Channel assignment and multicolouring of the induced subgraphs of the triangular lattice.

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Abstract

A basic problem in the design of mobile telephone networks is to assign sets of radio frequency bands (colours) to transmitters (vertices) to avoid interference. Often the transmitters are laid out like vertices of a triangular lattice in the plane. We investigate the corresponding colouring problem of assigning sets of colours of given size k to vertices of the triangular lattice so that the sets of colours assigned to adjacent vertices are disjoint. We prove here that every triangle-free induced subgraph of the triangular lattice is $\lceil \frac{7k}{3} \rceil$-$[k]$colourable. That means that it is possible to assign to each transmitter of such a network, k bands of a set of $\lceil \frac{7k}{3} \rceil$, so that there is no interference.

1 Introduction.

A basic problem in the design of mobile telephone networks is to assign sets of radio frequency bands (colours) to transmitters (vertices) to avoid interference. Here we consider that the number k of bands demanded at each transmitter is the same. We assume that the transmitters are located like vertices in a triangular lattice in the plane: this pattern is often used as it gives a good coverage. We assume also that adjacent vertices must not be assigned the same band, so as to avoid interference. We investigate the corresponding colouring problem of assigning sets of colours of given size k to vertices of the triangular lattice so that the sets of colours assigned to adjacent vertices are disjoints. There are more refined versions of this 'channel assignment problem', see for example [4], in which the number of bands demanded at transmitter may vary between transmitters, or [2, 3], in which we insist on a minimum separation between channels assigned to two transmitters (where this minimum separation depends on the proximity of the transmitters). But we consider only the most basic case here.

The channel assignment problem described above is a 'multicolouring' problem on the triangular lattice. Let us denote the set $\{1, 2, \ldots, n\}$ by $[1, n]$. A n-$[k]$*colouring* of a graph G is an application c from $V(G)$ into the set of the k-subset of $[1, n]$ such that for any adjacent vertices u and v, $c(u) \cap c(v) = \emptyset$. A graph is n-$[k]$*colourable* if it has a n-$[k]$colouring. The $[k]$*chromatic number* of a graph G, $\chi_k(G)$, is the smallest integer n such that G is n-$[k]$colourable. If $\chi_k(G) = n$, we say that G is n-$[k]$*chromatic*.

The [1]colouring is the usual colouring: to each vertex, we associate a colour in such a way that two adjacent vertices become different colours. In this paper, we call [2]colourings *bicolourings*, and [3]colourings *tricolourings*.

It is clear that $\chi_{k+k'}(G) \le \chi_k(G) + \chi_{k'}(G)$ $(\star)$. Moreover, n-$[pk]$colour a graph G is equivalent to n-$[p]$colour the graph G_k, obtained from G by blowing up each vertex with a clique on k vertices. This yields $\chi_{pk}(G) = \chi_p(G_k)$. In particular, $(K_p)_k = K_{pk}$. Then the triangle is $3k - [k]$chromatic.

Let C_{2m+1} be the cycle of order $2m+1$. It is easy to see that $\chi_k(C_{2m+1}) = \lceil \frac{2m+1}{m} k \rceil > 2k$.

We are interested in the $[k]$chromaticity of an induced subgraph of the triangular lattice, as this corresponds precisely to the basic channel assignment problem described above. This lattice graph may be described as follows. The vertices are all integer linear combinations $a\mathbf{p} + b\mathbf{q}$ of the two vectors $\mathbf{p} = (1, 0)$ and $\mathbf{q} = (\frac{1}{2}, \frac{\sqrt{3}}{2})$: thus we may identify the vertices with the pairs (a, b) of integer. Two vertices are adjacent when the Euclidean distance between them is 1. Thus each vertex $x = (a, b)$ has the six neighbours: its *left neighbour* $(a-1, b)$, its *right neighbour* $(a+1, b)$, its *leftup neighbour* $(a-1, b+1)$, its *rightup neighbour* $(a, b+1)$, its *leftdown neighbour* $(a, b-1)$ and its *rightdown neighbour* $(a+1, b-1)$.

It is easy to see that the triangular lattice is $3k - [k]$chromatic. Then any its subgraph is $3k - [k]$colourable. If it contains a triangle, then it is $3k - [k]$chromatic and if it is bipartite, it is $2k - [k]$chromatic.

So we just need to study the $[k]$chromaticity of non-bipartite triangle-free induced subgraphs of the triangular lattice. Let G be such a subgraph. It is easy to see that G contains no cycle of order 4, 5 or 7. So lettering $f(k)$, the maximum $[k]$-chromatic number of such a G, we have $\left\lceil \frac{9k}{4} \right\rceil \le f(k) \le 3k$.

Conjecture 1 (McDiarmid, Reed)
Every triangle-free induced subgraph of the triangular lattice is $\lceil \frac{9k}{4} \rceil$*-*$[k]$*colourable, i.e*

$$f(k) = \left\lceil \frac{9k}{4} \right\rceil$$

In this paper, we prove first that every triangle-free induced subgraph of the triangular lattice is 5-bicolourable (Theorem 1) and in the last section that every triangle-free induced subgraph of the triangular lattice is 7-tricolourable (Theorem 3). This implies that $f(k) \le \lceil \frac{7k}{3} \rceil$.

2 5-bicolouring of the triangle-free induced subgraphs of the triangular lattice.

Definition 1 Let G be a triangle-free induced subgraph of the triangular lattice. Each vertex of G is of degree at most three. The *nodes* of G are its vertices of degree three. There are two kinds of nodes: the *left node* whose neighbours are its left, rightup and rightdown neighbour, and the *right node*, whose neighbours are its right, leftup and leftdown neighbours.

Let $x = (a, b)$ and $y = (a', b')$ be two vertices of G, x is *upper* than y if $b > b'$. A node x is an *upmost node* of G if there is no node y of G that is upper than x. A *handle* of G is a subpath A of G such that its endvertices are nodes and its interior vertices of degree two. The set of the interior vertices of a handle A is denoted by $\dot{A}$.

Lemma 1 *Let* $P = (x_0, x_2, \ldots, x_m)$ *be a path of length* $m \ge 4$ *and* c_0 *and* c_m *two 2-subsets of* $[1, 5]$*. There exists a 5-bicolouring* C *of* P *such that* $C(x_0) = c_0$ *and* $C(x_m) = c_m$*.*

Proof. By induction. If $m = 4$, we have to consider three cases: If $c_0 = c_4 = \{1, 2\}$, let us take $C(x_1) = C(x_3) = \{3, 4\}$ and $C(x_2) = c_0$. If $c_0 = \{1, 2\}$ and $c_4 = \{1, 3\}$, let us take $C(x_1) = C(x_3) = \{4, 5\}$ and $C(x_2) = c_0$. If $c_0 = \{1, 2\}$ and $c_4 = \{3, 4\}$, let us take $C(x_1) = \{4, 5\}$, $C(x_2) = \{2, 3\}$ and $C(x_3) = \{1, 5\}$.

Suppose that it is true for $m-1$. Let c_{m-1} be a two subset of $[1,5]$ disjoint from c_m. By induction hypothesis, there exists a 5-bicolouring C of $(x_0, x_2, \dots, x_{m-1})$ such that $C(x_0) = c_0$ and $C(x_{m-1}) = c_{m-1}$. This gives a 5-bicolouring of P. ∎

Remark: This immediately implies that all cycles of order at least four are 5-bicolourable.

Theorem 1 *Each triangle-free induced subgraph of the triangular lattice is* 5*-bicolourable.*

Proof. By induction on the number of vertices. It is clearly true for a single vertex.

Let G be a triangle-free induced subgraph of the triangular lattice. We clearly may suppose that G is connected without vertices of degree 1. If G is a cycle the previous remark yields the result. So we may suppose that G has nodes. Let x be one of the upmost nodes of G. By symmetry, we may suppose that x is a left node. Let A be the handle containing the rightup neighbour x_1 of G and let y be the endvertex of A distinct of x. Since G is triangle-free and y is not upper than x, A is of length at least three. By induction, $G - \dot{A}$ admits a 5-bicolouring C. If A is of length at least four, by Lemma 1, we may extend C to G. If A is of length three, then we are in the configuration of Figure 1, and there exists a path (x, x_3, x_4, y) in $G - \dot{A}$. (This may not be a handle). Setting $C(x_3) = C(x_1)$ and $C(x_4) = C(x_2)$, we obtain a 5-bicolouring of G. ∎

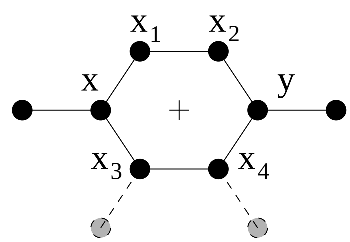

Figure 1: Handle A of length 3.

Corollary 1.1 *Every triangle-free induced subgraph of the triangular lattice is* $\lceil \frac{5k}{2} \rceil$*-*$[k]$*colourable. Thus* $f(k) \leq \left\lceil \frac{5k}{2} \right\rceil$.

We will now prove a generalization of Theorem 1.

Definition 2 Let H be a subgraph (not necessarly induced) of the triangular lattice and x be a vertex of degree three (a node) of H. We say that x is a *good node* if its neighbourhood and itself does not induced any triangle in the triangular lattice. A good node is either a left node or a right node described in Definition 1.

Theorem 2 *Let H be a triangle-free subgraph of the triangular lattice such that each vertex has degree at most three and the vertices of degree three are good nodes. Then H is 5-bicolourable.*

To prove this theorem, we need the following lemma:

Lemma 2 *Let $P = (a_1, a_2, a_3, a_4)$ be a path of length 3 and c_1 and c_4 two 2-subsets of* $[1,5]$.

i) There exists a 5-bicolouring C of P such that $C(a_1) = c_1$ and $C(a_4) = c_4$ if and only if $c_1 \neq c_4$.

ii) If $c_1 \neq c_4$, there exist two 5-bicolourings C and C' of P such that $C(a_1) = C'(a_1) = c_1$, $C(a_4) = C'(a_4) = c_4$ and $C(a_2) \neq C'(a_2)$.

Proof. We should check the statement when $c_1 \cap c_4 = \emptyset$ and $|c_1 \cap c_4| = 1$.

If $c_1 \cap c_4 = \emptyset$, say $c_1 = \{1,2\}$ and $c_4 = \{3,4\}$, then set $C(a_2) = \{3,4\}$ and $C(a_3) = \{1,2\}$, and $C'(a_2) = \{3,5\}$ and $C'(a_3) = \{1,2\}$.

If $|c_1 \cap c_4| = 1$, say $c_1 = \{1,2\}$ and $c_4 = \{1,3\}$, then set $C(a_2) = \{3,4\}$ and $C(a_3) = \{2,5\}$, and $C'(a_2) = \{3,5\}$ and $C'(a_3) = \{2,4\}$.

If $c_1 = c_4$, say $c_1 = \{1,2\}$, suppose, by way of contradiction, that there is a 5-bicolouring C of P with $C(a_1) = C(a_4) = \{1,2\}$. Then both $C(a_2)$ and $C(a_3)$ must contain two colours of $\{3,4,5\}$. Thus $C(a_2)$ and $C(a_3)$ have a colour in common, which is a contradiction. ■

Proof of Theorem 2. By induction on the number of vertices. As in Theorem 1, we may suppose that each vertex of H has degree at least two and there exist nodes. Let x be one of the upmost nodes. Without loss of generality, we may suppose that x is a left node and there is no left node to the right of x. Let A be the handle containing the rightup neighbour x_1 of H and let y be the endvertex of A distinct of x. By induction, $H - \dot{A}$ admits a 5-bicolouring C. If A is of length at least four, Lemma 1 yields a 5-bicolouring of H. If A is of length 2, the rightdown neighbour y_1 of x is the leftdown neighbour of y. Setting $C(x_1) = C(y_1)$, we have a 5-bicolouring of H. If A has length 3, let y_1 be the rightdown neighbour of x. If y_1 is a node, we are in the configuration of Figure 1 and we have the result in the same way as Theorem 1. If y_1 is not a node, let y_2 be its neighbour distinct from x. By Lemma 2 ii), there exists a 5-bicolouring C' of $H - \dot{A}$ such that $C'(x) \neq C'(y)$. Then by Lemma 2 i), we extend C' in a 5-bicolouring of H. ■

3 7-tricolouring of the triangle-free induced subgraphs of the triangular lattice.

In this section, we show that every triangle-free induced subgraph of the triangular lattice satisfies $\chi_3(G) \leq 7$. Here, we prove some preliminary lemmas that permit us to extend a 7-tricolouring of a graph to a bigger graph.

3.1 The extension lemmas.

Lemma 3 *Let $P = (a_1, a_2, a_3, a_4, a_5)$ be a path of length 4 and c_1 and c_5 two 3-subsets of $[1,7]$. There exists a 7-tricolouring C of P such that $C(a_1) = c_1$ and $C(a_5) = c_5$ if and only if $c_1 \cap c_5 \neq \emptyset$.*

Proof. If $|c_1 \cap c_5| = 1$, say $c_1 = \{1,2,3\}$ and $c_5 = \{1,4,5\}$, set $C(a_2) = \{5,6,7\}$, $C(a_3) = \{1,2,4\}$ and $C(a_4) = \{3,6,7\}$.

If $|c_1 \cap c_5| = 2$, say $c_1 = \{1,2,3\}$ and $c_5 = \{1,2,4\}$, set $C(a_2) = \{4,5,6\}$, $C(a_3) = \{1,2,3\}$ and $C(a_4) = \{5,6,7\}$.

If $c_1 = c_5$, say $c_1 = \{1,2,3\} = c_5$, set $C(a_2) = \{4,5,6\}$, $C(a_3) = \{1,2,3\}$ and $C(a_4) = \{4,5,6\}$.

If $c_1 \cap c_5 = \emptyset$, say $c_1 = \{1,2,3\}$ and $c_5 = \{4,5,6\}$ suppose, by way of contradiction, that there exists a 7-tricolouring of P such that $C(a_1) = c_1$ and $C(a_5) = c_5$. Then $C(a_2)$ and $C(a_4)$ may only have the colour 7 in common. Thus $|C(a_2) \cup C(a_4)| \geq 5$. Therefore $C(a_3)$ contains a colour of $C(a_2) \cup C(a_4)$, which is a contradiction. ■

Lemma 4 *Let $P = (a_1, a_2, a_3, a_4, a_5, a_6)$ be a path of length 5 and c_1 and c_6 two 3-subsets of $[1,7]$. There exists a 7-tricolouring C of P such that $C(a_1) = c_1$ and $C(a_6) = c_6$ if and only if $c_1 \neq c_6$.*

Proof. There exists a 3-subset c_5 of $[1,7]$ such that $c_5 \cap c_1 \neq \emptyset$ and $c_5 \cap c_6 = \emptyset$ if and only if $c_1 \neq c_6$. Lemma 3 yields the result. ■

Lemma 5 *Let $P = (a_1, a_2, \ldots, a_m)$ be a path of length $m-1 \geq 6$ and c_1 and c_m two 3-subsets of $[1,7]$. Then there exists a 7-tricolouring C of P such that $C(a_1) = c_1$ and $C(a_m) = c_m$.*

Proof. By induction. If $m = 7$, there exists a 3-subset c_6 of $[1,7]$ such that $c_6 \neq c_1$ and $c_7 \cap c_6 = \emptyset$. Lemma 4 yields the result. If $m \geq 8$, let us take c_{m-1} such that $c_{m-1} \cap c_m = \emptyset$. The induction hypothesis yields the result. ■

Lemma 6 *Let H be a graph having a path of length three, (a_1, a_2, a_3, a_4), whose interior vertices have degree 2. If H admits a 7-tricolouring C then there exists a 7-tricolouring C' of H such that $C(x) = C'(x)$, for $x \notin \{a_2, a_3\}$, and $C'(a_2) \neq C(a_2)$.*

Proof. We neccessarily have $|C(a_1) \cap C(a_4)| \leq 1$. If $|C(a_1) \cap C(a_4)| = 0$, say $C(a_1) = \{1,2,3\}$ and $C(a_4) = \{4,5,6\}$, then with $C(a_3) = C(a_1)$, we may have $C(a_2) = \{4,5,6\}$ or $C(a_2) = \{4,5,7\}$. If $|C(a_1) \cap C(a_4)| = 1$, say $C(a_1) = \{1,2,3\}$ and $C(a_4) = \{3,4,5\}$, we may have $C(a_2) = \{4,5,6\}$ and $C(a_3) = \{1,2,7\}$ or $C(a_2) = \{4,5,7\}$ and $C(a_3) = \{1,2,6\}$. ■

Lemma 7 *Let H be a graph having a path of length four, $(a_1, a_2, a_3, a_4, a_5)$, whose interior vertices have degree 2. If H admits a 7-tricolouring C then for any 3-subset d of $[1,7]$, there exists a 7-tricolouring C' of H such that $C(x) = C'(x)$, for $x \notin \{a_2, a_3, a_4\}$, and $d \cap C'(a_3) \neq \emptyset$.*

Proof.

By Lemma 3, we have $C(a_1) \cap C(a_5) \neq \emptyset$.

- If $C(a_1) = C(a_5) = \{1,2,3\}$.

  $C_1 : \{1,2,3\}, \{4,5,7\}, \{1,2,6\}, \{4,5,7\}, \{1,2,3\}$.

  If 1, 2 or $6 \in d$, C_1 gives the result. Interchanging 4 and 6 (or 5 and 6) in C_1, we have the result if $4 \in d$ or $5 \in d$.

- If $C(a_1) = \{1,2,3\}$ and $C(a_5) = \{1,2,4\}$.

  $C_2 : \{1,2,3\}, \{4,5,7\}, \{1,2,6\}, \{3,5,7\}, \{1,2,4\}$.

  If 1, 2 or $6 \in d$, C_2 gives the result. Interchanging 5 or 7 with 6 in C_2, we have the result if $5 \in d$ or $7 \in d$.

- If $C(a_1) = \{1,2,3\}$ and $C(a_5) = \{1,4,5\}$.

  $C_3 : \{1,2,3\}, \{5,6,7\}, \{1,2,4\}, \{3,6,7\}, \{1,4,5\}$.

  If 1, 2 or $4 \in d$, C_3 gives the result. Interchanging 4 and 5 in C_3, we have the result if $5 \in d$ and interchanging 2 and 3 in C_3, we have the result if $3 \in d$. ■

Lemma 8 *Let H be a graph having a path of length five, $(a_1, a_2, a_3, a_4, a_5, a_6)$, whose interior vertices have degree 2. If H admits a 7-tricolouring C then for any 3-subsets d, d' of $[1,7]$, there exists a 7-tricolouring C' of H such that $C(x) = C'(x)$, for $x \notin \{a_2, a_3, a_4, a_5\}$, $d \cap C'(a_3) \neq \emptyset$ and $d' \cap C'(a_4) \neq \emptyset$.*

Proof. By Lemma 4, $C(a_1) \neq C(a_6)$.

- If $C(a_1) = \{1,2,3\}$ and $C(a_6) = \{4,5,6\}$.

  $C_1 : \{1,2,3\}, \{4,5,6\}, \{1,2,3\}, \{4,5,6\}, \{1,2,3\}, \{4,5,6\}$.

  $C_2 : \{1,2,3\}, \{4,5,6\}, \{1,2,3\}, \{4,5,7\}, \{1,2,3\}, \{4,5,6\}$.

  $C_3 : \{1,2,3\}, \{4,5,6\}, \{1,2,7\}, \{3,4,5\}, \{1,2,7\}, \{4,5,6\}$.

  $C_4 : \{1,2,3\}, \{5,6,7\}, \{1,2,4\}, \{5,6,7\}, \{1,2,3\}, \{4,5,6\}$.

$C_5 : \{1,2,3\},\{5,6,7\},\{1,2,4\},\{3,5,6\},\{1,2,7\},\{4,5,6\}$.

$C_6 : \{1,2,3\},\{5,6,7\},\{1,3,4\},\{2,5,6\},\{1,3,7\},\{4,5,6\}$.

If $1 \in d$, we have the result if 3, 4, 5, 6 or 7$\in d'$ by C_1, C_2 and C_3. In the same way, we have the result if $2 \in d$ or $3 \in d$.

If $4 \in d$, we have the result if 2, 3, 5, 6 or 7$\in d'$ by C_4, C_5 and C_6. In the same way, we have the result if $5 \in d$ or $6 \in d$.

- If $C(a_1) = \{1,2,3\}$ and $C(a_6) = \{1,4,5\}$.

  $C_1 : \{1,2,3\},\{4,5,6\},\{1,2,3\},\{4,5,7\},\{2,3,6\},\{1,4,5\}$.

  $C_2 : \{1,2,3\},\{4,5,6\},\{1,2,3\},\{4,5,6\},\{2,3,7\},\{1,4,5\}$.

  $C_3 : \{1,2,3\},\{4,5,6\},\{2,3,7\},\{1,5,6\},\{2,3,7\},\{1,4,5\}$.

  $C_4 : \{1,2,3\},\{4,5,7\},\{1,3,6\},\{4,5,7\},\{2,3,6\},\{1,4,5\}$.

  $C_5 : \{1,2,3\},\{4,5,7\},\{1,3,6\},\{2,4,5\},\{3,6,7\},\{1,4,5\}$.

  $C_6 : \{1,2,3\},\{4,5,7\},\{2,3,6\},\{1,4,5\},\{3,6,7\},\{1,4,5\}$.

  If $2 \in d$, we have the result if 1, 4, 5, 6 or 7$\in d'$ by C_1, C_2 and C_3. In the same way, we have the result if $3 \in d$.

  If $1 \in d$, we have the result if 2, 4, 5, 6 or 7$\in d'$ by C_2, C_4 and C_5.

  If $6 \in d$, we have the result if 1, 2, 4, 5 or 7$\in d'$ by C_4, C_5 and C_6. In the same way, we have the result if $7 \in d$.

- If $C(a_1) = \{1,2,3\}$ and $C(a_6) = \{1,2,4\}$.

  $C_1 : \{1,2,3\},\{4,6,7\},\{2,3,5\},\{1,4,6\},\{3,5,7\},\{1,2,4\}$.

  $C_2 : \{1,2,3\},\{4,6,7\},\{1,3,5\},\{2,4,7\},\{3,5,6\},\{1,2,4\}$.

  $C_3 : \{1,2,3\},\{4,5,7\},\{2,3,6\},\{1,4,5\},\{3,6,7\},\{1,2,4\}$.

  $C_4 : \{1,2,3\},\{4,5,7\},\{2,3,6\},\{1,4,7\},\{3,5,6\},\{1,2,4\}$.

  If $5 \in d$, we have the result if 1, 2, 4, 6 or 7$\in d'$ by C_1 and C_2. In the same way, we have the result if $6 \in d$ and $7 \in d$.

  If $2 \in d$, we have the result if 1, 4, 5, 6 or 7$\in d'$ by C_1, C_3 and C_4. In the same way, we have the result if $1 \in d$. ■

Lemma 9 *Let H be a graph having a path whose interior vertices have degree 2, of length four, $(a_1, a_2, a_3, a_4, a_5)$. If H admits a 7-tricolouring C such that $|C(a_1) \cap C(a_5)| \geq 2$ then for any 3-subsets d, d' of $[1,7]$, there exists a 7-tricolouring C' of H such that $C(x) = C'(x)$ if $x \notin \{a_2, a_3, a_4\}$, $d \cap C'(a_3) \neq \emptyset$ and $d' \neq C'(a_4)$.*

Proof.

- If $C(a_1) = C(a_5) = \{1,2,3\}$, if $1 \in d$ then setting $C'(a_3) = \{1,2,3\}$ and $C'(a_2) = \{4,5,6\}$, we may choose $C'(a_4)$ among $\{4,5,6\}$ and $\{4,5,7\}$. Analogously, we conclude if $2 \in d$ or $3 \in d$.

  Hence we may suppose that $d = \{4,5,6\}$. If $d' \neq \{4,6,7\}$, set $C'(a_4) = C'(a_2) = \{4,6,7\}$ and $C'(a_3) = \{1,2,5\}$. If $d' = \{4,6,7\}$, set $C'(a_4) = C'(a_2) = \{5,6,7\}$ and $C'(a_3) = \{1,2,4\}$.

- If $|C(a_1) \cap C(a_5)| = 2$, say $C(a_1) = \{1,2,3\}$ and $C(a_5) = \{1,2,4\}$. If $d \cap \{1,2,4\} \neq \emptyset$, setting $C'(a_3) = \{1,2,4\}$ we may choose $C'(a_4)$ among $\{3,5,6\}$ and $\{3,5,7\}$. Hence we may suppose that $d \subset \{3,5,6,7\}$ and without loss of generality that $\{5,6\} \subset d$. If $d' = \{3,6,7\}$, set $C'(a_4) = \{3,5,7\}$, $C'(a_3) = \{1,2,6\}$ and $C'(a_2) = \{4,5,7\}$. If $d' \neq \{3,6,7\}$, set $C'(a_4) = \{3,6,7\}$, $C'(a_3) = \{1,2,5\}$ and $C'(a_2) = \{4,6,7\}$.

■

3.2 The main result.

Theorem 3 *Each triangle-free induced subgraph of the triangular lattice is 7-tricolourable.*

Proof. By taking the minimal counterexample G.

Claim 1: Every vertex of G has degree at least two and G has nodes.
[[Obvious since cycles of length at least 6 are 7-tricolourable.]]

Claim 2: G has no handle of length at least 6.
[[If G has a handle A of length at least 6, $G - \dot{A}$ admits a 7-tricolouring C and by Lemma 5, we may extend C to G.]]

Claim 3: G has no handle A of length 5 with an endvertex in the interior of a handle of length at least three in $G - \dot{A}$.
[[Let A be a handle with endvertices x and y such that x is on a handle of length at least three in $G - \dot{A}$. In $G - \dot{A}$, there exists a path (a_1, x, a_3, a_4) with x and a_3 of degree 2 in $G - \dot{A}$. Let C be a 7-tricolouring of $G - \dot{A}$. If $C(x) \neq C(y)$, then by Lemma 4, we extend it to G. Otherwise, by Lemma 6, there exists 7-tricolouring C' of $G - \dot{A}$ such that $C(x) \neq C'(x)$ et $C(y) = C'(y)$. So $C'(x) \neq C'(y)$, and by Lemma 4, we extend C' to G.]]

Claim 4: G has no handle of length five of type (A), (B), (C), (D), (E), (F) or (G) (see figure 2).
[[Let $A = (x, x_1, x_2, x_3, x_4, y)$ be a handle of one of these types. By Claim 3, the neighbours of x and y in $G - \dot{A}$ are nodes. This is a contradiction if A is of type (C), (D), (E), (F), or (G). If A is of type (A) or (B) and the neighbours of x and y in $G - \dot{A}$ are nodes, there is a path $(x, y_1, y_2, y_3, y_4, y)$ in $G - \dot{A}$. Let C be a 7-tricolouring in $G - \dot{A}$. Setting $C(x_i) = C(y_i)$, we have a 7-tricolouring of G.]]

Claim 5: G has no handle of length 4 of type (c). (see figure 2)
[[Let t be the common neighbour of x and y and C a 7-tricolouring of $G - \dot{A}$. Setting $C(x_1) = C(x_3) = C(t)$ and $C(x_2) = C(x)$, we have a 7-tricolouring of G.]]

Let x be one of the upmost nodes. By symmetry, we may suppose that x is a left node. And without loss of generality, we may suppose that there is no left node to the right of x.

Let A be the handle containing the rightup neighbour of x and y the endvertex of A distinct from x. By Claim 2, A is of length 3, 4 or 5.

Claim 6: A is not of length 3.
[[cf proof of Theorem 1]]

Claim 7: A is not of length 5.
[[According to Claim 4, and because there is no left node to the right of x, A is of type (H) (cf figure 3). Let t be the right neighbour of y; t is not a node. So y is on a handle of length at least three in $G - \dot{A}$ contradicting Claim 3.]]

Claim 8: A is not of length 4.
[[If $A = (x, x_1, x_2, x_3, y)$ is of length 4, there is two possible configurations: (a) and (b) (cf figure 4).

Let u_1 be the leftdown neighbour of y and t its right neighbour. If u_1 is node, then there exists a path (x, y_1, y_2, y_3, y) in $G - \dot{A}$. Let C be a 7-tricolouring of $G - \dot{A}$. Setting $C(x_i) = C(y_i)$, we have a 7-tricolouring of G. So we may suppose that u_1 has two neighbours: y and u_2. If t has degree 2, let t' be its neighbour distinct from y. In $G - \dot{A}$, (u_2, u_1, y, t, t') is a path. Thus by Lemma 7, there exists a 7-tricolouring C of $G - \dot{A}$ such that $C(x) \cap C(y) \neq \emptyset$. So by Lemma 3, we can extend C in a 7-tricolouring of G. So we may suppose that t is a node. This is impossible if A is of type (a), because there is no left node to the right of x. Thus A is of type (b) and t is a node. Let B be the handle containing the rightup

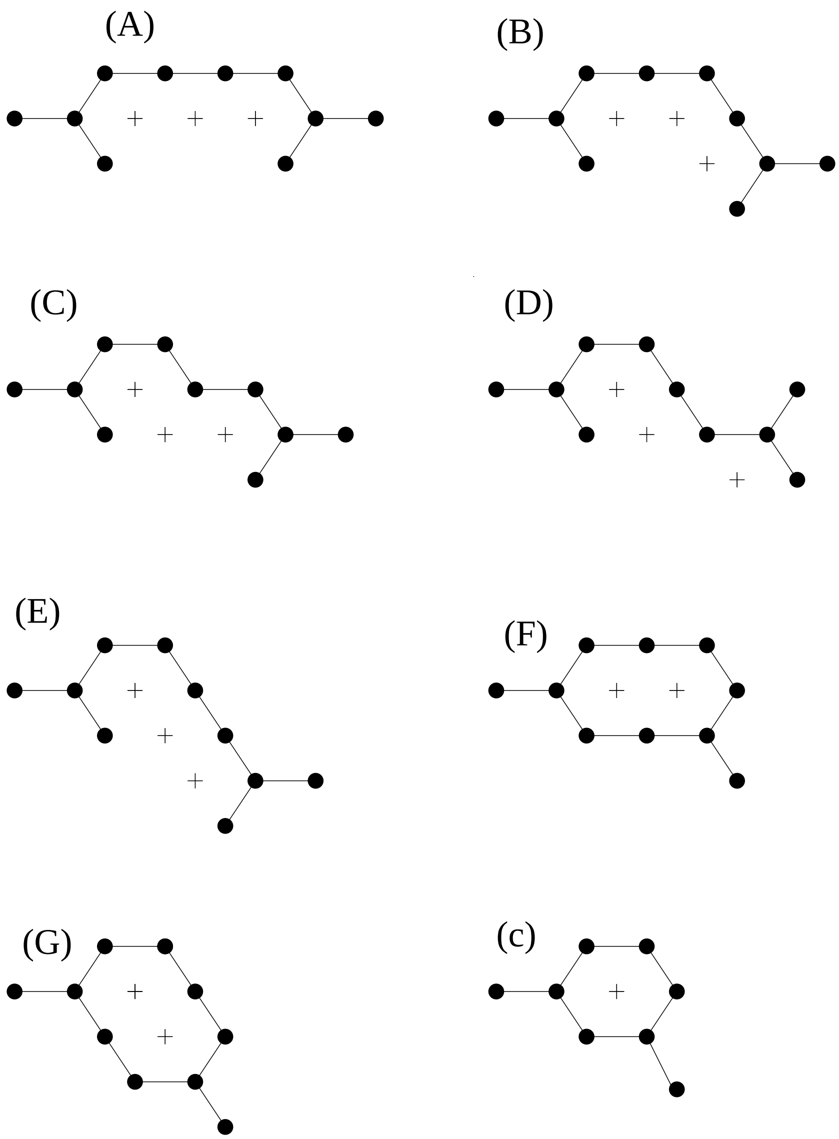

Figure 2: The forbidden handles.

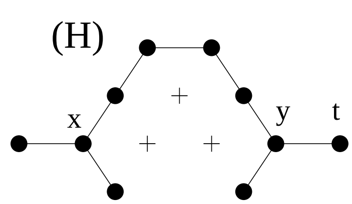

Figure 3: Handle of type (H).

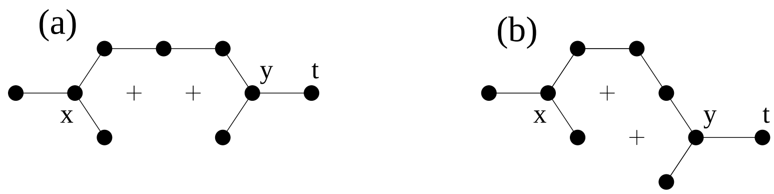

Figure 4: Handles of length 4 of type (a) and (b).

neighbour of t, z the endvertex of B distinct from t and v_1 the rightdown neighbour of t. Obviously, B has length at least 4 and at most 5 (by Claim 2).

Suppose first that v_1 has degree two. Let v_2 be the neighbour of v_1 distinct from t. $(v_2, v_1, t, y, u_1, u_2)$ is an induced path of length five in $G - (\dot{A}, \dot{B})$. Thus by Lemma 8, since $G - (\dot{A}, \dot{B})$ is 7-tricolourable there exists a 7-tricolouring C of $G - (\dot{A}, \dot{B})$ such that $C(x) \cap C(y) \neq \emptyset$ and $C(t) \cap C(z) \neq \emptyset$ if B has length four or $C(t) \neq C(z)$ if B has length five. So we may extend C in a 7-tricolouring of G by Lemma 3. Hence, we may assume that v_1 is a node.

If B has length four, there is a path of length four (see Figure 5) between t and z in $G - \dot{B}$ and we can extend a 7-tricolouring of $G - \dot{B}$ in a 7-tricolouring of G.

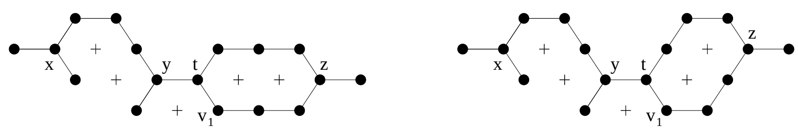

Figure 5: The possible extensions of (b) with a handle of length four.

If B has length five, because of claim 3, the neighbours of z which are not in B are nodes. Thus, we are in one of the five configurations B_i, $1 \leq i \leq 5$, depicted Figure 6. If we are in configuration B_1, B_2 or B_3, there is a path of length 5 between t and z in $G - \dot{B}$; so we can extend a 7-tricolouring of $G - \dot{B}$ into a 7-tricolouring of G. If we are in configuration B_4, let w be the rightup neighbour of z and consider the handle D containing the leftup neighbour of w. Since there is no left node to the right of x, then D has length at least six. This contradicts claim 2. So we must be in Configuration B_5.

Let y_1 be the leftdown neighbour of y. This vertex has no left neighbour otherwise there is a path of length four joining x and y in $G - \dot{A}$. Thus y_1 has a unique neighbour y_2 distinct from y. Moreover, the rightdown neighbour x' of x has degree two.

Let C be a 7-tricolouring C of $G - (\dot{A}, \dot{B})$. By Lemma 3, $|C(y_2) \cap C(v_1)| \geq 1$.

If $|C(y_2) \cap C(v_1)| \geq 2$, then by Lemma 9, there exists a 7-tricolouring C' of $G - (\dot{A}, \dot{B})$ such that $C'(x) \cap C'(y) \neq \emptyset$ and $C'(t) \neq C'(z)$. Hence, by Lemmas 3 amd 4, C' may be extended into a 7-tricolouring of G.

Hence we may suppose that $|C(y_2) \cap C(v_1)| = 1$. Without loss of generality, $C(y_2) = \{1, 2, 3\}$ and $C(v_1) = \{1, 4, 5\}$. Let C_i, $1 \leq i \leq 4$ be the colourings such that $C_i(u) = C(u)$ if $u \in G - (\dot{A}, \dot{B}, y_1, y, t)$ and:

$C_1(y_1) = \{5, 6, 7\}$, $C_1(y) = \{1, 2, 4\}$ and $C_1(t) = \{3, 6, 7\}$;
$C_2(y_1) = \{5, 6, 7\}$, $C_2(y) = \{1, 3, 4\}$ and $C_2(t) = \{2, 6, 7\}$;
$C_3(y_1) = \{4, 6, 7\}$, $C_3(y) = \{1, 2, 5\}$ and $C_3(t) = \{3, 6, 7\}$;
$C_4(y_1) = \{4, 6, 7\}$, $C_4(y) = \{1, 3, 5\}$ and $C_4(t) = \{2, 6, 7\}$.

If $C(x) \cap \{1, 4, 5\} \neq \emptyset$, there is $i \in \{1, 2, 3, 4\}$ such that $C_i(x) \cap C_i(y) \neq \emptyset$ and $C_i(t) \neq C_i(z)$. So, as previously, C_i may be extended into a 7-tricolouring of G.

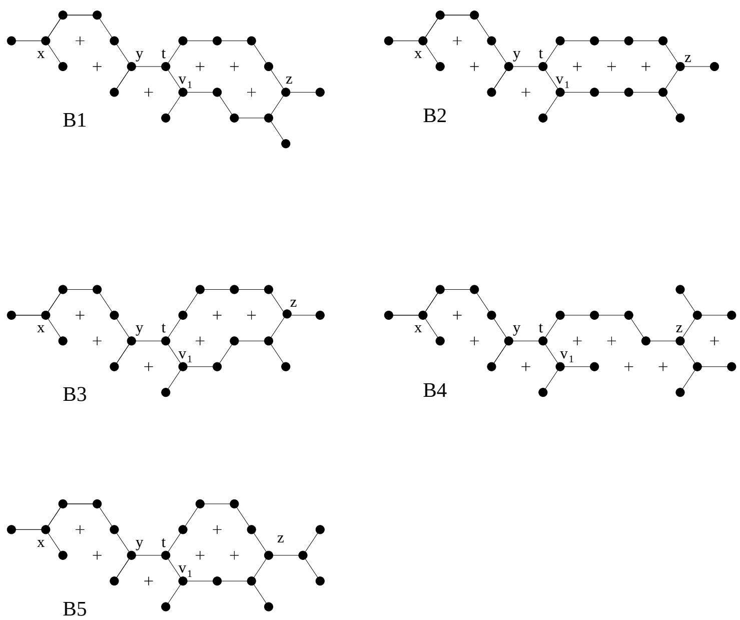

Figure 6: The five possible extensions of (b) with a handle of length five.

Now, since x' has degree two, in view of Lemma 6, there is a colouring C' such that $C'(u) = C(u)$, for any $u \in G - (\dot{A}, \dot{B}, x, x')$ and $C'(x) \neq C(x)$. Let the C'_i be defined from C' in the same way has the C_i are defined from C. If $C'(x) \cap \{1,4,5\} \neq \emptyset$, then we obtain a contradiction as presviously.

Thus we may suppose that $C(x) \cup C'(x) = \{2,3,6,7\}$. Without loss of generality, we may assume that $2 \in C(x)$ and $3 \in C'(x)$. Then, if $C(z) \neq \{3,6,7\}$, $C_1(x) \cap C_1(y) \neq \emptyset$ and $C_1(t) \neq C_1(z)$ so C_1 may be extended into a 7-tricolouring of G; and if $C(z) = \{3,6,7\}$, $C'_2(x) \cap C'_2(y) \neq \emptyset$ and $C'_2(t) \neq C'_2(z)$ so C'_2 may be extended into a 7-tricolouring of G.]] ■

Corollary 3.1 *Every triangle-free induced subgraph of the triangular lattice is* $\lceil \frac{7k}{3} \rceil$*-*$[k]$*colourable.*
Thus $\left\lceil \frac{9k}{4} \right\rceil \leq f(k) \leq \left\lceil \frac{7k}{3} \right\rceil$.

Remark 1: Following step by step the proof of Theorem 3, we obtain a recursive algorithm in $O(|V(G)|)$ that finds a $\lceil \frac{7k}{3} \rceil$-$[k]$colouring of a triangle-free induced subgraph G of the triangular lattice.

Remark 2: It may be possible to prove analogously to Theorem 3 that every triangle-free induced subgraph of the triangular lattice is $9-[4]$colourable, which would imply Conjecture 1. However such a proof would require the study of a huge number of cases. Indeed the analogous claim to the above Claim 2 states that the graph has no handle of length at least 8. Therefore, all the numerous possibilities of handle of length 4, 5, 6, or 7 should be investigated. Moreover, for lots of these cases, it is not sufficient to consider the very proximity of the handle A and to recolour it; so a wider part of the graph should be considered which requires numerous subcases to examine. For instance, suppose that we are in the configuration depicted Figure 7. Let C be a 9-[4]colouring of $G - \dot{A}$. It may be impossible to find a 9-[4]colouring $C'$of G such that $C'(v) = C(v)$ for every $v \in \{a,b,c,d,e,f\}$. For example, it may occur that $C(a) = C(e) = \{1,2,3,4\}$, $C(b) = C(f) = \{2,3,4,5\}$, $C(c) = \{1,2,3,6\}$ and $C(d) = \{1,2,3,9\}$. Therefore such a proof, if envisaged, would require a computer assistance.

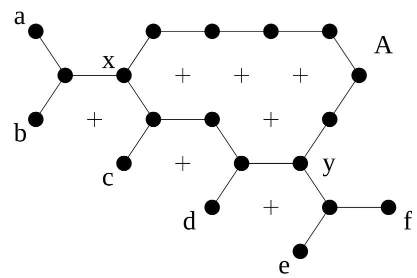

Figure 7: A problematical configuration.

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