are an origin of ${}^{**}P_2$. One of them, say x_3, has less than 5 outneighbours in Y, and thus has a inneighbour y_3 in $Y - y_1$. Hence y_3 is an origin of ${}^*P_2$ in $T[y_3, x_3, x_4, x_5]$. And $Y - [y_1, y_3]$ contains ${}^*P_1$ by Theorem 3. Therefore, t is an origin of C in T.

- Suppose that $b_1(P_2) \geq 2$ and $b_1(P_3) \geq 2$. Without loss of generality, we may suppose that ${}^{**}P_2 = (x_1, x_2)$ and ${}^{**}P_3 = (x_3, x_4)$. For degree reason, there exist two distinct vertices y_1 and y_3 that are dominated by x_1 and x_3 respectively. Then ${}^*P_2 = (y_1, x_1, x_2)$ and ${}^*P_3 = (y_3, x_3, x_4)$. And $Y - [y_1, y_3]$ contains ${}^*P_1$ by Theorem 3. Therefore, t is an origin of C in T.

- Suppose that $b_1(P_2) = b_1(P_3) = 1$.

  - If $b_1(P_1) \geq 2$, let $x_1 \rightarrow y_1$ be an arc with $x_1 \in X$ and $y_1 \in Y$. By Corollary 1, in $X - x_1$ at least 2 vertices are origins of ${}^{**}P_1$, so at least one of these origins, say x_2, has an inneighbour y_2 in $Y - y_1$. Let y_3 be a vertex of $Y - [y_1, y_2]$, and x_3 the sole vertex of $X - [{}^{**}P_1, x_1]$. By Theorem 2, in $T[x_1, x_3, y_1, y_3]$, y_1 or y_3 is an origin of ${}^*P_2$. Now $T - [{}^*P_1, {}^* P_2]$ has four vertices, three of which are in $Y - [y_1, y_2, y_3]$. So, by Corollary 1, a vertex of $Y - [y_1, y_2, y_3]$ is an origin of ${}^*P_3$ in $T - P_1 - P_2$. Hence t is an origin of C in T.
  - Suppose that $b_1(P_1) = 1$, and $b_2(P_2) = b_2(P_3) = 1$. It is easy to prove that in X, we can find 4 vertices $x_1 \rightarrow x_2$ and $x_3 \rightarrow x_4$ such that x_2 and x_4 have distinct outneighbours y_2 and y_4 respectively in Y. Then set $P_2 = (t, y_2, x_2, x_1)$ and $P_3 = (t, y_4, x_4, x_3)$. In the four remaining vertices of Y one can find ${}^*P_1$ by Theorem 3.
  - Suppose that $b_1(P_1) = 1$, $b_2(P_2) = 1$ and $b_2(P_3) = 2$. Find $x_1, x_2, x_3, x_4, x_5, x_6$ as previously. Set $P_2 = (t, y_2, x_2, x_1)$. Let $y_3 \rightarrow y_5$ in Y. By Theorem 2, in $\{y_5, y_4, y_3, x_4, x_3\}$, y_5 or y_4 is an origin of ${}^*P_1$. Then in the three remaining vertices of Y, one can find ${}^*P_3$ by Theorem 2.
  - Suppose that $b_1(P_1) = 1$ and $b_2(P_2) = b_2(P_3) = 2$.

    Let us prove first that there exist y_1, y_2 in Y and x_1, x_2, x_3, x_4 in X such that $y_1 \leftarrow x_1 \leftarrow x_2$ and $y_2 \leftarrow x_3 \leftarrow x_4$. (Note that at least 3 vertices in X have outneighbours in Y)

    * If exactly 3 vertices of X, say x_1 ,x_2 and x_3 have an outneighbour in Y, then by 2-strongness, one of them, say x_3 has an inneighbour in $X - \{x_1, x_2, x_3\}$. We may suppose $x_1 \leftarrow x_2$, and just remark that x_1 or x_3 has more than 1 outneighbour in Y.
    * If exactly 4 vertices of X, say z_1, z_2, z_3 and z_4 have an outneighbour in Y, with $z_1 \leftarrow z_2$ and $z_3 \leftarrow z_4$. If there are two distinct vertices y_1 and y_2 of Y dominated by z_1 and z_3 respectively, then setting $x_i = z_i$, $i = 1, 2, 3, 4$ we are done. If not, z_1 has a unique outneighbour y_1 in Y, which is also the sole outneighbour of z_3 in Y. Then let x_1 as the dominating vertex of $T[z_1, z_3]$ and x_3 the dominating vertex of $T[z_2, z_4]$. Because at least 9 arcs go from X to Y, we may find $y_1 \leftarrow x_1$ and $y_2 \leftarrow x_3$.

* If the five vertices of X have an outneighbour in Y, then consider $\{z_1, z_2, z_3, z_4\}$ the vertices of X having the maximum number of outneighbours in Y and apply the previous reasoning.

Now consider ${}^*P_2 = (t, y_1, x_1, x_2)$ and ${}^*P_3 = (t, y_2, x_3, x_4)$, and in the 4 remaining vertices of Y, find ${}^*P_1$ by Theorem 3.

■

Proposition 8 *Every tournament of order* 11 *contains every merging of order* 10 *of three outpaths.*

Proof.

Let T be a tournament of order 11 and C a tree of order 10 of three outpaths P_1, P_2 and P_3. By Lemmas 3 and 4, one of the following assertions holds :

1. $|P_1| = 6, |P_2| = |P_3| = 3$ and $\delta^-(T) = 5$;
2. $|P_1| = 5$, $|P_2| = 4$, $|P_3| = 3$ and $\delta^-(T) = 5$;
3. $|P_1| = |P_2| = |P_3| = 4$ and $\delta^-(T) = 5$;
4. $|P_1| = |P_2| = |P_3| = 4$ and $\delta^-(T) = 4$.

If $\delta^-(T) = 5$, then $v \in T$, $d_T^-(v) = d_T^+(v) = 5$. Let t be a vertex having minimal indegree. Let $X = N_T^-(t)$ and $Y = N_T^+(t)$.

1. Suppose that $|P_1| = 6, |P_2| = |P_3| = 3$ and $\delta^-(T) = 5$. There is at most one vertex in X, say x, that has no outneighbour in Y. Let x' be an origin of ${}^{**}P_1$ in $X - x$. In Y, there is one vertex that dominates x' and one vertex dominated by x'. This ensures that a vertex y of Y is an origin of ${}^*P_1$. Then in $Y - y$, we can find ${}^*P_2 \sqcup^* P_3$ by Corollary 2. Hence t is an origin of C in T.

2. Suppose that $|P_1| = 5$, $|P_2| = 4$, $|P_3| = 3$ and $\delta^-(T) = 5$. There is a vertex $x \in X$ having outdegree 1 or 2 in X. In $X - x$, there are two origins z, z' of ${}^{**}P_1$ by Corollary 1. z and z' have both inneighbours in Y, and for degree reason, one of them has an outneighbour in Y. So whatever orientation has the second arc of P_1, there is a vertex $y \in Y$ such that x is an origin of P_1 in $T[t, y, X - x]$. Now x has at least two outneighbours in Y and at least two inneighbours in Y, so there is a vertex y' in $Y - y$ such that $P_3 = (t, y', x)$.

   Finally, in $T - [P_1, P_3]$, by Corollary 1, at least two vertices are origins of ${}^*P_2$. So one of these origins is in Y. Thus t is an origin of C in T.

3. Suppose that $|P_1| = |P_2| = |P_3| = 4$ and $\delta^-(T) = 5$. Applying exactly the same reasoning as in the proof of Proposition 7, one can find $P_2 = (t, y, x, z)$ and $P_3 = (t, y', x', z')$ with $\{y, y'\} \subset Y$ and $\{x, x', z, z'\} \subset X$. Hence since ${}^*P_1$ is in $Y - \{y, y'\}$, t is an origin of C.

4. Suppose that $|P_1| = |P_2| = |P_3| = 4$ and $\delta^-(T) = 4$. By Corollary 3, one of the P_i has first block of length one. So, without loss of generality, we may assume that $b_1(P_1) = 1$.

   - Suppose that $b_1(P_2) \geq 2$. Let x_1, x_2, x_3, x_4 be the vertices of X ordered by increasing number of outneighbours in Y. Let y_4 be an outneighbour of x_4. Let x' be the vertex of $\{x_1, x_2\}$ that is the origin of ${}^{**}P_2$ in $T[x_1, x_2]$. At least two vertices of Y dominate x'. So one of them, say y_1, is distinct from y_4. Then y_1 is an origin of ${}^*P_2$ in $T[y_1, x_1, x_2]$. Let y_3 be a vertex of $Y - [y_1, y_4]$. Setting $T' = T[x_3, x_4, y_3, y_4]$, we have $s^-_{T'}(y_3, y_4) \geq 3$ and by Theorem 2, y_3 or y_4 is an origin of ${}^*P_1$ in T_1. Now, $T - [{}^*P_1, {}^*P_2, t]$ has four vertices, and at least three of them are in Y, so by Corollary 1, a vertex of Y is an origin of ${}^*P_3$ in $T - [{}^*P_1, {}^*P_2, t]$. Hence t is an origin of C in T.
   - Suppose now that $b_1(P_i) = 1$, $i = 1, 2, 3$.
     - Suppose that $b_2(P_1) = 2$ and $b_2(P_2) = 1$. In X, there are two vertices x_1 and x_2 such that $x_1 \leftarrow x_2$ and that x_1 has an outneighbour y_1 in Y. Then ${}^*P_1 = (y_1, x_1, x_2)$. Let $\{x_3, x_4\} = X - \{x_1, x_2\}$, and y_2, y_3 two vertices of Y distinct from y_1. By Theorem 2, in $T[y_2, y_3, x_3, x_4]$, y_2 or y_3 is an origin of ${}^*P_2$. Now, in $T - [{}^*P_1, {}^*P_2, t]$, by Corollary 1, there is a path ${}^*P_3$ with an origin in Y. Hence, t is an origin of C in T.
     - Suppose that $b_2(P_i) = 1$, $i = 1, 2, 3$. In $T_1 = T[x_1, x_2, y_1, y_2]$, y_1 or y_2 is an origin of ${}^*P_1$ by Theorem 2. Let $\{a_1\} = T_1 - [{}^*P_1]$. Again, by Theorem 2, in $T_2 = T[a_1, y_3, y_4, x_3]$, y_3 or y_4 is an origin of ${}^*P_2$. Let $\{a_2\} = T_2 - [{}^*P_2]$. In $T[a_2, x_4, y_5, y_6]$, y_5 or y_6 is an origin of ${}^*P_3$. Thus, t is an origin of C in T.
     - Suppose $b_2(P_i) = 2$, $i = 1, 2, 3$. We examine different cases depending on the number k of vertices of Y having inneighbours in X, and the number k' of vertices of X having outneighbours in Y.
       * If $k = 1$, let y_1 be the unique vertex of Y having an inneighbour in X. Set $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, y_3, y_4, y_5, y_6\}$ with $y_5 \leftarrow y_4$. We have $P_1 = (y_6, x_4, y_5, y_4)$, $P_2 = (y_6, x_3, y_3, t)$. And, by Theorem 2, x_1 or x_2 is an origin of ${}^*P_3$ in $T[x_1, x_2, y_1, y_2]$. Thus y_6 is an origin of C in T.
       * If $k' = 1$, say x_1 is the unique vertex of X that has an outneighbour in Y. Set $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, y_3, y_4, y_5, y_6\}$ with $y_2 \leftarrow y_3$

and $y_4 \leftarrow y_5$. Then we have $P_1 = (y_1, x_2, y_2, y_3)$, $P_2 = (y_1, x_3, y_4, y_5)$ and $P_3 = (y_1, x_4, y_6, t)$, Thus y_1 is an origin of C in T.

$*$ If $k \geq 2$ and $k' \geq 2$. It is easy to see that there exist x_1, x_2, x_3 in X and y_1, y_2 in Y such that $y_1 \leftarrow x_1 \leftarrow x_2$ and $y_2 \leftarrow x_3$. Set $P_1 = (t, y_1, x_1, x_2)$. By Theorem 2, y_2 or y_3 is an origin of ${}^*P_2$ in $T[y_2, y_3, x_3, x_4]$. Finally, by Theorem 3, $T[y_4, y_5, y_6]$ contains ${}^*P_3$. So t is an origin of C in T.

■

Proposition 9 *Every 2-strong tournament of order* 10 *contains every merging of order* 9 *of three outpaths.*

Proof.

Let T be a tournament of order 10 and C merging of order 9 of three outpaths P_1, P_2 and P_3. By Lemmas 3 and 4, we may assume that $\delta^-(T) = 4$ and

1. either $|P_1| = |P_2| = 4$ and $|P_3| = 3$,
2. or $|P_1| = 5, |P_2| = |P_3| = 3$.

Let t be a vertex having minimal indegree. Let $X = N_T^-(t)$ and $Y = N_T^+(t)$. For degree reason there are at least 10 arcs from a vertex in Y to a vertex in X, and at least 5 arcs from a vertex in X to a vertex in Y.

1. Suppose that $|P_1| = |P_2| = 4$ and $|P_3| = 3$.
   - If $b_1(P_1) \geq 2$, let x_1, x_2, x_3, x_4 the vertices of X ordered by increasing number of outneighbours in Y. x_4 has an outneighbour and an inneighbour in Y, so there is a vertex y_4 in Y so that ${}^*P_3 = (y_4, x_4)$. The vertices x_1 and x_2 have at least 2 inneighbours in Y. Thus, there is a vertex y_1 in $Y - y_4$ such that ${}^*P_1 = (y_1, x_1, x_2)$ or ${}^*P_1 = (y_1, x_2, x_1)$. Now $Y - [{}^*P_1, {}^*P_3]$ has four vertices, three of them being in Y. So, by Corollary 1, a vertex in $Y - [{}^*P_1, {}^*P_3]$ is an origin of ${}^*P_2$. Hence t is an origin of C in T.
   - Suppose that $P_3 = +(2)$ and $P_1 = +(1, 2)$. Because T is 2-strong, there is an ingenerator x of X that dominates a vertex $y \in Y$. Let $x' \in X$ be an inneighbour of x. We have $P_1 = (t, y, x, x')$. There is a vertex $x'' \in X - [x, x']$ that has at least two inneighbours in Y. So there is a vertex $y'' \in Y - y$ that dominates x''. Then $P_3 = (t, y'', x'')$. Now in $T - [P_1, P_3]$, by Corollary 1, there is a path ${}^*P_2$ with origin in $Y - [y, y'']$. Thus t is an origin of C in T.

- Suppose that $P_3 = +(2)$ and $P_1 = +(1,1,1)$. If there are three vertices $x, x' \in X$ and $y \in Y$ such that $x \to \{x', y\}$, we obtain the result as previously. Thus we may assume that there is a vertex $x \in X$ such that $X - x \to x$ and $X - x \leftarrow Y$. Now, by Theorem 3, Y contains a path P_2 with origin y. And y is also an origin of $P_3 = (y, x_1, x)$ and $P_1 = (y, x_2, y', x_3)$ where $X - x = \{x_1, x_2, x_3\}$ and y' is the sole vertex of $Y - P_2$. Thus y is the origin of C in T.
- Suppose that $P_3 = +(1,1)$.

  By looking at vertices of Y with minimum indegree in Y, it is simple matter to prove that there are $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ such that $x_1 \to y_1$ and $x_2 \to y_2$. Moreover the 2-strongness of T implies that there is a vertex x_3 such that $x_3 \to x_2$. Then $P_3 = (t, y_1, x_1)$.

  If $P_1 = +(1,2)$, then $P_1 = (t, y_2, x_2, x_3)$. Moreover, according to Corollary 1, there is a vertex of $Y - [y_1, y_2]$ which is an origin of ${}^*P_2$ in $T - [P_1, P_3]$. Thus t is an origin of C in T.

  In the same way, we have the result, if $P_2 = +(1,2)$, so we may assume that $P_1 = P_2 = +(1,1,1)$.

  Let y_3, y_4 and y_5 be the vertices of $Y - [y_1, y_2]$. By Theorem 2, in $T[X - x_1, y_2, y_3]$, y_2 or y_3 is an origin of ${}^*P_1$, and in $T - [{}^*P_1, P_3]$, y_4 or y_5 is an origin of ${}^*P_2$. Thus t is an origin of C in T.

2. Suppose that $|P_1| = 5, |P_2| = |P_3| = 3$.

   - If $b_1(P_1) \geq 2$. Find ${}^{**}P_1 = (x_1, x_2, x_3)$ in X. Every vertex in X has an inneighbour in Y, so let $y_1 \in Y$ such that ${}^*P_1 = (y_1, x_1, x_2, x_3)$. then by Theorem 3, in $Y - y_1$ we may find ${}^*P_2 \sqcup {}^*P_3$.
   - If $b_1(P_1) = 1$. Let x_1, x_2, x_3, x_4 the vertices of X ordered by increasing number of outneighbours in Y.
     - If $P_2 = +(2)$, let y_4 be a vertex of $N^{=}(x_3) \cap Y$ and y_5 a vertex of $N^+(x_4) \cap Y \setminus \{y_4\}$. Then $s^-_{T'}(y_4, y_5) \geq 4$ with $T' = \{x_2, x_3, x_4, y_4, y_5\}$ so that T' contains ${}^*P_1$ by Lemma 2. Then x_1 has at least 3 inneighbours in Y, so it has an inneighbour y_1 in $Y \setminus \{y_4, y_5\}$, take ${}^*P_2 = (y_1, x_1)$. Then in $Y - [P_1, P_2]$ we may find ${}^*P_3$ by Theorem 3.
     - If $b_1(P_1) = b_1(P_2) = 1$. As in the case $|P_1| = |P_2| = 4$, $|P_3| = 3$, $P_3 = +(1,1)$, we may find x_1, x_2, x_3 in X and y_1, y_2 in Y such that $y_2 \leftarrow x_2 \leftarrow x_3$ and $y_1 \leftarrow x_1$. By Theorem 2, in $T' = T[y_2, y_3, x_2, x_3, x_4]$, y_2 or y_3 is an origin of ${}^*P_1$. Then set $P_2 = (t, y_1, x_1)$, and find ${}^*P_3$ in $T[y_4, y_5]$.

■

Proposition 10 *Every 2-strong tournament of order* 9 *contains every merging of order* 8 *of three outpaths.*

Proof.

Let T be a tournament of order 9 and C a merging of order 8 of three outpaths P_1, P_2 and P_3. By Lemmas 5 and 6, we may assume that $\delta^-(T) = 4$. So for every vertex $v \in T$, $d_T^-(v) = d_T^+(v) = 4$. By Lemmas 3 and 4, we may assume that one of the following assertions holds:

1. $|P_1| = |P_2| = 4$ and $|P_3| = 2$;
2. $|P_1| = 5$, $|P_2| = 3$ and $|P_3| = 2$;
3. $|P_1| = 4$ and $|P_2| = |P_3| = 3$.

Let t be a vertex and set $X = N_T^-(t)$ and $Y = N_T^+(t)$.

1. Suppose that $|P_1| = |P_2| = 4$ and $|P_3| = 2$. Without loss of generality, we may assume that $b_1(P_1) \leq b_1(P_2)$.

   - Assume first that $b_1(P_1) = 1$ and $b_1(P_2) \geq 2$. Let $\{y_1, y_2, y_3, y_4\}$ be the vertices of Y ordered by increasing outdegree in Y. Then $d_Y^-(y_4) \leq 1$, so y_4 has at least 2 inneighbours x_3 and x_4 in X. Then $P_1 = (t, y_4, x_3, x_4)$ or $P_1 = (t, y_4, x_4, x_3)$. Moreover, $d_Y^+(y_1) \leq 1$, so y_1 has an outneighbour in $\{x_1, x_2\} = X \setminus \{x_3, x_4\}$. We have $s_{T'}^+(y_1, y_2) \geq 3 \geq b_1(^*P_2) + 1$, with $T' = \{y_1, y_2, x_1, x_2\}$. Then by Theorem 2 y_1 or y_2 is an origin of $^*P_2$ in T'. Finally we have $^*P_3 = (y_3)$. So t is an origin of C in T.

   - Assume now that $b_1(P_1) \geq 2$. Set $Y = \{y_1, y_2, y_3, y_4\}$ with y_1 having minimum outdegree in Y. The vertex y_1 has at least 2 outneighbours say x_1 and x_2 in X, so that $P_1 = (t, y_1, x_1, x_2)$ or $P_1 = (t, y_1, x_2, x_1\}$. At least one vertex in $\{x_3, x_4\} = X \setminus \{x_1, x_2\}$ has an inneighbour, say y_2, in Y. Let $T' = T[y_2, y_3, x_3, x_4]$. We have $s_{T'}^+(y_2, y_3) \geq 3$. Thus, by Theorem 2 y_2 or y_3 is an origin $^*P_2$ in T'. Finally, we have $P_3 = (t, y_4)$. So t is an origin of C in T.

   - Assume that $b_1(P_1) = b_1(P_2) = 1$. It is easy to prove that we can find $\{y_2, y_4\}$ in Y and $\{x_2, x_3, x_4\}$ in X so that $y_4 \rightarrow \{x_3, x_4\}$ and $y_2 \rightarrow x_2$. Then either $P_1 = (t, y_4, x_3, x_4)$ or $P_1 = (t, y_4, x_4, x_3)$. Let $T' = T[y_1, y_2, x_1, x_2]$. We have $s_{T'}^-(y_1, y_2) \geq 3$. Hence, by Theorem 2, y_1 or y_2 is an origin of $^*P_2$ in T'. Finally, since $^*P_3 = (y_3)$, t is an origin of C in T.

2. Suppose that $|P_1| = 5, |P_2| = 3$ and $|P_3| = 2$. If $b_1(P_1) \geq 2$, there is a vertex x which is an origin of $^{**}P_1$ in X and which has an inneighbour y in Y. Now, by Corollary 2, $Y - y$ contains $^*P_2 \sqcup\ ^*P_3$ thus t is an origin of C. If $b_1(P_1) = 1$, by Corollary 1 at least two vertices in X are origin of $^{**}P_1$ in X, and for degree reason, one of these origins, say x, has an outneighbour y in Y. Then we conclude as previously.

3. Suppose that $|P_1| = 4$ and $|P_2| = |P_3| = 3$. In $X - x_4$, there are two vertices (say x_1 and x_2) of indegree in X at most two. For $i = 1, 2$, x_i has an inneighbour y_i in Y (possibly $y_1 = y_2$).

   Assume first that $b_1(P_1) \geq 2$. Then $P_1 = (t, y_1, x_1, x_2)$ or $P_1 = (t, y_2, x_2, x_1)$. By Corollary 3, we may suppose that $b_1(P_2) = 1$, so $P_2 = (t, y_4, x_4)$. Finally, $Y - [P_1, y_4]$ contains $^*P_3$. So t is an origin of C.

   Assume now that $b_1(P_1) = 1$. It is easy to check that there are $\{y_2, y_4\}$ in Y and $\{x_2, x_3, x_4\}$ in X so that $y_4 \leftarrow \{x_3, x_4\}$ and $y_2 \leftarrow x_2$. Thus, $P_1 = (t, y_4, x_3, x_4)$ or $P_1 = (t, y_4, x_4, x_3)$. Moreover, we can find $x \in \{x_1, x_2\}$ et $y \in Y - y_4$ with $y \rightarrow x$. If $P_2 = +(2)$, take $P_2 = (t, y, x)$ and $^*P_3$ in $Y - [y, y_4]$. In the same way, we have the result if $P_3 = +(2)$. Henceforth, we may suppose that $P_2 = P_3 = +(1, 1)$. Then take $P_2 = (t, y_2, x_2)$, and $^*P_3 = (y_1, y_3)$ or $^*P_3 = (y_3, y_1)$. Hence t is an origin of C in T. ■

Proof of Theorem 1: Propositions 3 to 10 together with Theorem 8 shows that Theorem 1 holds if T is a 2-strong tournament. Then, using Theorem 4, Corollary 3 and Theorem 7, we prove that Theorem 1 holds if T is not a 2-strong tournament, which completes the proof. ■

8 Sumner's conjecture for trees with four leaves

The following assertions derive directly from Theorem 1 and Theorem 3:

Corollary 5 *Let Q_1 and Q_2 be two paths of order n_1 and n_2 respectively and D a tree of order n_3 with three leaves.*
a) $D \sqcup Q_1 \sqcup Q_2$ is $(n_1 + n_2 + n_3)$-unavoidable.
b) If D is not a star then $D \sqcup Q_2$ is $(n_2 + n_3 + 1)$-unavoidable.
c) If D is not a star and Q_2 is not a Grünbaum's path then $D \sqcup Q_2$ is $(n_2 + n_3)$-unavoidable.

We will now prove that Sumner's conjecture holds for trees with four leaves :

Theorem 9 *Every tree of order n with four leaves is $(2n - 2)$-unavoidable.*

Proof. Let T be a tournament of order $2n - 2$ and A a tree of order n.

Essentially, there are two kinds of trees with four leaves: the ones having two nodes (vertices with sum of out- and indegree 3) and the merging of four paths with origin x.

I) Suppose first that A is a merging of four paths. $A = P_1 \vee P_2 \vee P_3 \vee P_4$. For $1 \leq i \leq 4$, set $n_i = |P_i| - 1$. By duality, we may suppose that P_1 and P_2 are outpaths. If P_3 and P_4 are also outpaths, let t be a vertex with outdegree at least $n - 1$. By Corollary 2, one can find $\bigsqcup_{i=1}^{4} {}^*P_i$ in $T[N^+(t)]$. Thus t is an origin of A in T. If P_3 and P_4 are both inpaths, let us pick a vertex t with outdegree at least n_1+n_2 and indegree at least n_3+n_4. Such a vertex exists because there are at most $2(n_1+n_2)-1$ vertices with outdegree less than n_1+n_2 and at most $2(n_3+n_4)-1$ vertices with indegree less than n_3+n_4. By Corollary 2, one can find $^*P_1 \sqcup {}^*P_2$ in $T[N^+(t)]$ and $^*P_3 \sqcup {}^*P_4$ in $T[N^-(t)]$. Thus t is an origin of A in T. Suppose now that P_3 is an outpath and P_4 is an inpath. Let t be a vertex with outdegree at least $n_1+n_2+n_3$ and indegree at least n_4. By Corollary 2, one can find $^*P_1 \sqcup {}^*P_2 \sqcup {}^*P_3$ in $T[N^+(t)]$. Consider now a subtournament T' of $T-[t,^* P_1, {}^*P_2, {}^*P_3]$ of order n_4+1 containing at least n_4 inneighbours of t. Then by Corollary 1, an inneighbour of t is an origin of $^*P_4$ in T'. Hence, t is an origin of A in T.

II) Suppose now that A has two nodes x and x'. A is the union of two paths P_1 and P_2 with origin x and a tree C with three leaves (one of them being x) with origin x'. Let y be the vertex adjacent to x in C. By directional duality, we may suppose that $x \to y$. Let $Q_1 = P_1 - x$, $Q_2 = P_2 - x$ and $D = C - x$ and let n_1, n_2 and n_3 be their respective orders.

A) Suppose that P_1 and P_2 are both outpaths. Let t be a vertex of outdegree at least $n-1$ in T. By Corollary 5, $Q_1 \sqcup Q_2 \sqcup D$ is contained in $T[N^+(t)]$, so A is contained in T.

   Note that if x' has out- or indegree 3 in A, we obtain the result analogously. Therefore, we may now suppose that C is not a star.

B) Suppose that P_1 is an inpath and P_2 is an outpath.

   – Suppose first that Q_1 is not a Grünbaum's path. If there exists a vertex t in T such that $d^-(t) \geq n_1$ and $d^+(t) \geq n_2+n_3+1$, then T contains A because Q_1 is n_1-unavoidable and $Q_2 \sqcup D$ is (n_2+n_3+1)-unavoidable. Therefore we may assume that the vertex set of T is partitioned in two subsets $V_1 = \{v, d^-(v) \leq n_1\}$ and $V_2 = \{v, d^+(v) \leq n_2+n_3\}$. Note that $|V_1| \leq 2n_1 - 1$ and $|V_2| \leq 2(n_2+n_3)+1$. Since $|V_1|+|V_2| = 2n-2 = 2(n_1+n_2+n_3)$, we have $|V_1| = 2n_1-1$ and $|V_2| = 2(n_2+n_3)+1$. Thus $V_1 \to V_2$.
   Now, by Theorem 1, P_1 is contained in $T[V_1]$ and by Corollary 5, $Q_2 \sqcup D$ is contained in $T[V_2]$. Thus T contains A.
   Analogously, we have the result if Q_2 is not a Grünbaum's path. Indeed this implies that $Q_2 \sqcup D$ is (n_2+n_3)-unavoidable, and Q_1 is (n_1+1)-unavoidable.

   – Suppose now that Q_1 and Q_2 are Grünbaum's paths. If there exists a vertex t in T such that $d^-(t) \geq n_1+1$ and $d^+(t) \geq n_2+n_3+1$, then T contains A because Q_1 is (n_1+1)-unavoidable and $Q_2 \sqcup D$ is (n_2+n_3+1)-unavoidable. Therefore we may assume that the vertex set of T is partitioned in two subsets $V_1 = \{v : d^-(v) \leq n_1\}$ and $V_2 = \{v : d^+(v) \leq n_2+n_3\}$. Note that $|V_1| \leq 2n_1+1$ and $|V_2| \leq 2(n_2+n_3)+1$.

As $|V_1| + |V_2| = 2(n_1 + n_2 + n_3)$, we are in one of the three following cases:
Case 1: $|V_1| = 2n_1 + 1$ and $|V_2| = 2(n_2 + n_3) - 1$. Since $|V_1| = 2n_1 + 1$, $T[V_1]$ is a regular tournament and $V_1 \to V_2$. Now, by Corollary 1, P_1 is contained in $T[V_1]$ and by Corollary 5, $Q_2 \sqcup D$ is contained in $T[V_2]$. Thus T contains A.
Case 2: $|V_1| = 2n_1 - 1$ and $|V_2| = 2(n_2+n_3)+1$. Since $|V_2| = 2(n_2+n_3)+1$, $T[V_2]$ is a regular tournament and $V_1 \to V_2$. Now, by Corollary 1, Q_1 is contained in $T[V_1]$ and by Theorem 1, $A - Q_1$ is contained in $T[V_2]$. Thus T contains A.
Case 3: $|V_1| = 2n_1$ and $|V_2| = 2(n_2 + n_3)$. Suppose that $n_1 \leq n_2 + n_3$. In V_1 one can find P_1 (because $2n_1 \geq n_1 + 2$) with origin t. Now t has at most n_1 inneighbours, one of which is in P_1 and so in V_1. Thus $|N^+(t) \cap V_2| \geq 2(n_2 + n_3) - n_1 + 1 \geq n_2 + n_3 + 1$. Hence, one can find $P_2 \sqcup D$ in $T[N^+(t) \cap V_2]$ and T contains A.

Analogously, we have the result if $n_1 \geq n_2 + n_3$, by first finding C in V_2 and Q_1 in $N^-(x) \cap V_1$.

C) Suppose now that P_1 and P_2 are both inpaths. If there exists a vertex t in T such that $d^-(t) \geq n_1 + n_2$ and $d^+(t) \geq n_3 + 1$, then T contains A because $Q_1 \sqcup Q_2$ is $(n_1 + n_2)$-unavoidable and D is $(n_3 + 1)$-unavoidable. Therefore we may assume that the vertex set of T is partitioned in two subsets $V_1 = \{v : d^-(v) \leq n_1 + n_2\}$ and $V_2 = \{v : d^+(v) \leq n_3\}$. As previously, we have $|V_1| = 2(n_1 + n_2) - 1$ and $|V_2| = 2n_3 + 1$. Thus $V_1 \to V_2$. Now, by Theorem 1, C is contained in $T[V_2]$ and by Corollary 5, $Q_2 \sqcup Q_1$ is contained in $T[V_1]$. Thus T contains A. ■

It would be nice to prove Conjecture 3 for trees with three leaves. In particular, it is likely that it could be showed using the same scheme of proof for mergings of four paths. Indeed, using a result of [2], Corollary 1 and Theorem 1, we can prove that a merging of two outpaths and two inpaths or of three oupaths and one inpath is $(n+2)$-unavoidable. Hence, to prove Conjecture 3 for merging of outpaths, we only need to prove it for merging of four outpaths.

In the same way as for mergings of three outpaths, one can prove that a merging of order n of four outpaths is contained in every reducible tournament of order $n+1$ and then in every non 2-strong tournament of order $n+2$.

Moreover, identically to Lemma 4, one can prove the following result:
Let T be a 2-strong tournament of order $n+1$ and $C = P_1 \vee P_2 \vee P_3 \vee P_4$ a merging of order n of four outpaths. If $|P_1| + |P_2| - 5 \geq \delta^-(T)$ and $b_1(P_1) = b_1(P_2) = 1$ then T contains C.
According to this result and Theorem 3, it only remains to prove that a merging D of order n of the four paths having the same length are contained in every 2-strong tournament of order $n+2$ unless D is a star.

References

[1] R. Häggkvist and A. G. Thomason. Trees in tournaments. *Combinatorica*, 11:123–130, 1991.

[2] F. Havet. Trees in tournaments. *Discrete Math.* to appear, 2001.

[3] F. Havet. On unavoidability of trees with k leaves. Submitted.

[4] F. Havet and S. Thomassé. Oriented hamiltonian paths in tournaments: a proof of Rosenfeld's conjecture. *J. Combinatorial Theory Ser. B*, 78:243–273, 2000.

[5] F. Havet and S. Thomassé. Median order: a tool for the second neighborhood problem and Sumner's conjecture. *J. Graph Theory*, 35(4):244–256, 2000.

[6] X. Lu. On avoidable and unavoidable trees. *J. Graph Theory*, 22(4):335–346, 1996.

[7] X. Lu, D. Wang, and C. K. Wong. On avoidable and unavoidable claws. *Discrete Math.*, 184(1-3):259–265, 1998.

[8] K. B. Reid and N. C. Wormald. Embedding oriented n-trees in tournaments. *Studia Sci. Math. Hungaria*, 18:377–387, 1983.

[9] A. G. Thomason. Paths and cycles in tournaments. *Trans. Amer. Math. Soc.*, 296(1):167–180, 1986.