Graph colouring and applications

F. Havet
Projet Mascotte, CNRS/INRIA/UNSA,
INRIA Sophia-Antipolis,
2004 route des Lucioles BP 93,
06902 Sophia-Antipolis Cedex,
France
fhavet@sophia.inria.fr

Foreword

The work presented here has been done with various coauthors. For sake of clarity, I decided to use "we" as subject when a set of coauthors (that varies during the monograph) should be understood. However it should be clear in the reader's mind that this work is "ours" i.e. it is the work of my coauthors and me and certainly not exclusively mine. Working with other people is one of the most pleasant aspect of research and I would like to thank all my coauthors without whom this work would never have been possible: Omid AMINI, Jean-Claude BERMOND, Stéphane BESSY, Etienne BIRMELÉ, Ricardo CORREA, Jan VAN DEN HEUVEL, Florian HUC, Ross KANG, Colin MCDIARMID, Claudia LINHARES-SALES, Bruce REED, Jean-Sébastien SERENI, Riste ŠKREKOVSKI, Stéphan THOMASSÉ, Min-Li (Joseph) YU, Janez ZEROVNIK.

1 Introduction to colouring

1.1 Basic definitions

All the definitions given in this section are mostly standard and may be found in several books on graph theory like [21, 40, 163].

1.1.1 Graphs

A *graph* G is a pair (V,E) of sets satisfying $E \subset [V]^2$, where $[V]^2$ denotes the set of all 2-element subsets of V. We also assume tacitly that $V \cap E = \emptyset$. The elements of V are the *vertices* of the graph G and the elements of E are its *edges*. The vertex set of a graph G is referred to as $V(G)$ and its edge set as $E(G)$. An edge $\{x,y\}$ is usually written as xy. A vertex v is *incident* with an edge e if $v \in e$. The two vertices incident with an edge are its *endvertices*. Note that in our definition of graphs, there is no *loops* (edges whose endvertices are equal) nor *multiple edges* (two edges with the same endvertices).

Sometimes we will need to allow multiple edges. So we need the notion of *multigraph* which generalizes the one of graph. A *multigraph* G is a pair (V,E) where V is the vertex set and E is a collection of elements of $[V]^2$. In a multigraph G, we say that xy is an edge of *multiplicity* m if there are m edges with endvertices x and y. We write $\mu(x,y)$ for the multiplicity of xy, and write $\mu(G)$ for the maximum of the edges multiplicities in G.

A *subgraph* of a graph G is a graph H such that $V(H) \subset V(G)$ and $E(H) \subset E(G)$. Note that since H is a graph we have $E(H) \subset E(G) \cap [V(H)]^2$. If H contains all the edges of G between vertices of $V(H)$, that is $E(H) = E(G) \cap [V(H)]^2$, then H is the subgraph *induced* by $V(H)$. The notion of *submultigraph* and *induced submultigraph* are defined similarly. If S is a set of vertices, we denote by $G[S]$ the (multi)graph induced by S and by $G - S$ the (multi)graph induced by $V(G) \setminus S$. For simplicity, we write $G - v$ rather than $G - \{v\}$. For a collection F of elements of $[V^2]$, we write $G \setminus F = (V(G), E(G) \setminus F)$ and $G \cup F = (V(G), E(G) \cup F)$. As above $G \setminus \{e\}$ and $G \cup \{e\}$ are abbreviated to $G \setminus e$ and $G \cup e$ respectively.

Let G be a multigraph. When two vertices are the endvertices of an edge, they are *adjacent* and are *neighbours*. The set of all neighbours of a vertex v in G is the *neighbourhood* of G and is denoted $N_G(v)$, or simply $N(v)$. The *degree* $d_G(v) = d(v)$ of a vertex is the number of edges to which it is incident. If G is a graph, then this is equal to the number of neighbours of v. The *maximum degree* of G is $\Delta(G) = \max\{d_G(v) \mid v \in V(G)\}$. The *minimum degree* of G is $\delta(G) = \max\{d_G(v) \mid v \in V(G)\}$. If the graph G is clearly understood, we often write Δ and δ instead of $\Delta(G)$ and $\delta(G)$. A graph is *k-regular* if every vertex has degree k. The *average degree* of G is $Ad(G) = \frac{1}{|V(G)|} \sum_{v \in V(G)} d(v) = \frac{2|E(G)|}{|V(G)|}$. The *maximum average degree* of G is $Mad(G) = \max\{Ad(H) \mid H \text{ is a subgraph of } G\}$.

The *complete graph* K_n on n vertices is the graph in which any two vertices are linked by an edge. A *clique* in a graph is a set of pairwise adjacent vertices. In other words, it is a set of vertices inducing a complete graph. The *clique number* $\omega(G)$ is the maximum size of a clique in G. A *stable* or *independent* set is a set of vertices pairwise non-adjacent. The *stability number* of a graph G, denoted by $\alpha(G)$, is the maximum size of a stable set in G,

A *path* is a non-empty graph $P = (V, E)$ of the form $V = \{x_0, x_1, \dots, x_k\}$ and $E = \{x_0x_1, x_1x_2, \dots, x_{k-1}x_k\}$ where the vertices x_i are all distinct. The vertices x_0 and x_k are the *endvertices* of P and the vertices $x_1, \dots, x_{k-1}$ the *internal vertices*. The path P is denoted by the succession of its vertices $(x_0, x_1, \dots, x_k)$. A path with endvertices u and v is called (u,v)*-path*. A *cycle* is a non-empty graph $C = (V, E)$ of the form $V = \{x_0, x_1, \dots, x_k\}$ and $E = \{x_0x_1, x_1x_2, \dots, x_{k-1}x_k, x_kx_0\}$ where the x_i are all distinct. It is denoted by $(x_0, x_1, \dots, x_k, x_0)$. The *length* of a path or a cycle is its number of edges. The minimum length of a cycle contained in a multigraph is its *girth*; if the graph contains no cycles, the *girth* is defined to be $+\infty$. Note that in a graph the girth is at least 3. An edge which joins two vertices of a cycle but is not itself an edge of the cycle is a *chord* of that cycle.

A graph G is *connected* if any two of its vertices are linked by a path in G. It is *k-connected* (for $k \in \mathbb{N}$), if $|G| > k$ and $G - X$ is connected for every set $X \subset V$ with $|X| < k$. The *connected components* or simply *components* of a graph are its maximal (with respect to the inclusion) connected subgraphs. The *blocks* of a graph are its maximal 2-connected subgraphs. A *cutvertex* is a vertex v such that $G - v$ has more connected components than G.

1.1.2 Digraphs

A *multidigraph* D is a pair $(V(D), E(D))$ of disjoint sets (of *vertices* and *arcs*) together with two maps $tail : E(D) \to V(D)$ and $head : E(D) \to V(D)$ assigning to every arc e a *tail*, $tail(e)$, and a *head*, $head(e)$. The tail and the head of an arc are its *endvertices*. An arc with tail u and head v is denoted by uv and is said to *leave* u and to *enter* v; we say that u *dominates* v and write $u \to v$; we also say that u and v are adjacent. Note that a directed multidigraph may have several arcs with same tail and same head. Such arcs are called *multiple arcs*. A *multidigraph* without multiple arcs is a *digraph*. It can be seen as a pair (V, E) with a $E \subset V^2$. An arc whose head and tail are equal is a *loop*. All the digraphs we will consider in this monograph have no loops.

The multigraph G *underlying* a multidigraph D is the multigraph obtained from D by replacing each arc by an edge. Note that the multigraph underlying a digraph may not be a graph: there are edges uv of multiplicity 2 whenever uv and vu are arcs of D. *Subdigraphs* and *submultidigraphs* are defined similraly to subgraphs and submultigraphs.

Let D be a multidigraph. If uv is an arc, we say that u is an *inneighbour* of v and that v is an *outneighbour* of u. The *outneighbourhood* of v in D, is the set $N_D^+(v) = N^+(v)$ of outneighbours of v in G. Similarly, the *inneighbourhood* of v in D, is the set $N_D^-(v) = N^-(v)$ of inneighbours of v in G. The *outdegree* of a vertex v is the number $d_D^+(v) = d^+(v)$ of arcs leaving v and the *indegree* of v is the number $d_D^-(v) = d^-(v)$ of arcs entering v. Note that if D is a digraph then $d^+(v) = |N^+(v)|$ and $d^-(v) = |N^-(v)|$. The *degree* of a vertex v is $d(v) = d^-(v) + d^+(v)$. It corresponds to the degree of the vertex in the underlying multigraph. The *maximum outdegree* of D is $\Delta^+(D) = \max\{d^+(v), v \in V(D)\}$, the *maximum indegree* of D is $\Delta^-(D) = \max\{d^-(v), v \in V(D)\}$, and the *maximum degree* of D is $\Delta(D) = \max\{d(v), v \in V(D)\}$. When D is clearly understood from the context, we often write Δ^+, Δ^- and Δ instead of $\Delta^+(G)$, $\Delta^-(G)$ and $\Delta(G)$ respectively.

A *directed path* or *dipath* is a non-empty digraph $P = (V,E)$ of the form $V = \{x_0, x_1, \dots, x_k\}$ and $E = \{x_0x_1, x_1x_2, \dots, x_{k-1}x_k\}$ where the x_i are all distinct. The vertex x_0 is the *origin* of P and x_k its *terminus*. A *circuit* is a non-empty digraph $C = (V,E)$ of the form $V = \{x_0, x_1, \dots, x_k\}$ and $E = \{x_0x_1, x_1x_2, \dots, x_{k-1}x_k, x_kx_0\}$ where the x_i are all distinct. The *length* of a dipath or a circuit is its number of edges.

A multidigraph D is *strongly connected* or *strong* if for every two vertices u and v, there is a dipath in D with origin u and terminus v. It is *connected* if its underlying multigraph is connected.

1.2 Vertex colouring

A *(vertex) colouring* of a graph G is a mapping $c : V(G) \to S$. The elements of S are called *colours*; the vertices of one colour form a *colour class*. If $|S| = k$, we say that c is a *k-colouring* (often we use $S = \{1, \dots, k\}$). A colouring is *proper* if adjacent vertices have different colours. A graph is *k-colourable* if it has a proper k-colouring. The *chromatic number* $\chi(G)$ is the least k such that G is k-colourable. Obviously, $\chi(G)$ exists as assigning distinct colours to vertices yields a proper $|V(G)|$-colouring. An *optimal colouring* of G is a $\chi(G)$-colouring. A graph G is *k-chromatic* if $\chi(G) = k$.

Obviously, the complete graph K_n requires n colours, so $\chi(K_n) = n$. Then $\chi(G) \geq \omega(G)$. This bound can be tight, but it can also be very loose. Indeed for any given integers $k \leq l$, there are graphs with clique number k and chromatic number l. For example, Mycielski [125] gave a simple construction of triangle-free graphs (i.e. with clique number 2) with arbitrarily large chromatic number.

In a proper colouring, each colour class is a stable set, so $\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$.

Most upper bounds on the chromatic number come from algorithms that produce colourings. The most widespread one is the greedy algorithm. A *greedy colouring* relative to a vertex ordering $v_1, \dots, v_n$ of $V(G)$ is obtained by colouring the vertices in the order $v_1, \dots, v_n$, assigning to v_i the smallest-indexed colour not already used on its lowered-indexed neighbourhood. In a vertex-ordering, each vertex has at most $\Delta(G)$ earlier neighbours, so the greedy colouring cannot be forced to use more than $\Delta(G) + 1$ colours.

Proposition 1. $\chi(G) \leq \Delta(G) + 1$.

The bound $\Delta(G) + 1$ is the worst upper bound that greedy colouring could produce (although optimal for complete graphs and odd cycles). However choosing the vertex odering is the main problem. Indeed there is a vertex ordering relative to which the greedy algorithm yields an optimal colouring. If c is an optimal colouring of G, then any ordering $v_1, \dots, v_n$ such that for any $i < j$, $c(v_i) \leq c(v_j)$ will be. But there are $n!$ possible orderings and it is difficult to fing a good one. Actually, for any $k \geq 3$ it is NP-complete to decide if a graph is k-colourable See [53]. Deciding if a graph is 1-colourable is easy since such a graph as no edges and deciding if a graph is 2-colourable can be done in polynomial time (using bread-first search for instance). Furthermore, it is NP-hard to approximate the chromatic number within $|V(G)|^{\varepsilon_0}$ for some positive constant ε_0 as shown by Lund and Yannakakis [111].

However, one can easily determine if the chromatic number is equal to $\Delta(G)+1$ as shown by Brooks Theorem [34].

Theorem 2 (Brooks [34]). *Let G be a connected graph. Then $\chi(G) \leq \Delta(G)$ unless G is either a complete graph or an odd cycle.*

In order to prove this theorem, we need the following easy result.

Proposition 3. *Let G be a connected graph which is not a complete graph. Then there exists three vertices u, v and w such that $uv \in E(G)$, $vw \in E(G)$ and $uw \notin E(G)$.*

Proof. Since G is not complete there exists two vertices u and u' which are not linked by an edge. Since G is connected there is a path between u and u'. Let P be a shortest (u,u')-path and let v and w be respectively the second and third vertices on P. Then uv and vw are edges of the paths and uw is not an edge otherwise it would shortcut P. □

Proof of Theorem 2. We may assume that $\Delta = \Delta(G) \geq 3$, since G is complete if $\Delta \leq 1$ and G is an odd cycle or bipartite when $\Delta = 2$, in which cases the bound holds.

We shall find an ordering of the vertices so that the greedy colouring relative to it yields the desired bound.

Assume first that G is not Δ-regular. Let v_n be a vertex of degree less than Δ. Since G is connected, one can grow a spanning tree of G from v_n, assigning indices in decreasing order as we reach vertices. We obtain an ordering $v_1,\dots,v_n$ such that every vertex other than v_n has a higher-indexed neighbour. Therefore the greedy colouring uses at most Δ colours.

Assume now that G is Δ-*regular*. If G has a cut-vertex x, we may apply the above nethod on each component of $G-x$ plus x. Then permuting the names of the colours, one can make the colourings agree on x, to complete a proper Δ-colouring of G.

Hence we may assume that G is 2-connected. In such a case, for G is not complete, by Proposition 3 some vertex v_n has neighbours v_1 and v_2 such that $v_1v_2 \notin E(G)$. Moreover considering such vertices for which the component of v_n in $G-\{v_1,v_2\}$ has maximum size, one easily shows that $G-\{v_1,v_2\}$ is connected. Then indexing the vertices of a spanning tree rooted in v_n in a decreasing order, with $\{3,\dots,n\}$, we obtain an ordering $v_1,\dots,v_n$ such that every vertex other than v_n has a higher-indexed neighbour. Now the greedy colouring will assign colour 1 to both v_1 and v_2. So when colouring v_n at most $\Delta-1$ colours will be assigned to its neighbours. Hence the greedy colouring will use at most Δ colours.

□

In other words, Brooks Theorem states that for $\Delta(G) > 2$, $\chi(G) = \Delta(G)+1$ if and only if G contains a clique of size $\Delta(G)+1$. It is natural to ask whether this extends further. E. g. if $\chi(G) \geq \Delta+1-k$ does G contain a large clique? One cannot expect a clique of size $\Delta+1-k$ if k is large. Indeed consider the graph $H_{\Delta,p}$ formed by adding all the edges between a $(\Delta+1-5p)$-clique and p disjoint 5-cycles. It is easy to see that $H_{\Delta,p}$ has maximum degree Δ, chromatic number $\Delta+1-2p$ and clique number $\Delta+1-3p$. Reed [133] conjectured that if $\chi(G) \geq \Delta+1-k$ then G contains a clique of size at least $\Delta+1-2k$.

Conjecture 4 (Reed [133]). *Let G be a graph.*
If $\chi(G) \geq \Delta(G)+1-k$ *then* $\omega(G) \geq \Delta(G)+1-2k$. *In other words,*

$$\chi(G) \leq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil.$$

Note that this value $2k$ is best possible. Indeed consider random graph R on n vertices with edge probability $(1-n^{-3/4})$. The expected number of cliques of size i is

$$\begin{aligned}\binom{n}{i}\left(1-n^{-3/4}\right)^{\binom{i}{2}} &\leq 2^{i\log n}\left(1-n^{-3/4}\right)^{\frac{i^2}{4}} \\ &\leq 2^{i\log n}e^{-n^{-3/4}\frac{i^2}{4}}.\end{aligned}$$

For $i > n^{3/4}\log n$, this is $o(1)$ so (with high probability) $\omega(R) \leq n^{3/4}\log n$. Now the expected number of stable sets of size 3 is $\binom{n}{3} \times \left(n^{-3/4}\right)^3 = O(n^{3/4})$. Hence removing one vertex per such stable set, we obtain a graph H with $n - O(n^{3/4})$ vertices and stability number $\alpha(H) = 2$. Hence its chromatic number is at least $n/2 - O(n^{3/4})$. Let G be the graph obtained by connecting all the vertices of H to a clique of size $\Delta - n$. Then $\Delta(G) = \Delta$, $\chi(G) = \Delta - n + \chi(H) \geq \Delta - n/2 - O(n^{3/4})$ and $\omega(G) \leq \Delta - n + \omega(G) \leq \Delta - n + n^{3/4}\log n$.

As an evidence for Conjecture 4, Reed [133] showed that there is an $\varepsilon > 0$ such that $\chi(G) \leq \varepsilon\omega(G) + (1-\varepsilon)(\Delta(G)+1)$. Johannson [87] settled Conjecture 4 for $\omega = 2$ and Δ sufficiently large. In fact, he proved that there is a constant c such that if $\omega(G) = 2$ then $\chi(G) \leq c\frac{\Delta(G)}{\log\Delta(G)}$.

When $k = 1$ Conjecture 4 asserts that if $\chi(G) = \Delta(G)$ then $\omega(G) \geq \Delta - 1$. In fact, Reed [134] showed that when Δ is large if $\chi(G) = \Delta(G)$ then $\omega(G) = \Delta(G)$, thus settling a conjecture of Beutelspacher and Hering [18]. Borodin and Kostochka [28] conjectured that it is true for $\Delta \geq 9$; counterexamples are known for each $\Delta \leq 8$.

Conjecture 5 (Borodin and Kostochka [28]). *Let G be a graph of maximum degree* $\Delta \geq 9$. *If* $\chi(G) = \Delta$ *then* $\omega(G) = \Delta$.

When $k = 2$, one cannot expect all $(\Delta-1)$-chromatic graphs to have a clique of size $\Delta - 1$. Indeed $H_{\Delta,1}$ has chromatic number $\Delta - 1$ but clique number $\Delta - 2$. However, Farzad, Molloy and Reed [50] showed that for Δ sufficiently large if $\chi(G) \geq \Delta - 1$ then G contains either a $(\Delta-1)$-clique or $H_{\Delta,1}$. They also proved similar results for $k = 3$ and $k = 4$; in these cases, G must contain one of five or thirty eight graphs respectively.

Let k_Δ the maximum integer such that $(k+1)(k+2) \leq \Delta$. Thus, $k_\Delta \approx \sqrt{\Delta} - 2$. Molloy and Reed [121] showed that k_Δ is a threshold to Brooks-like theorems. Indeed if $k < k_\Delta$ then, if Δ is large enough, if $\chi(G) \geq \Delta - k + 1$ then G must contain a graph H that is close to a $(\Delta+1-k)$-clique, in that H has small size ($|H| \leq \Delta + 1$) and cannot be $(\Delta-k)$-coloured. As a consequence, one can check polynomially if $\chi(G) \geq \Delta - k$ or not. On the opposite, if $k < k_\Delta$ then there are arbitrarily large $(\Delta+k-1)$-critical (i.e $(\Delta+1-k)$-chromatic graphs such that every proper subgraph is $(\Delta-k)$-colourable graphs with maximum degree Δ. Furthermore, Embden-Weinert, Hougardy and Kreuter [46], proved that for any constant Δ and $\Delta - 3 \leq k < k_\Delta$, determining whether a graph of maximum degree Δ is $(\Delta-k)$-colourable is NP-complete.

A graph G is k-*degenerate* if each of its subgraph has a vertex of degree at most k. The *degeneracy* of G, denoted $\delta^*(G)$, is the smallest k such that G is k-degenerate. It is easy to see that a graph is k-degenerate if and only if there is an ordering $v_1, v_2, \ldots, v_n$ of the vertices such that for every $1 < i \leq n$, the vertex v_i has at most k neighbours in $\{v_1,\dots,v_{i-1}\}$. Hence the greedy colouring relative to this ordering uses at most $\delta^*(G)+1$ colours.

Proposition 6.

$$\chi(G)\leq \delta^*(G)+1.$$

Note that finding an ordering as above (and thus the degeneracy of a graph) is easy. It suffices to recursively take a vertex v_n of minimum degree in the graph and to put it at the end of the ordering $v_1,\dots,v_{n-1}$ of $G-v_n$.

1.2.1 Fractional colouring

A *fractional (vertex) colouring* is a set $\{S_1,\dots,S_l\}$ of stable sets and corresponding positive real weights, $\{w_1,\dots,w_l\}$ such that the sum of the weights of the stable sets containing each vertex is one. I.e. $\forall v\in V, \sum_{\{S_i\ \mid\ v\in S_i\}} w_i = 1$. Of course the weight of each stable set in a fractional colouring will be at most one. We note that a colouring is simply a fractional colouring in which every weight is an integer (to be precise, each weight is one). Letting $\mathcal{S}(G)$ be the set of stable sets in G, one can also describe a fractional colouring by an assignment of a non-negative real weight w_S to each stable set S in $\mathcal{S}(G)$ so that the sum of the weights of the stable set containing each vertex is 1.

A fractional colouring is a *fractional c-colouring* if $\sum_{S\in\mathcal{S}(G)} w_S = c$. The *fractional chromatic number* of G, denoted $\chi_f(G)$, is the minimum c such that G has a fractional c-colouring. Obviously $\chi_f(G)\geq \omega(G)$ and since every colouring is a fractional colouring, $\chi_f(G)\leq\chi(G)$. Molloy and Reed showed (Chapter 21 of [122]) that Conjecture 4 holds fractionally, that is $\chi_f(G)\leq \frac{\omega(G)+\Delta(G)+1}{2}$.

If G has n vertices then $\chi_f(G)\geq \frac{n}{\alpha(G)}$, because for any fractional colouring we have:

$$n=\sum_{v\in V(G)}\ \sum_{\{S\in\mathcal{S}(G)\ \mid\ v\in S\}} w_S = \sum_{S\in\mathcal{S}(G)}\sum_{v\in S} w_S \leq \sum_{S\in\mathcal{S}(G)} \alpha(G) w_S.$$

The Linear Programming duality implies directly that the fractional chromatic number is equal to the maximum of $\sum_{v\in V(G)} x(v)$ such that $\forall S\in\mathcal{S}(G), \sum_{v\in S} x(v)\leq 1$ over all non-negative weightings x on $V(G)$. (For more details on fractional graph theory see [141]). Despite this pleasing result, it is still NP-complete to compute the fractional chromatic number of a graph. In fact, it is even difficult to approximate the fractional chromatic number of graphs with n vertices to within a factor of n^{ε} for some positive ε [11].

1.2.2 List colouring and choosability

List colouring is a generalisation of vertex colouring in which the set of colours available at each vertex is restricted. This model was introduced independently by Vizing [158] and Erdős-Rubin-Taylor [48].

A *list-assignment* of a graph G is an application L which assigns to each vertex $v\in V(G)$ a prescribed list of colours $L(v)$. A list-assignment is a *k-list-assignment* if each list is of size at least k. An *L-colouring* of G is a colouring such that $\forall v\in V(G), v\in L(v)$. A graph G is *L-colourable* if there exists a proper colouring of G. It is *k-choosable* if it is L-colourable for every k-list-assignment L. The *choice number*, *choosability* or *list chromatic number* $ch(G)$ is the least k such that G is k-choosable.

Since the lists could be identical, $ch(G)\geq\chi(G)$. One the opposite, it is not possible to pace an upper bound on $ch(G)$ in terms of $\chi(G)$ because there are bipartite graphs with arbitrarily large choice number.

Proposition 7 (Erdős, Rubin and Taylor [48]). *If $m=\binom{2k-1}{k}$, then $K_{m,m}$ is not k-choosable.*

Proof. Let (A,B) be the bipartition of $K_{m,m}$. Let L be a list assignment such that every k-subset I of $\{1,2,\dots,2k-1\}$ there exists a vertex $a_I \in A$ and a vertex $b_I \in B$ such that $L(a_I) = L(b_I) = I$. Consider an L-colouring of $K_{m.m}$. Let S_A be the set of colours used on A and S_B be the set of colours used on B. Then $|S_A| \geq k$ otherwise a vertex a_I with $I \subset \{1,2,\dots,2k-1\} \setminus S_A$ would be assigned no colour. Similarly, $|S_B| \geq k$. Hence $S_A \cap S_B \neq \emptyset$, so the colouring is not proper. □

One can decide in polynomial time if a graph is 2-colourable. One can also decide in polynomial time if a graph is 2-choosable as one can characterize such graphs.

Definition 8. The graph consisting of two vertices connected with three independent paths (that is with no internal vertices in common) of length i, j and k is denoted $\theta_{i,j,k}$.

The *heart* of a connected graph G is its maximum subgraph of minimum degree different from 1. So if G is 1-degenerate, that is a tree, then its heart is K_1 otherwise it is the largest induced subgraph with minimum degree at least 2.

It is easy to show that a graph is 2-choosable if and only if its heart is. Hence to characterize 2-choosable graphs it is sufficient to characterize their hearts.

Theorem 9 (Erdős, Rubin and Taylor [48]). *A connected graph G is 2-choosable if and only if its heart is either K_1 or an even cycle or a $\theta_{2,2,2m}$ for some postive integer m.*

If the lists have size $\delta^*(G)+1$, then colouring greedily the vertices according to a degenerate ordering leaves an available colour at each vertex. So

Proposition 10.

$$ch(G) \leq \delta^*(G) \leq \Delta(G)+1.$$

Brooks Theorem may be extended to list colouring. It is directly implied by the following theorem proved independently by Borodin [22] and Erdős, Rubin and Taylor [48]. A graph G is *degree choosable* if for any assignment L such that for all $v \in V(G)$, $|L(v)| \geq d_G(v)$, then G is L-colourable. A connected graph is said to be a *Gallai tree* if each of its blocks is either a complete graph or an odd cycle.

Theorem 11 (Borodin [22], Erdős, Rubin and Taylor [48]). *Let G be a connected graph. Then G is degree choosable if and only if G is not a Gallai tree.*

To prove this theorem we need the following definitions and preliminary results.

Definition 12. Let G be a graph and u a vertex of G. A list assignment L on $V(G)$ is *u-nice* if for all $v \in V(G)$, $|L(v)| \geq d_G(v)$ and $|L(u)| \geq d_G(u)+1$. A list assignment is *nice* if it is u-nice for some $u \in V(G)$.

Proposition 13. *Let G be a connected graph. For any nice assignment L, G is L-colourable.*

Proof. Let u be the vertex such that L is u-nice. Let $v_1,\dots,v_n = u$ be an ordering of the vertices of G such that every vertex but u has a higher-indexed neighbour. Such an ordering exists: take for example the reverse of a bread-first search ordering from u. Colour the vertices greedily according to this ordering. When we are about to colour v_i, $i < n$, there are at most $d(v_i)-1$ neighbours of v_i already coloured because the higher-indexed neighbour is uncoloured. Since $|L(v_i)| \geq d(v_i)$, one can assign to v_i a colour in $L(v_i)$ which does not appear on a vertex of its neighbourhood. Thus the greedy algorithm proceeds until the last vertex u. Then since $|L(u)| \geq d(u)+1$, one can colour u with a colour in $L(u)$ which does not appear on its neighbourhood. □

Lemma 14. *Let G be a 2-connected graph which is neither complete nor a cycle. Then G has an induced $\theta_{i,j,k}$ for some i,j,k.*

Proof. By Proposition 3, there exists three vertices u, v and w such that $uv \in E(G)$, $vw \in E(G)$ and $uw \notin E(G)$. Now since G is 2-connected, there is a (u,w)-path in $G-v$. Let $P = (u = u_0, u_1, \dots, u_{l-1}, u_l = w)$ be a shortest such path.

Suppose first that there is an edge vu_i for some $1 \le i \le l-1$. Let i_1 and i_2 be the smallest and second smallest integers $i > 0$ such that vu_i is an edge. Then the graph induced by $\{v, u_0, \dots, u_{i_2}\}$ is a $\theta_{1,i_1+1,i_2-i_1+1}$.

Suppose now that vu_i is not an edge for all $1 \le i \le l-1$. Then $C = (v, u_0, \dots, u_l, v)$ is an induced cycle of length at least 4. Since G is not a cycle and connected, there is a C-path in G, that is a path with both ends distinct and in $V(C)$ and the internal vertices in $V(G) \setminus V(C)$. Let $P = (x_0, \dots, x_m, x_{m+1})$ be such a path with minimal length. W.l.o.g. we may assume that $v = x_0$. By minimality of P, P is an induced path and the sole internal vertices that are adjacent to a vertex of C are x_1 and x_m.

Assume first that x_1 has at least three neighbours in C. Let j_1 and j_2 be the smallest and second smallest integers $j \ge 0$ such that vu_j is an edge. By rotation around C, we may assume that $j_2 \ne l$. Then the graph induced by $\{x_1, v, u_0, \dots, u_{j_2}\}$ is a $\theta_{1,j_1+2,j_2-j_1+1}$.

If x_1 has exactly two neigbours in C, v and u_i for some $1 \le i \le l$ then the graph induced by $\{x_1\} \cup V(C)$ is a $\theta_{1,i+2,l-i+2}$. Similarly, we get the result if x_m has at least two neighbours in C.

Hence, we may assume that x_1 and x_m have both a unique neigbour in C. It follows that $V(C) \cup V(P)$ induces a $\theta_{i,j,k}$ for some i,j,k. □

Proposition 15. *For any positive integers i, j and k, the graph $\theta_{i,j,k}$ is degree choosable.*

Proof. Let x and y be twe two vertices of degree 3 in $\theta_{i,j,k}$ and $xu_1 \dots u_{i-1}y$, $xv_1 \dots v_{j-1}y$ and $xw_1 \dots w_{k-1}y$ be the three paths joining them.

Let L be a list assignment of $\theta_{i,j,k}$ such that $|L(v)| = d(v)$. Let $c(x)$ be a colour of $L(x) \setminus L(u_1)$. Extend c greedily to an L-colouring of $\theta_{i,j,k}$ with respect to the ordering $v_1, \dots, v_{j-1}, w_1, \dots, w_{k-1}, y, u_{i-1}, \dots, u_1$. This is possible since $c(x) \notin L(u_1)$. □

Proof of Theorem 11. We proceed by induction on the number of vertices of the graph G.

Let G be a graph which is not a Gallai tree and let L be a list assignment on $V(G)$ such that $|L(v)| = d(v)$ for any vertex v. We shall prove that there exists an L-colouring of G.

Suppose first that G has a cutvertex x. Let C_1 be a connected component of $G-v$ and $C_2 = V(G) \setminus (C_1 \cup \{x\})$. For $i = 1,2$ set $G_i = G[C_i \cup \{x\}]$. Then G_1 and G_2 are connected and there is no edge between vertices of C_1 and C_2. Moreover $d_G(x) = d_{G_1}(x) + d_{G_2}(x)$. Since G is not a Gallai tree, one of the G_is is not a Gallai tree. Without loss of generality, we may assume that G_1 is not a Gallai tree. Let $L'(x)$ be the set of colours α such that there exists an L-colouring of G_2 with x coloured α. By Proposition 13, for any subset S of $L(x)$ of size $d_{G_2}(x)+1$, there is an L-colouring of G_1 with x coloured in S, so $|L'(x)| \ge d_G(x) - d_{G_2}(x) = d_{G_1}(x)$. Now since G_1 is not a Gallai tree then by induction hypothesis it is degree choosable, so there is an L-colouring c_1 of G_1 such that $c_1(x) \in L'(x)$. Since $c_1(x) \in L'(x)$, there is an L-colouring c_2 of G_2 such that $c_2(x) = c_1(x)$. The union of c_1 and c_2 is an L-colouring of G.

Suppose now that G is 2-connected. If G is an even cycle, we have the result by Theorem 9. Hence we may assume that G is not a cycle. So by Lemma 14, G contains an induced $\theta_{i,j,k}$ say T. Let $v_1, \dots, v_q$ be an ordering of $V(G) \setminus V(T)$ such that for every $1 \le i \le q$, v_i has a higher-indexed neighbour or a neighbour in T. (Such an ordering may be obtained by reversing a bread-first search ordering from T.) Let c be a greedy L-colouring of $V(G) \setminus V(T)$ according to this ordering. Now for each vertex

$v \in V(T)$, let $L'(v)$ be the set of available colours at v, that is $L'(v) = L(v) \setminus \{c(u) \mid u \in N(v) \setminus V(T)\}$. Then $|L'(v)| \geq d_T(v)$ for all $v \in V(T)$, so by Proposition 15, T admits an L'-colouring whose union with c forms an L-colouring of G. □

1.2.3 Colouring planar graphs

A graph G is *planar* if it has a drawing in the plane without crossing. Such a drawing is called a *planar embedding* of G. A *plane graph* is a particular planar embedding of a planar graph.

The girth and the maximum average degree of a planar graph are related to each other:

Theorem 16. *Let G be a planar graph of girth g.*

$$Mad(G) < 2 + \frac{4}{g-2}.$$

Proof. The assertion is easy if G has no cycle, so we may assume that g is finite. We recall the Euler's formula for a planar graph H: $|V(H)| - |E(H)| + |F(H)| = 2$ with $|F(H)|$ the number of faces of H. Note that every subgraph H of G has girth at least g, so $g|F(H)| \leq 2|E(H)|$. Thus $2g - g|V(H)| + g|E(H)| = g|F(H)| \leq 2|E(H)|$. Hence $\frac{2|E(H)|}{|V(H)|} \leq \frac{2g}{g-2} - \frac{4g}{(g-2)|V(H)|} < \frac{2g}{g-2}$ for every subgraph H of G. □

In particular, planar graphs have girth at least 3, so have maximum degree less than 6. Thus they are 5-degenerate and consequently are 6-choosable by Proposition 10. In fact, Thomassen [152] showed that they are 5-choosable.

Theorem 17 (Thomassen [152]). *Every planar graph is* 5*-choosable.*

Proof. Free to add some extra edges, we may assume that G is an *inner triangulation*, that is that every inner face of G is bounded by a triangle and its outer face by a cycle $F = (v_1, v_2, \dots, v_k, v_1)$.

We shall prove by induction a stronger assertion. To do so we need the following definition. A list assignment L of quasi-triangulation G is *suitable* if

- $|L(v_1)| = 1$ and $|L(v_2)| = 2$,
- for every $v \in V(F) \setminus \{v_1, v_2\}$, $|L(v)| \geq 3$, and
- for every $v \in V(G) \setminus V(F)$, $|L(v)| \geq 5$.

We shall prove by induction on the number of vertices that *if L is a suitable list assignment of a quasi-triangulation G then G is L-colourable.*

The results holds trivially if G has three vertices. Now let $|G| \geq 4$.

Suppose first that F has a chord vw. Then vw lies in two unique cycles in $F \cup vw$, one C_1 containing v_1v_2 and the other C_2. For $i = 1, 2$, let G_i denote the subgraph induced by the vertices lying on C_i or in its inner face. See Figure 1. By induction hypothesis, there exists a proper L-colouring c_1 of G_1. Let L_2 be the list assignment on G_2 defined by $L_2(v) = \{c_1(v)\}$, $L_2(w) = \{c_1(v), c_1(w)\}$ and $L_2(u) = L(u)$ if $u \in V(G_2) \setminus \{v, w\}$. Then L_2 is suitable for G_2 so G_2 admits a proper L_2-colouring c_2 by induction hypothesis. The union of c_1 and c_2 is a proper L-colouring of G.

Suppose now that F has no chord. Let $v_1, u_1, u_2, \dots, u_m, v_{k-1}$ be the neighbours of v_k in their natural cyclic order around v_k. All the u_i are not in F since F has no chord. Furthermore, as G is a quasi-triangulation, $(v_1, u_1, u_2, \dots, u_m, v_{k-1}) = P$ is a path. Hence the graph $G - v_k$ has $F' = P \cup (F - v_k)$ as outer face. See Figure 2.

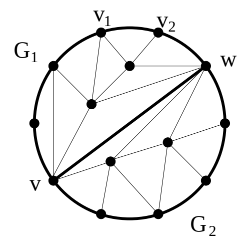

Figure 1: Configuration when F has a chord

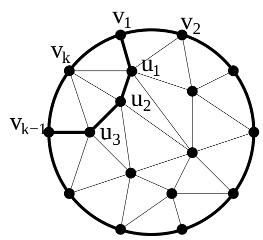

Figure 2: Configuration when F has no chord

Let c_3 and c_4 be two distinct colours of $L(v_k) \setminus \{c_1\}$. Set $L_1(u_i) = L(u_i) \setminus \{c_3, c_4\}$ for $1 \leq i \leq m$ and $L_1(v) = L(v)$ if $v \notin \{u_1, u_2, \dots, u_m\}$. L_1 is suitable for $G - v_k$. So by induction hypothesis, there is a proper L_1-colouring c of $G - v_k$. At least one of the colours of $\{c_3, c_4\}$ is not used for v_{k-1}. Hence we may assign it to v_k to obtain a proper L-colouring of G. □

Voigt [159] showed planar graphs which are not 4-choosable. Indeed consider the graph H and the list assignment $L_{i,j}$ depicted Figure 3. It is simple matter to check that for $j > i \geq 5$, H is not $L_{i,j}$-colourable.

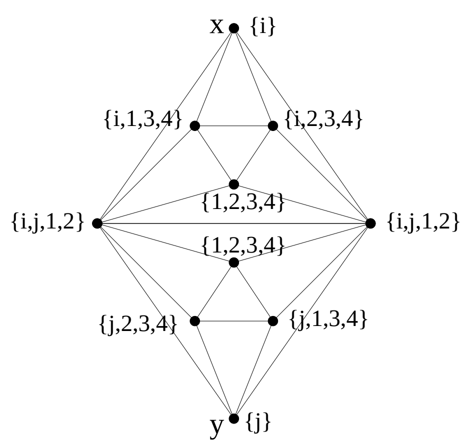

Figure 3: The graph H and its list assignment $L_{i,j}$

Let G be the graph obtained from 16 copies $H_{i,j}$ of H, $5 \leq i \leq 8$, $9 \leq j \leq 12$, by identifying all the vertices x of all copies and the vertices y of all copies. Let L be the list assignment defined by $L(x) = \{5, 6, 7, 8\}$, $L(y) = \{9, 10, 11, 12\}$ and $L(v) = L_{i,j}(v)$ if $v \in V(H_{i,j}) \setminus \{x, y\}$. Then G is not L-colourable and thus not

4-choosable.

However, the celebrated Four Colour Theorem by Appel and Haken [8, 9, 10] states that every planar graph is 4-colourable.

Theorem 18 (Appel and Haken [8, 9, 10]). *Every planar graph is* 4*-colourable.*

It is NP-Complete (see [53]) to decide if the chromatic number of a planar graph is 3 or 4, even if the maximum degree does not exceed 4.

But, the problem becomes easier for planar graphs with girth at least 4. According to Theorem 16, planar graphs with girth at least 4 are 3-degenerate and thus 4-choosable. The celebrated Grötzsch Theorem [61] asserts that such graphs are also 3-colourable.

Theorem 19 (Grötzsch [61]). *Every planar graph with girth at least* 4 *is* 3*-colourable.*

Thomassen [154] gave a short elegant proof of this theorem. Note that it implies that one can find in polynomial time the chromatic number of a planar graphs of girth at least 4 as it is polynomial to decide if a graph is 2-colourable.

Voigt [160] showed a planar graph with girth 4 which is not 3-choosable. But if its girth is at least 5 then a planar graph is 3-choosable.

Theorem 20 (Thomassen [153]). *Every planar graph with girth at least* 5 *is* 3*-choosable.*

The general idea of the proof of this theorem has the same flavour as the proof of Theorem 17. A stronger result is shown by induction: if the lists of the vertices have size three except on the outer face where the lists satisfy some particular conditions, then the graph is list colourable.

1.3 Edge-colouring

An *edge-colouring* of G is a mapping $f : E(G) \rightarrow S$. The element of S are *colours*; the edges of one colour form a *colour class*. If $|S| = k$ then f is a *k-edge-colouring*. An edge-colouring is *proper* if incident edges have different colours; that is, if each colour class is a matching. A graph is *k-edge-colourable* if it has a proper k-edge-colouring. The *chromatic index* or *edge-chromatic number* $\chi'(G)$ of a graph G is the least k such that G is k-edge-colourable.

Since edges sharing an endvertex need different colours, $\chi(G) \geq \Delta(G)$. Furthermore if a subgraph H of G is odd then a matching contains at most $\frac{|V(H)|-1}{2}$ edges. Hence at least $\frac{2|E(H)|}{|V(H)|-1}$ colours are needed to edge-colour H and thus G. It follows that

$$\chi'(G) \geq \max\left\{\Delta(G), \max\left\{\frac{2|E(H)|}{|V(H)|-1} \mid H \text{ odd subgraph of } G\right\}\right\}. \tag{1}$$

As an edge is incident to at most $2\Delta(G)-2$ other edges ($\Delta - 1$ at each endvertex), colouring the edges greedily we use at most $2\Delta(G)-1$ colours. However, one needs less colours. Vizing [156] and Gupta [63] independently showed that $\chi'(G) \leq \Delta(G)+1$.

Theorem 21 (Vizing [156], Gupta [63]). *If G is a graph then* $\chi'(G) \leq \Delta(G)+1$.

Proof. We prove the result by induction on $|E(G)|$. For $|E(G)| = 0$, it is trivial.

Suppose now that $|E(G)| \geq 1$ and that the assertion holds for graphs with fewer edges than G. Set $\Delta(G) = \Delta$.

Let xy_0 be an edge of G. By induction hpothesis, $G \setminus xy_0$ admits a $(\Delta+1)$-edge-colouring. As y_0 is incident to at most $\Delta-1$ edges in $G \setminus xy_0$, there exists a colour $c_1 \in \{1,2,\ldots,\Delta+1\}$ *missing* at y_0, i.e. such that no edge incident to y_0 is coloured c_1. If c_1 is also missing at x, then colouring xy_0 with c_1, we obtain a $(\Delta+1)$-edge-colouring of G. So we may assume that there is an edge xy_1 coloured c_1.

Because y_1 is incident to at most Δ edges, a colour $c_2 \in \{1,2,\ldots,\Delta+1\}$ is missing at y_1. If c_2 is missing at x then recolouring xy_1 with c_2 and colouring xy_0 with c_1, we obtain a $(\Delta+1)$-edge-colouring of G. So we may assume that there is an edge xy_2 coloured c_2.

And so on, we construct a sequence $y_1, y_2, \ldots$ of neighbours of x and a sequence of colours $c_1, c_2, \ldots$ such that: xy_i is coloured c_i and c_{i+1} is missing at y_i. Since the degree of x is bounded, there exists a smallest l such that for an integer $k < l, c_{l+1} = c_k$.

Now, for $0 \le i \le k-1$, let us recolour the edge xy_i with c_{i+1}.

There exists a colour $c_0 \in \{1,2,\ldots,\Delta+1\}$ missing at x. In particular, $c_0 \neq c_k$. Let P be the maximal path starting at y_{k-1} with edges alternatively coloured c_0 and c_k. Let us interchange the colour c_0 and c_k on $P + xy_{k-1}$. If P does not contain y_k, we have a $(\Delta+1)$-edge-colouring of G. If P contains (and thus ends in) y_k, recolouring the edge xy_i with c_{i+1} for $k \le i \le l$, we obtain a $(\Delta+1)$-edge-colouring of G. □

Hence $\chi'(G) \in \{\Delta(G), \Delta(G)+1\}$. A graph is said to be *Class 1* if $\chi'(G) = \Delta(G)$ and *Class 2* if $\chi'(G) = \Delta(G)+1$. Determining whether a graph is Class 1 or Class 2 is NP-complete [84]. However there are classes of graphs for which we know if they are Class 1 or Class 2. For example, a regular graph of odd order, say $2n+1$, is Class 2, as a matching contains at most $n/2$ edges. König Theorem [96] states that every bipartite graph is Class 1. Planar graphs with sufficiently large maximum degree Δ are Class 1. Sanders and Zhao [140] showed that planar graphs with maximum degree $\Delta \ge 7$ are Class 1. Vizing edge-colouring conjecture asserts that planar graphs of maximum degree 6 are also Class 1.

1.3.1 Line-graph and edge-colouring

Edge-colouring may be seen as a vertex colouring of a special class of graphs, namely the line graphs. The *line graph* of a graph G, denoted $L(G)$, is the graph whose vertices are the edges of G, with $ef \in E(L(G))$ whenever e and f share an endvertex. Then $\chi'(G) = \chi(L(G))$. Obviously, not all graphs are line graphs. Indeed a set of pairwise-intersecting edges of G form a clique in $L(G)$. Thus the neighbourhood of every vertex is covered by two cliques in $L(G)$. So $L(G)$ may be partitionned into cliques such that every vertex is in at most two of these cliques. Krausz [105] showed that this necessary condition to be a line graph is also sufficient. However, this characterization does not directly yield an efficient test for line graphs, because there are too many possible decompositions into cliques to test. Beineke [13] gave a forbidden subgraphs characterization that provides a polynomial algorithm to test if a graph is a line graph.

Theorem 22 (Beineke [13]). *A graph is a line graph of some graph if and only if it does not have any of the nine graphs depicted Figure 4 as a subgraph.*

As the forbidden subgraphs have at most six vertices, Theorem 22 yields a polynomial algorithm to test if a graph G is a line graph that runs in time $O(n^6)$. In fact, there is such an algorithm that runs in linear time (Lehot [107]) and produces a graph H such that $L(H) = G$ if it exists.

1.3.2 Edge-colouring multigraphs

In case of multigraph, the chromatic index may exceed $\Delta(G)+1$. Indeed the multigraph, called *fat triangle* (see Figure 5), with three vertices and k edges between each pair of vertices has maximum

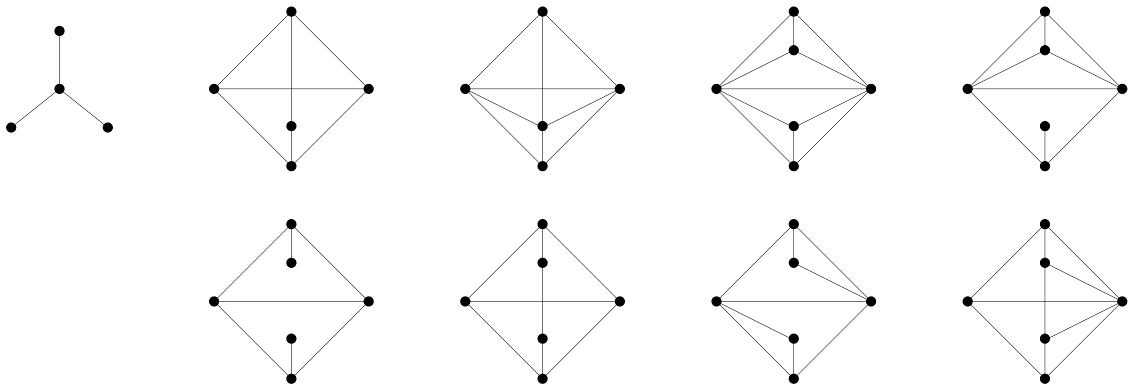

Figure 4: Forbidden subgraphs in a line graph

degree $2k$ and chromatic index $3k$. This example is actually extremal as Shannon [145] proved that for

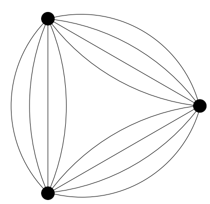

Figure 5: The fat triangle

every multigraph G, $\chi(G') \leq \frac{3}{2}\Delta(G)$. Vizing [156] and Gupta [63] improved this result by proving the following:

Theorem 23 (Vizing [156]; Gupta [63]).

$$\chi(G') \leq \Delta(G) + \mu(G).$$

These bounds follow from that of Andersen [7] and Goldberg [56, 57]:

$$\chi'(G) \leq \max\left\{\Delta(G), \max_T \left\lfloor \frac{1}{2}d(x) + \mu(x,y) + \mu(y,z) + d(z) \right\rfloor \right\}$$

where $T = \{x,y,z \in V(G) \mid z \in N(x) \cap N(y)\}$.

1.3.3 List edge-colouring

Analogously to list colouring, one can define list edge-colouring. In this variant, we assign lists to the edges and must choose a proper edge-colouring. The *list chromatic index* $ch'(G)$ is the minimum k such that for every assignment L of lists of size k to the edges, G admits a proper edge-colouring f such that $f(e) \in L(e)$ for every edge e. Equivalently, $ch'(G) = ch(L(G))$, where $L(G)$ is the line graph of G.

Analogously as for χ', the greedy algorithm yields $ch'(G) \leq 2\Delta(G) - 1$. So $ch'(G)$ is bounded in terms of $\chi'(G)$. It was suggested independently by many researchers including Vizing, Gupta, Albertson, Collins, and Tucker and appeared first in Bollobás and Harris [20] that the list chromatic index equals the chromatic index.

List Colouring Conjecture *The chromatic index is equal to the list chromatic index, that is* $\chi' = ch'$.

This conjecture and Vizing's Theorem (Theorem 21) would yield $ch'(G) \leq \Delta(G)+1$. Bollobás and Harris [20] proved that $ch'(G) < c\Delta(G)$ when $c > 11/6$ for sufficiently large Δ. Using probabilistic methods, Kahn [89] proved the conjecture asymptotically: $ch'(G) \leq (1+o(1))\Delta(G)$. The error term was sharpened by Häggkvist and Janssen [65]: $ch'(G) \leq \Delta(G) + O(\Delta(G)^{2/3}\sqrt{\log \Delta(G)}$ and further on by Molloy and Reed [120]: $ch'(G) \leq \Delta(G) + O(\Delta(G)^{1/2}(\log \Delta(G))^4)$.

Galvin [52] proved the List Colouring Conjecture for bipartite graph.

1.3.4 Fractional edge-colouring

Fractional edge-colourings and the *fractional chromatic index* $\chi'_f(G)$ are defined similarly to fractional colourings and the fractional chromatic number. Of course, we need to assign weights to the set $\mathcal{M}(G)$ of matchings of G.

The lower bound of (1) for the chromatic index is also a lower bound for the fractional chromatic index by the same argument. The seminal result on fractional edge-colourings, due to Edmonds [43], shows that it is indeed the fractional chromatic index.

Theorem 24 (Edmonds [43]). *The fractional chromatic index of a graph G is*

$$\chi'_f(G) = \max\left\{\Delta(G), \max\left\{\frac{2|E(H)|}{|V(H)|-1} \mid H \text{odd subgraph of } G\right\}\right\}.$$

Using this characterization, Padberg and Rao [129] obtained a polynomial algorithm for computing the fractional chromatic index of a graph and indeed an optimal fractional edge-colouring.

One of the most celebrated conjectures concerning edge-colouring is the Goldberg-Seymour Conjecture.

Conjecture 25 (Goldberg [55]; Seymour [144]). *Let G be a multigraph. Then*

$$\chi'(G) \leq \max\left(\Delta(G)+1, \lceil \chi'_f(G) \rceil\right).$$

The main results towards this conjecture and the List Colouring Conjecture are due to Kahn [90, 91] using the probabilistic method and hard core distributions.

Theorem 26 (Kahn [90]). $\chi'(G) \leq (1+o(1))\chi'_f(G)$.

Theorem 27 (Kahn [91]). $ch'(G) \leq (1+o(1))\chi'_f(G)$.

In these two theorems the $o(1)$ should be understood as a function tending to 0 as $\chi'_f(G)$ tends to infinity.

1.4 Total colouring

A *total colouring* of G is a mapping f from $V(G) \cup E(G)$ into a set S of *colours* such that:

- adjacent vertices have different colours;
- incident edges have different colours;

- each edge and its endvertices have different colours.

If $|S| = k$ then f is a *k-total colouring*. A graph is *k-total colourable* if it has a k-total colouring. The *total chromatic number* $\chi^T(G)$ of a graph G is the least k such that G is k-total-colourable. The colour classes in a total colouring are called *total stable sets*.

Since a vertex and the edges incident to it need different colours, then $\chi(G) \geq \Delta(G)+1$.

Total Colouring Conjecture $\chi^T(G) \leq \Delta(G)+2$.

Total colouring was introduced by Vizing [156, 157] and independently by Behzad [14]. They both formulated the Total Colouring Conjecture. A generalisation of the Total Colouring Conjecture for multigraph exists. It states that $\chi^T(G) \leq \Delta(G)+\mu(G)+1$ for every multigraph G.

The List Colouring Conjecture would yield the upper bound of $\Delta(G)+3$. Indeed consider a (vertex) colouring c of G in $\{1,\dots,\Delta+3\}$; such a colouring exists by Proposition 1. For any edge $e = xy$, set $L(e) = \{1,\dots,\Delta+3\} \setminus \{c(x),c(y)\}$. Then $|L(e)| = \Delta+1$. Now if $ch'(G) = \chi'(G)$, then G is $(\Delta+1)$-choosable and thus G is L-edge colourable and one can extend c into a proper $(\Delta+3)$-total colouring of G.

Trivially using distinct colours for edges and vertices, we obtain $\chi^T(G) \leq \chi'(G)+\chi(G) \leq 2\Delta+2$. Lots of better upper bounds on the total chromatic number have been obtained. We just give here the more significant ones. An interested reader is referred to the Section 4.9 of [86]. Hind [79, 80] showed that $\chi^T(G) \leq \chi'(G)+2\left\lceil\sqrt{\chi(G)}\right\rceil$. Häggkvist and Chetwynd [64] and independlty McDiarmid and Reed [116] proved that given a graph G with n vertices, $\chi^T(G) \leq \chi'(G)+k+1$ where k is an integer such that $k! > n$. In [82], it is proved that if $\Delta(G)$ is large enough, then $\chi^T(G) \leq \Delta(G)+O(\log^{10}\Delta(G))$ (see also Chapter 9 of [122]). Finally Molloy and Reed [119] proved that there is a constant c such that the total chromatic number is at most $\Delta+c$ as long as Δ is sufficiently large, where $c \leq 10^{26}$.

McDiarmid and Reed [116], showed that the Total Colouring Conjecture is true for almost all graphs: there is a constant $c < 1$ such that only a fraction $o(c^{n^2})$ of all graphs on n vertices are potential counterexample to the conjecture.

The Total Colouring Conjecture has been proved for many classes of graphs. (See Section 4.9 of [86].) For example, it has been settled for graphs with maximum degree 5 by Kostochka [97, 98, 99], for graphs with minimum degree $\delta(G) \geq 3|V(G)|/4$ by Hilton and Hind [78], for r-partite graphs by Yap [166] and planar graphs with the exception of $\Delta = 6$ and $\Delta = 7$ by Borodin (see Section 4.9 of [86]).

The Total Colouring Conjecture asserts that the graphs fall into two classes. *Type 1 graphs* have $\chi^T(G) = \Delta(G)+1$, whereas *type 2 graphs* have $\chi^T(G) = \Delta(G)+2$. Sánchez-Arroyo [135] proved that it is NP-complete to decide for a given graph if it is type 1 or type 2. McDiarmid and Sánchez-Arroyo [118] showed that it remains NP-complete for a given k-regular (for $k \geq 3$) bipartite graph. Planar graphs with sufficently high maximum degree Δ are of Type 1. Borodin [24] showed this for planar graphs with maximum degree $\Delta \geq 14$. This bound 14 then been decreased succesively to 12 [29] and then 11 [30] by Borodin, Kostochka and Woodall and then to 10 by Wang [161]. Very recently, Kowalik, Sereni and Škrekovski [101] showed that every planar graph of maximum degree 9 is 10-total colourable. Sanders and Zhao [138] showed that planar graphs of maximum degree 7 are type 1.

Interested readers in total colouring are referred to the book of Yap [167].

1.4.1 Fractional total colouring

Fractional total colourings and the *fractional total chromatic number* $\chi^T_f(G)$ are defined similarly to fractional colourings and the fractional chromatic number. Of course, we need to assign weights to the set $\mathcal{T}(G)$ of total stable sets of G.

An approach to prove the Total Colouring Conjecture is to prove that it holds fractionally. It is not difficult to show that $\chi^T_f(G) \le \Delta(G)+3$. Indeed let $S_1,\dots,S_{\Delta+3}$ be the stable sets of a proper $(\Delta+3)$-colouring of G and $M_1,\dots,M_{\Delta+1}$ be the matchings of a proper $(\Delta+1)$-edge-colouring of G. For every $1 \le i \le \Delta+3$ and $1 \le j \le \Delta+1$, let T_i, j be the total stable set $S_i \cup (M_j \setminus \{e \mid e \cap S_i \ne \emptyset\})$ and set $w(T_{i,j}) = \frac{1}{\Delta+1}$. Then for each vertex v, $w(v) = 1$ as it is in the $\Delta+1$ $T_{i,j}$ such that $v \in S_i$ and for each edge $e = uv$, $w(e) = 1$ as it is in the $\Delta+1$ $T_{i,j}$ such that $e \in M_j$, $u \notin S_i$ and $v \notin S_i$. In [94], Kilakos and Reed proved that the Total Colouring Conjecture holds fractionally: $\chi^T_f \le \Delta(G)+2$. However the complexity of computing the fractional total chromatic number remains unclear.

2 Colouring with contraints at distance two or more

2.1 Motivation: channel assignment

The channel assignment problem in radio or cellular phone networks is the following : we need to assign radio frequency bands to transmitters (each station gets one channel which corresponds to an integer). In order to avoid interference, if two stations are very close, then the separation of the channels assigned to them has to be large enough. Moreover, if two stations are close (but not very close), then they must also receive channels that are sufficiently far apart.

Such a problem may be modelled by $L(p,q)$-labellings of a graph G. The vertices of this graph correspond to the transmitters and two vertices are linked by an edge if they are very close. Two vertices are then considered close if they are at distance 2 in the graph. Let $dist(u,v)$ denote the distance between the two vertices u and v. An $L(p,q)$-*labelling of* G is an integer assignment f to the vertex set $V(G)$ such that :

- $|f(u) - f(v)| \ge p$, if $dist(u,v) = 1$, and
- $|f(u) - f(v)| \ge q$, if $dist(u,v) = 2$.

As the separation between channels assigned to vertices at distance 2 cannot be smaller than the separation between channels assigned to vertices at distance 1, it is often assumed that $p \ge q$.

The *span* of f is the difference between the largest and the smallest labels of f plus one. The $\lambda_{p,q}$-*number* of G, denoted by $\lambda_{p,q}(G)$, is the minimum span over all $L(p,q)$-labellings of G.

Moreover, very often, because of technical reasons or dynamicity, the set of channels available varies from transmitter to transmitter. Therefore one has to consider the list version of $L(p,q)$-labellings. Recall that a k-*list-assignment* L of a graph is a function which assigns to each vertex v of the graph a list $L(v)$ of k prescribed integers. Given a graph G, the *list* $\lambda_{p,q}$-*number*, denoted $\lambda^l_{p,q}(G)$ is the smallest integer k such that, for every k-list-assignment L of G, there exists an $L(p,q)$-labelling f such that $f(v) \in L(v)$ for every vertex v. Surprisingly, list $L(p,q)$-labellings have been studied only very little explicitely and seems to appear only very recently in the literature [95]. However, some of the proofs for $L(p,q)$-labellings also work for list $L(p,q)$-labellings.

Generalisations of $L(p,q)$-labellings in which for each $i \ge 1$, a minimum gap of p_i is required for channels assigned to vertices at distance i, have also been studied (see for example [109] or [102]).

2.2 Bounds on the $\lambda_{p,q}$-number

Note that $L(1,0)$-labellings of G correspond to ordinary vertex colourings of G. Hence the $\lambda_{1,0}$-number of a graph G equals its *chromatic number* $\chi(G)$, and its $\lambda^l_{1,0}$-number equals its *choice number* $ch(G)$.

Moreover $L(1,1)$-labellings of G correspond to the vertex colourings of the square of G. The *square* of a graph G, denoted G^2, is the graph with vertex set $V(G)$ such that two vertices u,v are linked by an edge in G^2 if and only if u and v are at distance at most 2 in G. Formally, $E(G^2) = \{ uv \mid dist_G(u,v) \leq 2 \}$. So $\lambda_{1,1}(G) = \chi(G^2)$ and $\lambda^l_{1,1}(G) = ch(G^2)$.

Recall that $\omega(G) \leq \chi(G) \leq ch(G) \leq \Delta(G)+1$. Similar easy inequalities may be obtained for $L(p,q)$-labellings : $q\,\omega(G^2) - q + 1 \leq \lambda_{p,q}(G) \leq \lambda^l_{p,q}(G) \leq p\Delta(G^2)+1$. As $\omega(G^2) \geq \Delta(G)+1$, the previous inequality gives that $\lambda_{p,q} \geq q\Delta + 1$. However, a straightforward argument shows that $\lambda_{p,q} \geq q\Delta + p - q + 1$. In the same way, $\Delta(G^2) \leq \Delta^2(G)$ so $\lambda^l_{p,q}(G) \leq p\Delta^2(G)+1$ and the greedy algorithm shows that $\lambda^l_{p,q}(G) \leq (2q-1)\Delta^2(G) + (2p-1)\Delta(G) + 1$. Taking an $L(\lceil p/k \rceil, \lceil q/k \rceil)$-labelling and multiplying each label by k, we obtain an $L(p,q)$-labelling. This proves the following easy observation.

Proposition 28. *For all graph G and positive integers k,p,q we have*

$$\lambda_{p,q}(G) \leq k(\lambda_{\lceil p/k \rceil, \lceil q/k \rceil}(G) - 1) + 1.$$

In general, determining the $\lambda_{p,q}$-number of a graph is NP-hard [54]. In their seminal paper, Griggs and Yeh [60] observed that a greedy algorithm yields $\lambda_{2,1}(G) \leq \Delta^2 + 2\Delta + 1$. Moreover, they conjectured that this upper bound can be decreased to $\Delta^2 + 1$.

Conjecture 29 (Griggs and Yeh [60]). *For every $\Delta \geq 2$ and every graph G of maximum degree Δ,*

$$\lambda_{2,1}(G) \leq \Delta^2 + 1.$$

This upper bound would be tight: there are graphs with maximum degree Δ, diameter 2 and Δ^2+1 vertices, namely the 5-cycle, the Petersen graph and the Hoffman-Singleton graph. Thus, their square is a clique of order Δ^2+1, so the span of every $L(2,1)$-labelling is at least Δ^2+1.

However, such graphs exist only for $\Delta = 2,3,7$ and possibly 57, as shown by Hoffman and Singleton [83]. So one can ask how large may be the $\lambda_{2,1}$-number of a graph with large maximum degree. As it should be at least as large as the largest clique in its square, one can ask what is the largest clique number $\gamma(\Delta)$ of the square of a graph with maximum degree Δ. If Δ is a prime power plus 1, then $\gamma(\Delta) \geq \Delta^2 - \Delta + 1$. Indeed, in the projective plane of order $\Delta - 1$, each point is in Δ lines, each line contains Δ points, each pair of distinct points is in a line and each pair of distinct lines has a common point. Consider the *incidence graph* of the projective plane: it is the bipartite graph with vertices the set of points and lines of the projective plane, and every line is linked to all the points it contains. The properties of the projective plane implies that the set of points and the set of lines form two cliques in the square of this graph, and there are $\Delta^2 - \Delta + 1$ vertices in each.

Jonas [88] improved slightly on Griggs and Yeh's upper bound by showing that every graph of maximum degree Δ admits a $(2,1)$-labelling with span at most $\Delta^2 + 2\Delta - 3$. Subsequently, Chang and Kuo [35] provided the upper bound $\Delta^2 + \Delta + 1$ which remained the best general upper bound for about a decade. Král' and Škrekovski [104] brought this upper bound down by 1 as the corollary of a more general result. And, using the algorithm of Chang and Kuo [35], Gonçalves [58] decreased this bound by 1 again, thereby obtaining the upper bound $\Delta^2 + \Delta - 1$. Note that Conjecture 29 is true for planar graphs of maximum degree $\Delta \neq 3$. For $\Delta \geq 7$ it follows from a result of van den Heuvel and McGuiness [77], and Bella et al. [15] proved it for the remaining cases.

In [69], we show Conjecture 29 for sufficiently large Δ, and generalises it to $L(p,1)$-labelling.

Theorem 30 (Havet, Reed and Sereni [69]). *Let p be a positive integer. There is a Δ_p such that for every graph G of maximum degree $\Delta \geq \Delta_p$,*

$$\lambda_{p,1}(G) \leq \Delta^2 + 1.$$

The proof of this theorem makes an intensive use of probabilistic methods. In particular, it uses dozen of times the Lovász Local Lemma [47] together with some concentration bounds like the Chernoff Bound [3, 113], Azuma's inequality [12], Talagrand's inequality [151] and McDiarmid's inequality [114]. All this tools are presented in the book of Molloy and Reed [122] to which we refer the reader interested in colouring via the probabilistic method. It is possible that using a similar proof, one can show that $\lambda_{p,1}(G) \leq \gamma(\Delta)$. Note however that it does not mean that for every graph G $\lambda_{p,1}(G) \leq \omega(G^2)$, which in fact is false. For example, consider a graph consisting of five vertices x_i, $1 \leq i \leq 5$ together with $5k$ additional vertices of degree two, such that x_i has k common neighbours with x_{i+1} (indices should be understood modulo 5). Then $\omega(G^2) = 2k+1$ and $\chi(G^2) = \frac{5k+5}{2}$.

Problem 31. *What is the smallest function f such that for every graph G, $\chi(G^2) \leq f(\omega(G^2))$?*

The problem of determining $\lambda_{p,q}(G)$ has been studied for lots of specific classes of graphs (see the survey of Yeh [168]). We will now only discuss the few classes that we studied.

2.3 $L(p,q)$-labellings of planar graphs

Because the transmitters are laid out on the surface of the earth, $L(p,q)$-labellings of planar graphs are of particular interest. There are planar graphs for which $\lambda_{p,q} \geq \frac{3}{2} q\Delta + c(p,q)$, where $c(p,q)$ is a constant depending on p and q. For example, consider a graph consisting of three vertices x, y and z together with $3k-1$ additional vertices of degree two, such that z has k common neighbours with x and k common neighbours with y, x and y are connected and have $k-1$ common neighbours (see Figure 6).

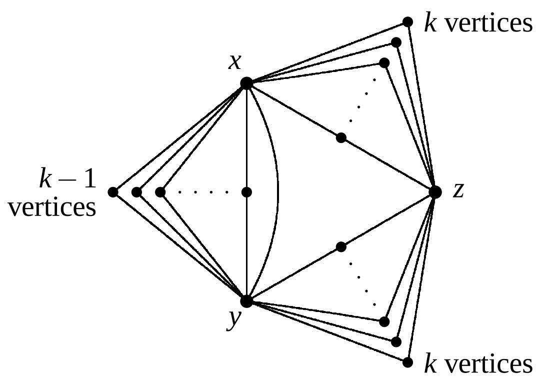

Figure 6: The planar graphs G_k

This graph has maximum degree $2k$ and yet its square contains a clique with $3k+1$ vertices (all the vertices except z).

A first upper bound on $\lambda_{p,q}(G)$, for planar graphs G and positive integers $p \geq$ has been proved by Van den Heuvel and McGuinness [77] : $\lambda_{p,q}(G) \leq 2(2q-1)\Delta + 10p + 38q - 24$. Molloy and Salavatipour [123] improved this bound by showing the following :

Theorem 32 (Molloy and Salavatipour [123]). *For a planar graph G and positive integers p,q,*

$$\lambda_{p,q}(G) \leq q\left\lceil \frac{5}{3}\Delta \right\rceil + 18p + 77q - 18.$$

Moreover, they described an $O(n^2)$ time algorithm for finding an $L(p,q)$-labelling whose span is at most the bound in their theorem.

The Four Colour Theorem (Theorem 18) states that $\lambda_{1,0}(G) = \chi(G) \leq 4$ for planar graphs. Regarding the chromatic number of the square of a planar graph, Wegner [162] posed the following conjecture which is mentioned in Jensen and Toft [86, Section 2.18].

Conjecture 33 (Wegner [162]). *For a planar graph G of maximum degree* Δ *:*

$$\lambda_{1,1}(G) = \chi(G^2) \leq \begin{cases} 7, & \text{if } \Delta = 3, \\ \Delta+5, & \text{if } 4 \leq \Delta \leq 7, \\ \left\lfloor \frac{3}{2}\Delta \right\rfloor + 1, & \text{if } \Delta \geq 8. \end{cases}$$

Wegner also gave examples showing that these bounds would be tight. For $\Delta \geq 8$, these are the same examples is in Figure 6.

Kotoschka and Woodall [100] conjectured that, for every square of a graph, the list-chromatic number equals the choice number. This conjecture and Wegner's one imply directly the following :

Conjecture 34. *For a planar graph G of maximum degree* Δ *:*

$$\lambda^l_{1,1}(G) = ch(G^2) \leq \begin{cases} 7, & \text{if } \Delta = 3, \\ \Delta+5, & \text{if } 4 \leq \Delta \leq 7, \\ \left\lfloor \frac{3}{2}\Delta \right\rfloor + 1, & \text{if } \Delta \geq 8. \end{cases}$$

Wegner also showed that if G is a planar graph with $\Delta = 3$, then G^2 can be 8-coloured. Very recently, Thomassen [155] solved Wegner's conjecture for $\Delta = 3$ and Cranston and Kim [38] showed that the square of every connected graph (non necessarily planar) which is subcubic (i.e., with $\Delta \leq 3$) is 8-choosable, except for the Petersen graph. However, the 7-choosability of the square of subcubic planar graphs is still open. The first upper bound on $\chi(G^2)$ in terms of Δ was obtained by Jonas [88] who showed $\chi(G^2) \leq 8\Delta - 22$. This bound was later improved by Wong [164] to $\chi(G^2) \leq 3\Delta + 5$ and then by van den Heuvel and McGuinness [77] to $\chi(G^2) \leq 2\Delta + 25$. Better bounds were then obtained for large values of Δ. It was shown that $\chi(G^2) \leq \lceil \frac{9}{5}\Delta \rceil + 1$ for $\Delta \geq 749$ by Agnarsson and Halldórsson [1], and that $\chi(G^2) \leq \lceil \frac{9}{5}\Delta \rceil + 1$ for $\Delta \geq 47$ by Borodin et al. [27]. Finally, the best known upper bound so far has been obtained by Molloy and Salavatipour [123] as a special case of Theorem 32 :

Theorem 35 (Molloy and Salavatipour [123]). *For a planar graph G,*

$$\lambda_{1,1}(G) = \chi(G^2) \leq \left\lceil \frac{5}{3}\Delta \right\rceil + 78.$$

As mentioned in [123], the constant 78 can be reduced for sufficiently large Δ. For example, it was improved to 24 when $\Delta \geq 241$.

In [70], we prove the following theorem :

Theorem 36 (Havet et al. [70]). *The square of every planar graph G of maximum degree* Δ *has list chromatic number at most* $(1+o(1))\,\frac{3}{2}\Delta$*. Moreover, given lists of this size, there is an acceptable colouring in which the colours on every pair of adjacent vertices of G differ by* $\Delta^{1/4}$.

The idea of the proof of this theorem is to reduce to a problem of colouring a line graph. We follow the approach developed by Kahn [91] for the proof of Theorem 27. We need to modify the proof so it can handle our situation in which we have a graph which is slightly more than a line graph and in which we have lists with fewer colours than he permitted. This reduction to line graphs for which asymptotically the chromatic number equals the fractional chromatic number suggests the following question.

Problem 37. *Let G be a planar graph. Is is true that $\chi(G^2) = (1+o(1))\chi_f(G^2)$?*

Theorem 36 yields that for every planar graph G and any fixed p, $\lambda^l_{p,1}(G) \le (1+o(1))\,\frac{3}{2}\Delta(G)$. Together with Proposition 28, this yields:

Corollary 38 (Havet et al. [70]). *Let $p \ge q$ be two fixed integers. Then for any planar graph G we have $\lambda_{p,q}(G) \le (1+o(1))\,\frac{3}{2}\,q\Delta(G)$.*

Note that using exactly the same proof as for Theorem 36, one can show that for any fixed $p \ge q$, for every planar graph G, $\lambda^l_{p,q}(G) \le (1+o(1))\,\frac{3}{2}\,(2q-1)\,\Delta(G)$.

Problem 39. *Is it true that $\lambda^l_{p,q}(G) \le (1+o(1))\,\frac{3}{2}\,q\Delta(G)$ for planar graphs G and fixed $p \ge q$?*

Finally, Theorem 36 implies that $\omega(G^2) \le (1+o(1))\,\frac{3}{2}\Delta(G)$ for any planar graph G. But does there exists a simple proof of this inequality. Furthermore, a step forward to Wegner's Conjecture would be to prove that $\omega(G^2) \le \left\lfloor \frac{3}{2}\Delta \right\rfloor + 1$. This inequality is tight as shown by the graph G_k depicted Figure 6.

2.3.1 Connections with frugal and cyclic colouring

The size (number of vertices in its boundary) of a largest face of a plane graph G is denoted by $\Delta^*(G)$. A *cyclic colouring* of a plane graph G is a vertex colouring of G such that any two vertices incident to the same face have distinct colours. This concept was introduced by Ore and Plummer [128], who also proved that a plane graph has a cyclic colouring using at most $2\Delta^*$ colours. Borodin [23] (see also Jensen and Toft [86, page 37]) conjectured the following.

Conjecture 40 (Borodin [23]). *Any plane graph has a cyclic colouring with $\left\lfloor \frac{3}{2}\Delta^* \right\rfloor$ colours.*

The best general known upper bound in the general case is due to Sanders and Zhao [139], who proved that any plane graph has a cyclic colouring with $\left\lceil \frac{5}{3}\Delta^* \right\rceil$ colours. Denote by $f_c(x)$ the minimum number of colours needed to cyclically colour every plane graph of maximum face size m. The value of $f_c(m)$ is known for $m \in \{3,4\}$: $f_c(3) = 4$ (the problem of finding $f_c(3)$ being equivalent to the Four Colour Theorem proved in [10]) and $f_c(4) = 6$ (see [23, 26]). It is also known that $f_c(5) \in \{7,8\}$ and $f_c(6) \le 10$ [31], and that $f_c(7) \le 12$ [25].

As noted by Amini et al. [5], there are connections between Conjecture 40 and Conjecture 34 through *frugal colouring*. For an integer $m \ge 1$, an *m-frugal colouring* of a graph G is a proper vertex colouring of G (i.e., adjacent vertices get a different colour) such that no colour appears more than m times in the neighbourhood of any vertex. The least number of colours in a m-frugal colouring of G is called the *m-frugal chromatic number*, denoted $\chi^m(G)$. Clearly, $\chi^1(G)$ is the chromatic number of the square of G; and for m at least the maximum degree of G, $\chi^m(G)$ is the usual chromatic number of G. This type of colouring was introduced by Hind, Molloy and Reed in [81].

Inspired by Wegner's Conjecture, Amini, Esperet and van den Heuvel conjectured the following bounds for the m-frugal chromatic number of planar graphs.

Conjecture 41 (Amini, Esperet and van den Heuvel [5]).
For any integer $m \ge 1$ and planar graph G with maximum degree $\Delta(G) \ge \max\{2m, 8\}$ we have

$$\chi^m(G) \le \begin{cases} \left\lfloor \frac{\Delta(G)-1}{m} \right\rfloor + 3, & \text{if } m \text{ is even;} \\ \left\lfloor \frac{3\Delta(G)-2}{3m-1} \right\rfloor + 3, & \text{if } m \text{ is odd.} \end{cases}$$

Note that the graphs G_k in Figure 6 also show that the bounds in this conjecture are best possible. The graph G_k has maximum degree $2k$. First consider an m-frugal colouring with $m = 2\ell$ even. We can use the same colour at most $\frac{3}{2}m$ times on the vertices of G_k, and every colour that appears exactly $\frac{3}{2}m = 2\ell$ times must appear exactly ℓ times on each of the three sets of common neighbours of x and y, of x and z, and of y and z. So we can take at most $\frac{1}{\ell}(k-1) = \frac{1}{m}(\Delta(G_k)-1)$ colours that are used $\frac{3}{2}m$ times. The graph that remains can be coloured using just three colours.

If $m = 2\ell+1$ is odd, then each colour can appear at most $3\ell+1 = \frac{1}{2}(3m-1)$ times, and the only way to use a colour so many times is by using it on the vertices in $V(G_k)\setminus\{x,y,z\}$. Doing this at most $\frac{3k-1}{(3m-1)/2} = \frac{3\Delta(G)-2}{3m-1}$ times, we are left with a graph that can be coloured using three colours.

In [5], Amini, Esperet and van den Heuvel showed that if there is an even $m \geq 4$ so that Borodin's Conjecture 40 holds for all plane graphs with $\Delta^* \leq m$, and Conjecture 41 is true for the same value m, then Wegner's conjecture is true up to an additive constant factor. More precisely, if $m \geq 4$ is an even integer such that every plane graph G with $\Delta^*(G) \leq m$ has a cyclic colouring using at most $\frac{3}{2}m$ colours, then, if G is a planar graph satisfying $\chi^m(G) \leq \left\lfloor \frac{\Delta(G)-1}{m} \right\rfloor + 3$, we also have $\chi(G^2) = \chi^1(G) \leq \left\lfloor \frac{3}{2}\Delta(G) \right\rfloor + \frac{9}{2}m - 1$.

The concept of facial colourings, introduced in [103], extends the concept of cyclic colourings. A *facial segment* of a plane graph G is a sequence of vertices in the order obtained when traversing a part of the boundary of a face. The *length* of a facial segment is the number of its edges. Two vertices u and v of G are *ℓ-facially adjacent*, if there exists a facial segment of length at most ℓ between them. An *ℓ-facial colouring* of G is a function which assigns a colour to each vertex of G such that any two distinct ℓ-facially adjacent vertices are assigned with distinct colours. A graph admitting an ℓ-facial colouring with k colours is called *ℓ-facially k-colourable*. The following conjecture, called $(3\ell+1)$-Conjecture, is proposed in [103]:

Conjecture 42 (Král', Madaras and Škrekovski). *Every plane graph is ℓ-facially colourable with $3\ell+1$ colours.*

Observe that the bound offered by Conjecture 42 is tight: as shown by Figure 7, for every $\ell \geq 1$, there exists a plane graph which is not ℓ-facially 3ℓ-colourable.

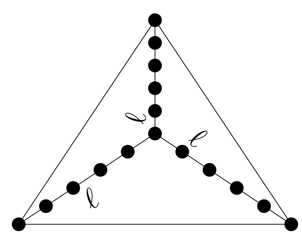

Figure 7: The plane graph $G_\ell = (V,E)$: each thread represents a path of length ℓ. The graph G_ℓ is not ℓ-facially 3ℓ-colourable: every two vertices are ℓ-facially adjacent, therefore any ℓ-facial colouring must use $|V| = 3\ell+1$ colours.

Conjecture 42 can be considered as a counterpart for ℓ-facial colouring of Conjecture 40. Note that Conjecture 42 implies Conjecture 40 for odd values of Δ^*.

Conjecture 42 is trivially true for $\ell = 0$, and is equivalent to the Four Colour Theorem for $\ell = 1$. It is open for all other values of ℓ. As noted in [103], if Conjecture 42 were true for $\ell = 2$, it would have several interesting corollaries. Besides giving the exact value of $f_c(5)$ (which would then be 7), it would allow to decrease from 16 to 14 (by applying a method from [103]) the upper bound on the number of colours needed to 1-diagonally colour every plane quadrangulation (for more details on this problem, consult [85, 136, 137, 103]). It would also imply Conjecture 34 restricted to plane cubic graphs since colourings of the square of a plane cubic graph are precisely its 2-facial colourings.

Let $f_f(\ell)$ be the minimum number of colours needed to ℓ-facially colour every plane graph. Clearly, $f_c(2\ell+1) \leq f_f(\ell)$. So far, no value of ℓ is known for which this inequality is strict. The following problem is offered in [103].

Problem 43 (Král, Madaras and Škrekovski [103]). *Is it true that, for every integer* $\ell \geq 1$*,* $f_c(2\ell+1) = f_l(\ell)$*?*

Another conjecture that should be maybe mentioned is the so-called 3ℓ-Conjecture proposed in [41], stating that every plane triangle-free graph is ℓ-facially 3ℓ-colourable. Similarly as the $(3\ell+1)$-Conjecture, if this conjecture were true, then its bound would be tight and it would have several interesting corollaries (see [41] for more details).

It is proved in [103] that every plane graph has an ℓ-facial colouring using at most $\lfloor \frac{18}{5}\ell \rfloor + 2$ colours (and this bound is decreased by 1 for $\ell \in \{2,4\}$). So, in particular, every plane graph has a 3-facial 12-colouring. In [72], we improve this last result by proving the following theorem.

Theorem 44 (Havet, Sereni and Škrekovski [72]). *Every plane graph is* 3*-facially* 11*-colourable.*

The proof of this theorem uses a rather complicated discharging method involving faces and vertices.

2.3.2 Planar graphs with given girth

In [41], Dvořák, Škrekovski and Tancer proved that the choice number of the square of a subcubic graph G is at most 4 if $Mad(G) < 24/11$ and G has no 5-cycle, at most 5 if $Mad(G) < 7/3$ and at most 6 if $Mad(G) < 5/2$. By Theorem 16, it implies that the choice number of the square of a planar graph with girth g is at most 6 if $g \geq 10$, at most 5 if $g \geq 14$ and at most 4 if $g \geq 24$. The two latter results had been previously proved by Montassier and Raspaud [124].

In [67], we improve some of these results.

Theorem 45 (Havet [67]). *The choice number of the square of a subcubic graph with maximum average degree less than* $18/7$ *is at most* 6*.*

As a corollary, by Theorem 16, we get that the choice number of the square of a subcubic planar graph with girth at least 9 is at most 6. (Note that this corollary has been proved later and independently by Cranston and Kim [38].) We show that if the girth is large enough then the square of a subcubic planar graph is 5-choosable.

Theorem 46 (Havet [67]). *The choice number of the square of a subcubic planar graph with girth at least* 13 *is at most* 5*.*

The idea of the proofs of Theorems 45 and 46 is to consider a minimum counterexample, to show that it does not contain some configurations (i.e. induced subgraphs) and then to use a discharging method to get a contradiction.

2.4 $(p,1)$-total labelling

The *first subdivision* of a graph G is the graph $s_1(G)$ obtained from G by inserting one vertex along each edge of G. An $L(p,1)$-labelling of $s_1(G)$ corresponds to an assignment of positive integers to $V(G) \cup E(G)$ such that:

(i) any two adjacent vertices of G receive distinct integers,

(ii) any two adjacent edges of G receive distinct integers, and

(iii) a vertex and an edge incident receive integers that differ by at least p in absolute value.

We call such an assignment a *(p,1)-total labelling of* G. It is a total colouring strenghtened with an extra condition by insisting on a minimal separation of p between incident vertices and edges.

The *span* of a $(p,1)$-total labelling is the maximum over all the labels. The $(p,1)$-*total number* of a graph G, denoted by $\lambda_p^T(G)$, is the minimum span of a $(p,1)$-total labelling of G. Note that a $(1,1)$-total labelling is a total colouring so $\lambda_1^T = \chi^T$, where χ^T is the total chromatic number.

Looking at the label of a vertex with maximum degree and its incident edges, it is easy to see that $\lambda_p^T \geq \Delta + p$. This lower bound may be increased to $\Delta + p + 1$ if G is Δ-regular or if $p \geq \Delta$.

Colouring the vertices with $\{1, \dots, \chi(G)\}$ and edges with $\{\chi(G)+p, \dots \chi(G)+\chi'(G)+p-1\}$, we obtain a $(p,1)$-total labelling of G. So,

$$\lambda_p^T(G) \leq \chi(G) + \chi'(G) + p - 1. \tag{2}$$

As a corollary, if G be a bipartite graph then $\Delta + p \leq \lambda_p^T(G) \leq \Delta + p + 1$. Moreover if $p \geq \Delta$ then $\lambda_p^T(G) = \Delta + p + 1$. If $p < \Delta$ then there are bipartite graphs with maximum degree Δ such that $\lambda_p^T = \Delta + p$ and others such graphs that $\lambda_p^T = \Delta + p$. In [73], we proved that for any fixed $\Delta > p$ then it is NP-complete to decide for a bipartite graph G with maximum degree Δ if $\lambda_p^T(G) = \Delta + p$ unless $\Delta = 3$ and $p = 2$ in which case, we gave a polynomial algorithm. However, for any fixed Δ, it is polynomial to decide if a tree with maximum degree Δ has $(p,1)$-total number $\Delta + p + 1$ using dynamic programming. But the complexity of the determining the $(p,1)$-total number of a tree remains unclear.

Moreover using Brooks and Vizing Theorems (and a short proof for odd cycles and odd complete graphs), we derived [74] from (2) that $\lambda_p^T \leq 2\Delta + p$. However, this upper bound is not supposed to be tight when $\Delta \geq p$: As a natural extension of the Total Colouring Conjecture to $(p,1)$-total labelling, we [74] conjectured the following.

$(\mathbf{p},\mathbf{1})$-Total Labelling Conjecture: $\lambda_p^T \leq \Delta + 2p - 1$ *or* $\lambda_p^T \leq \min\{\Delta + 2p - 1, 2\Delta + p - 1\}$.

As noted in [74], lots of upper bounds on the total chromatic number (see Subsection 1.4) may be generalised to λ_p^T. For example, generalising the result of [82], one can show that if $\Delta(G)$ is large enough then $\lambda_p^T(G) \leq \Delta(G) + O(\log^{10}\Delta(G))$ and extending a result of [116], also show that as $n \to \infty$, the proportion of graphs on n vertices with $(p,1)$-total number $\lambda_p^T > \Delta + 2p$ is very small. More precisely, if q and c are constants with $0 < q < 1$ and $0 < c < \min\{\frac{1}{3}, \frac{q}{2}\}$, then

$$P\{\lambda_p^T(G_{n,q}) > \Delta + 2p\} = o(n^{-cn/2}).$$

It is also very likely, that mimicing the proof of [119], one can prove the existence of a constant c_p such that the $(p,1)$-total number is at most $\Delta + c_p$ provided that Δ is large enough. It has been verified for sparse graphs by Esperet, Montassier and Raspaud [49].

In [74], we also show that $\lambda_p^T \leq 2\Delta - 2\log(\Delta+2) + 2\log(16p-8) + p$ which gives a better upper bound on the $(p,1)$-total number when Δ is not too large.

We also study [74] the $(p,1)$-total number of complete graphs and determine the exact values of the $(p,1)$-total numbers for almost all complete graphs : If n is odd then $\lambda_p^T(K_n) = \min\{n+2p-2, 2n+$

$p-2\}$; if n is even then $\lambda_p^T(K_n) = \min\{n+2p-2, 2n+p-2\}$ if $n \leq p+5$, $\lambda_p^T(K_n) = n+2p-1$ if $n > 6p^2-10p+4$ and $\lambda_p^T(K_n) \in \{n+2p-2, n+2p-1\}$ otherwise.

We then focused $(2,1)$-total labelling. We show that if $\Delta \geq 2$, then $\lambda_2^T \leq 2\Delta+1$ and therefore the $(p,1)$-Total Labelling Conjecture is true when $p=2$ and $\Delta = 3$. In fact, the bound for this special case is tight as $\lambda_2^T(K_4) = 7$. However, we conjecture that K_4 is essentially the unique such graph.

Conjecture 47 (Havet and Yu [74]). *Let G be a connected graph. If $\Delta(G) \leq 3$ and $G \neq K_4$ then $\lambda_2^T(G) \leq 6$.*

One approach to prove the $(p,1)$-Total Labelling Conjecture is to obtain a small function $a(p)$ such that a $(p,1)$-total labelling with span $\Delta + a(p)$ of a graph can be constructed by extending a vertex colouring with a suitable edge colouring.

Conjecture 48. *Let $p \geq 1$. There is an integer $a(p)$, such that for any vertex colouring c_v of a non-complete graph G with colours in $[1,\Delta]$, there is an edge colouring c_e of G with colours in $[1, \Delta + a(p)]$ such that $c_v \cup c_e$ is a $(p,1)$-total labelling of G.*

Conjecture 48 for $a(p) = 4p-1$ is implied by the List Colouring Conjecture. Since every graph is $(\Delta+1)$-edge colourable (Vizing's Theorem), the List Colouring Conjecture implies that it also is $(\Delta+1)$-edge choosable. Let c_v be a vertex colouring of a non-complete graph with colours in $[1,\Delta]$. For any edge $e=(x,y)$, there is a set $L(e) \subset [0, \Delta+4p-2]$ of $\Delta+1$ colours such that $L(e) \cap ([c_v(x)-p+1, c_v(x)+p-1] \cup [c_v(y)-p+1, c_v(y)+p-1] = \emptyset$. Then since G is $(\Delta+1)$-choosable, there exists a desired edge colouring. In [74], we relax the constraints and show that one can extend the vertex colouring with a $(\Delta+3p+1)$-fractional edge colouring.

3 Improper colouring

3.1 Motivation

Our sudy is motivated by a problem posed by Alcatel Space Technologies (see [4]). A satellite sends informations to receivers on earth, each of which is listening several frequencies, one for each signal it needs to receive. Technically it is impossible to focus a signal sent by the satellite exactly on the destination receiver. So part of the signal is spread in an area around it creating noise for the other receivers displayed in this area and listening the same frequency. Each receiver is able to distinguish the signal directed to it from the extraneous noises it picks up if the sum of the noises does not become too big, i.e. does not exceed a certain threshold T. The problem is to assign frequencies to the receivers in such a way that each receiver gets its dedicated signals properly, while minimizing the total number of frequencies used.

Generally the "noise relation" is symmetric, that is if a receiver u is in the noise area of a receiver v then v is in the noise area of u. Hence, interferences may be modelled by a *noise graph* whose vertices are the receivers and in which two vertices are joined by an edge if and only if they interfer. Moreover, the graph is attached a *weight function* $p : V \to \mathbb{N}$, where the weight $p(v)$ of the vertex v is equal to the number of signals it has to receive. The *maximum weight* of a graph G under p is $p_{\max} = \max\{p(v) \mid v \in V(G)\}$.

In a simplified version, the intensity I of the noise created by a signal is independent of the frequency and the receiver. Hence to distinguish its signal from noises, a receiver must be in the noise area of at most $k = \lfloor \frac{T}{I} \rfloor$ receivers listening signals on the same frequency. Hence the problem comes to find a weighted colouring of the graph which is k-improper. Let G be a graph and p a weight function on G. A $[p]$*colouring* of G is a function $C : V \to \mathcal{P}(S)$ such that $|C(v)| \geq p(v)$. As usual, the set S is called the set of *colours* and is usually $\{1, 2, \ldots, l\}$ for some integer l as we are only interested in its cardinality. A $[p]$colouring into a set S of cardinality l is called an l-$[p]$*colouring* of G. For any integer q, we denote by $\mathbf{q}$ the constant weight function equal to q. Note that a $[\mathbf{1}]$colouring is a usual colouring. A $[p]$colouring C of G is *k-improper* if for any colour i, the set of vertices coloured i induces a graph of degree at most k. The *k-improper* $[p]$*chromatic number* of G, denoted $\chi_k(G, p)$, is the smallest ℓ such that G admits a k-improper ℓ-$[p]$colouring. The *k-improper chromatic number* of G is $\chi_k(G) = \chi_k(G, \mathbf{1})$. Note that 0-improper colouring corresponds to proper colouring, so $\chi_0(G) = \chi(G)$.

Similarly to channel assignment, often due to technical reasons or dynamicity, each receiver is given a list of available frequencies. This can be modelled by weighted improper list colouring. The *k-improper* $[p]$*choice number* of G is denoted by $ch_k(G, p)$ and we write $ch_k(G)$ for $ch_k(G, \mathbf{1})$.

As in the usual channel assignment problem, planar graphs and more generally graphs with bounded maximum average degree are of particular interest. Moreover, usually the noise areas of the receivers are modelled by disks of the same radius. Hence, the noise graph is a unit disk graph. A *unit disk graph* is the intersection graph of equal-sized disks in the the plane. In other words, given a set of points fixed in the plane and a positive quantity d, we construct a unit disk graph by joining edges between any two points within distance d of one another. Clearly, we may assume $d = 1$ or otherwise rescale. An important restricted class of unit disk graphs (specific to radio channel assignment and Alcatel's problem in particular) is that of weighted induced subgraphs of the triangular lattice, or *hexagonal graphs*. This class is related to a common placement pattern of radio transmission towers in a cellular communications network: for efficient coverage, the transmitters are only placed on points of a the *triangular lattice* graph R that may be described as follows. The vertices are all integer linear combinations $a\vec{e_1} + b\vec{e_2}$ of the two vectors $\vec{e_1} = (1, 0)$ and $\vec{e_2} = (\frac{1}{2}, \frac{\sqrt{3}}{2})$: thus we may identify the vertices with the pairs (a, b) of integers. Two vertices are adjacent when the Euclidean distance between them is 1. Thus each vertex $x = (a, b)$ has the six neighbours: its *left neighbour* $(a-1, b)$, its *right neighbour* $(a+1, b)$, its *leftup neighbour* $(a-1, b+1)$, its *rightup neighbour* $(a, b+1)$, its *leftdown neighbour* $(a, b-1)$ and its *rightdown neighbour* $(a+1, b-1)$.

In this part, we first study (non weighted) improper colouring of graphs, and more specifically, improper colouring of planar graphs. We then study (weighted) improper colouring of unit disk graphs, and more particularly *unit interval graphs* (A *unit interval graph* is a graph whose vertices are intervals of length one on a straight line, in whose two vertices are joined if and only if the correspondig intervals intersect.) and hexagonal graphs.

3.2 Improper colouring

One can generalise Propositions 1 and 10 to k-improper colouring.

Proposition 49 (Lovász [110]). *Let k be a non-negative integer. Then for any graph G,*

$$\chi_k(G) \leq ch_k(G) \leq \left\lceil \frac{\Delta(G)+1}{k+1} \right\rceil .$$

Proof. Set $\ell = \left\lceil \frac{\Delta(G)+1}{k+1} \right\rceil$. Let L be an ℓ-list assignment on G. Without loss of generality, we may assume that $\bigcup_{v \in V(G)} L(v) = \{1, \ldots, q\}$ for some integer q. Consider a partition of $V(G)$ into q sets $V_1, \ldots, V_q$ such that for every vertex v, if $v \in V_i$ then $i \in L(v)$, and that minimizes the number of internal edges, i.e. $\sum_{i=1}^{\ell} E(G[V_i])$, under this condition. Let v be a vertex and V_i the set containing v. Then for any

$j \in L(v) \setminus \{i\}$, v has at least as many vertices in V_j as in V_i otherwise moving v from V_i to V_j would give a partition with less internal edges. Hence if v has $k+1$ neighbours in V_i it has at least $(k+1) \times \ell \geq \Delta(G)+1$ neighbours in G, which is impossible by the definition of $\Delta(G)$. Thus v has at most k neighbours in V_i. So the sets $V_1, \ldots, V_q$ correspond to the colour classes of a k-improper L-colouring of G. □

However, one cannot generalise Propositions 1 and 10 by replacing the maximum degree by the degeneracy in the above proposition, if $k \geq 1$.

Proposition 50. *Let k and d be two positive integers. There is a d-degenerate graph G such that* $\chi_k(G) = d+1$.

Proof. Let us prove by induction on l that if $0 \leq l \leq d$, there is a d-degenerate graph G_l such that $\chi_k(G_l) \geq l$. The result is true for $l = 0$ with G_0 the graph with a unique vertex. Suppose now that the result is true for l. Let G_{l+1} be the graph constructed from G_l as follows : For every subset A of l vertices, add a set S_A of $kl+1$ new vertices and connect all the vertices of S_A to all the vertices of A. It is easy to check that G_{l+1} is d-degenerate since every new vertex has degree at most $l \leq d$. Let us now prove that $\chi_k(G_{l+1}) \geq l+1$. Suppose to the contrary that there exists a k-improper l-colouring. Then since $\chi_k(G_l) \geq l$, the l colours must be used on G_l. Hence there is a set A of l vertices of G_l, one of each colour. Now consider the set S_A. There is a colour c that is assigned to at least $k+1$ of them. So the vertex of A which is coloured c has $k+1$ neighbours of its colour which is a contradiction. □

Let $\alpha_k(G)$ be the largest size of an induced subgraph of G with maximum degree at most k. As every colour class induces a sibgraph of maximum degree at most k, we obtain:

$$\chi_k(G) \geq |V(G)|/\alpha_k(G). \tag{3}$$

3.2.1 Improper colouring of planar graphs

Improper colourings of planar graphs have been widely studied. In particular p_k and p_k^*, the smallest integers l such that every planar graph is k-improper l-colourable and k-improper l-choosable respectively, are known for almost all k. Recall that Thomassen [152] showed that every planar graph is 5-choosable (Theorem 17) and there are planar graphs which are not 4-choosable [159] so $p_0^* = 5$. Moreover, every planar graph is 4-colourable [8, 9] and there are planar graphs which are not 1-improper 3-colourable, so $p_0 = p_1 = 4$. But we do not know the exact value of p_1^* which is either 4 or 5. However, it is conjectured that it is 4:

Conjecture 51 (Eaton and Hull [42], Škrekovski [146]). *Every planar graph is* 1*-improper* 4*-choosable.*

As shown independently by Eaton and Hull [42] and Škrekovski [146], every planar graph is 2-improper 3-choosable and for every k, there are planar graphs which are not k-improper 2-colourable. Hence $p_k = p_k^* = 3$ for any $k \geq 2$. The two proofs of this result are very similar and use an induction hypothesis in the same flavour as the one of the proof of Theorem 17.

Moreover improper colourings of planar graphs have also been studied under some girth restrictions. As mentionned in Section 1.2.3, the Grötzsch Theorem states that every planar graph of girth at least 4 is 3-colourable. Voigt [160] showed a planar graph of girth 4 which is not 3-choosable and Thomassen [153] proved that every planar graph of girth at least 5 is 3-choosable. Škrekovski [147] showed that every planar graph of girth at least 4 is 1-improper 3-choosable. Škrekovski [148] investigated k-improper 2-choosability of planar graphs in relation with their girth. Denoting by g_k be the smallest integer such that every planar graph of girth at least g_k is k-improper 2-choosable, he proved that $6 \leq g_1 \leq 9$, $5 \leq g_2 \leq 7$, $5 \leq g_3 \leq 6$ and $\forall k \geq 4, g_k = 5$. Hence the only unknown values are g_1, g_2 and g_3.

In [71], we study the k-improper l-choosability of graphs in relation with their maximum average degree. Let $M(k,l)$ be the greatest real such that every graph of maximum average degree less than $M(k,l)$ is k-improper l-choosable. Obviously, $M(k_1,l) \leq M(k_2,l)$ if $k_1 \leq k_2$. A graph is k-improper 1-choosable if and only if it has maximum degree at most k. So $M(k,1) = \frac{2k+2}{k+2}$. We then give relatively close lower and upper bounds on $M(k,2)$.

Theorem 52 (Havet and Sereni [71]). *For every $k \geq 0$,*

$$4 - \frac{4}{k+2} \leq M(k,2) \leq 4 - \frac{2k+4}{k^2+2k+2}.$$

As a corollary, we obtain the following upper bounds for g_k which are better than Škrekovski's ones: $g_1 \leq 8$, $g_2 \leq 6$, $g_3 \leq 6$ and $\forall k \geq 4, g_k \leq 5$.

We then extend the lower bound to any value of l:

Theorem 53 (Havet and Sereni [71]). *For every $l \geq 2$ and $k \geq 0$,*

$$l + \frac{lk}{l+k} \leq M(k,l).$$

Finally, we provide for any value of l and k a graph which is not k-improper l-choosable, and we deduce that $M(k,l) \xrightarrow[k\to\infty]{} 2l$.

3.2.2 Complexity and Brooks-like theorems

For all integers k and ℓ, let k-IMP ℓ-COL be the following problem:

INSTANCE: a graph G.

QUESTION: is G k-improper ℓ-colourable?

Cowen et al. [39] showed that the problem k-IMP ℓ-COL is NP-complete for all pairs (k,ℓ) of integers with $k \geq 1$ and $\ell \geq 2$. When $\ell \geq 3$, this is not a very surprising result since it is already hard to determine whether a given graph is properly 3-colourable. On the contrary, determining if a graph is 2-colourable, i.e. bipartite, can be done in polynomial-time, whereas it is NP-complete to know if it is k-improper 2-colourable as soon as $k > 0$.

Of more interest is the question of complexity of k-IMP ℓ-COL when restricted to graphs with maximum degree $(k+1)\ell$. Recall that determining whether a graph with large constant maximum degree Δ is $(\Delta-k)$-colourable can be done in linear time if $(k+1)(k+2) \leq \Delta$ [121] and is NP-complete if $(k+1)(k+2) > \Delta$ [46]. It is natural to ask whether analogous results can be found for improper colouring. This first problem to grapple with is the existence, or not, of a Brooks-like theorem for improper colouring : does there exist a polynomial-time algorithm that decides whether a graph G of maximum degree Δ has k-improper chromatic number at most $\lceil \frac{\Delta+1}{k+1} \rceil - 1$? Proving that k-IMP ℓ-COL is NP-complete when restricted to graphs with maximum degree $(k+1)\ell$ would provide a negative answer to this question (unless P = NP). Cowen et al. [39] proved that k-IMP 2-COL is NP-complete for the class of graphs with maximum degree $2(k+1)$, and asked what happens when $\ell \geq 3$. In [37], we provide a negative answer if ℓ is not to large compared to a function of k.

Theorem 54 (Corrêa, Havet and Sereni [37]). *Let $k \geq 1$ and $\ell \geq 3$ be two integers. If $l + \sqrt{l} \leq 2k+3$ then k-IMP ℓ-COL restricted to graphs with maximum degree $(k+1)\ell$ is NP-complete.*

But we do not know the complexity of this problem for larger value of ℓ:

Problem 55 (Corrêa, Havet and Sereni [37]). *What is the complexity of k-IMP ℓ-COL restricted to graphs with maximum degree $(k+1)\ell$ when $\ell+\sqrt{\ell} > 2k+3$?*

We conjecture that it is always NP-complete. As an evidence, we prove [37] the NP-completeness when $k = 1$ and $\ell = 4$.

In view of these negative results, one may ask what happens for planar graphs. Recall that every planar graph is 4-colourable [10], and 2-improper 3-colourable [42, 146]. Cowen et al. [39] proved that it is NP-complete to know whether or not a planar graph is 1-improper 3-colourable, but without any restriction on the maximum degree. Cowen et al. [39] proved that k-IMP 2-COL is NP-complete for planar graphs, again without any restriction on the degree. In particular, they asked if 1-IMP 2-COL is still NP-complete for planar graphs with maximum degree four – they could prove it only for maximum degree five. More generally what is the complexity of k-IMP 2-COL for planar graphs of maximum degree $2k+2$?

Problem 56. *What is the complexity of k-IMP 2-COL restricted to planar graphs of maximum degree $2k+2$?*

We show [37] that it is NP-complete when $k \in \{1,2\}$. Note that for $k = 1$, it settles Cowen et al. [39] question. However, we conjecture that if k is sufficiently large k-IMP 2-COL can be polynomially decided, as the answer is always affirmative:

Conjecture 57 (Corrêa, Havet and Sereni [37]). *There exists an integer $k_0 \geq 3$ such that for any $k \geq k_0$, any planar graph with maximum degree at most $2k+2$ is k-improper 2-colourable.*

Problem 58 (Correa, Havet and Sereni [37]). *What is the complexity of 1-IMP 3-COL restricted to planar graphs with maximum degree at most 6?*

3.2.3 Improper colouring of unit disk graphs

As k-IMP ℓ-COL is NP-complete ($k \geq 2$), a natural question is to ask whether it remains NP-complete when restricted to unit disk graphs. For all integers k and ℓ, let UD k-IMP ℓ-COL be the following problem:

INSTANCE: a unit disk graph G.

QUESTION: is G k-improper ℓ-colourable?

Clark *et al.* [36] demonstrated NP-completeness of UD 0-IMP 3-COL. Then Gräf, Stumpf and Weißenfels [59] extend this for all value of ℓ: for any fixed integer $\ell \geq 3$, the problem UD 0-IMP ℓ-COL is NP-complete. In [68], we generalise this result to improper colouring.

Theorem 59 (Havet, Kang and Sereni [68]). *UD k-IMP ℓ-COL is NP-complete for any fixed integers k and ℓ such that $k \geq 0$ and $\ell \geq 3$.*

Our approach generalises that of Gräf *et al.* [59] and our key contribution is to produce auxiliary graphs — in particular, the auxiliary crossing graph — that are more general than those of the proof for unit disk ℓ-colourability.

At first sight, it is not clear if we should expect k-improper 2-colourability for unit disk graphs to be NP-complete, as 2-colourability is polynomial-time in general, while the planar k-improper 2-colourability problem, for any fixed positive integer k, is NP-complete [39]. We can in fact reduce from the latter problem to show NP-completeness

Theorem 60 (Havet, Kang and Sereni [68]). *Unit disk k-improper 2-colourability is NP-complete for any fixed positive integer k.*

The reduction from planar k-improper 2-colourability requires no crossing auxiliary graphs. However, the auxiliaries must transmit information about impropriety; also, we need to take care of high-degree vertices. The task of constructing such auxiliary graphs is the crux of the reduction.

These two results show that, like for unit disk (proper) colourability, the unit disk improper colourability problem is NP-hard in a relatively strong sense. In light of these negative results, our next question is to consider approximability. The following proposition implies that the chromatic number of a unit disk graph is approximable to within a factor of 3.

Proposition 61 (Peeters [130]). *There is a polynomial-time algorithm that, for any unit disk graph G, finds a proper colouring of G using at most $3\omega(G)-2$ colours.*

The relatively simple proof of this uses geometric ideas to show that the degeneracy $\delta^*(G)$ is at most $3\omega(G)-3$. (Consider the vertex with the largest first coordinate). Hence, a colouring using at most $3\omega(G)-2$ colours can be found inductively. Note that there exist $(3\omega-3)$-regular unit disk graphs [112]. There has been no tangible improvement of this approximation result since then. Gräf *et al.* [59] provide a more sophisticated heuristic called the STRIPE algorithm, but it also has performance guarantee of 3.

Since there is no improper analogue for colouring graphs with bounded degeneracy, the only known positive approximation result is the following which gives a performance guarantee of 6.

Proposition 62. *For any fixed non-negative integer k, there is a polynomial-time approximation algorithm that, given a unit disk graph G, finds a k-improper colouring of G using at most $\left\lceil \frac{6\omega(G)-6}{k+1} \right\rceil$ colours.*

This is a direct consequence of the bound $\Delta(G) \leq 6\omega(G)-7$ for unit disk graphs, and Proposition 49. It is unknown whether the best approximation ratio for computing χ_k of unit disk graphs is closer to $3/2$ or 6, if k is a fixed positive integer.

A related problem is to consider the best upper bound for unit disk graphs on the ratio between the k-improper chromatic number and the trivial lower bound, i.e. $\left\lceil \frac{\omega}{k+1} \right\rceil$. Malesińska *et al.* [112] showed that there are classes of unit disk graphs with $\chi(G) \geq \frac{3}{2}\omega(G)$; however, the question of whether this parameter is closer to $3/2$ or 3 is an enticing open problem. Results on colouring of random unit disk graphs show that this parameter is lower than 3 for "most" unit disk graphs [115]. By the last proposition, we know that for positive integers k this bound is at most 6.

Proposition 63 (Havet, Kang and Sereni [68]). *There exist unit disk graphs G_n such that*

$$\frac{\chi_k(G_n)}{\omega(G_n)/(k+1)} \geq \begin{cases} 2 & \textit{if } k \textit{ is odd, and} \\ 2(k+1)/(k+2) & \textit{if } k \geq 2 \textit{ is even.} \end{cases}$$

Proof. Fix an arbitrary integer $n > k/2+1$. Consider the graph G_n whose vertices are the $2n$ points equally spaced on a circle. Join each point to all other points on the circle except for the one directly opposite it. It can be verified that G_n is a unit disk graph and that $\omega(G_n) = n$.

Since each vertex is adjacent to all but one vertex in G_n, then $\alpha_k(G_n) \leq k+2$. When k is odd, we can reduce this estimate by one: suppose k is odd and there is an induced subgraph S of size $k+2$ with maximum degree at most k. Since k is odd, there must be two opposite (non-adjacent) vertices u and v such that only one of them, say v, is in S. Then v must be adjacent to all other vertices in S and hence have degree $k+1$, a contradiction.

Equation (3) yields the result. □

These examples are inspired by the unit disk graphs that show the ratio $3/2$ can be attained in the case $k=0$ (cf. Malesińska *et al.* [112]) — these are also formed by equally spaced points around a circle. That we can obtain higher ratios for all other cases (except $k=3$) gives us further evidence to believe that, for unit disk graphs, the improper chromatic numbers (i.e. when $k \geq 1$) are harder to approximate than the chromatic number.

On the other hand, we note that random analysis for this problem, generalising McDiarmid [115], has been performed [103]. In the standard model for random unit disk graphs, it is shown, for nearly all asymptotic choices for the distance parameter $r(n)$, that, as $n \to \infty$, the k-improper chromatic number tends to a value at most $2\sqrt{3}/\pi \approx 1.103$ times the optimal. One interpretation of this is that, given large instances G of randomly generated unit disk graphs, returning $2\sqrt{3}/\pi \cdot \frac{\omega(G)}{k+1}$ is a reasonable approximation for χ_k.

3.2.4 Improper colouring of unit interval graphs

As unit interval graphs are particular cases of unit disk graphs, we investigate [68] improper colouring for unit interval graphs.

Proposition 64 (Havet, Kang and Sereni [68]). *For any fixed non-negative integers k and l, there exists a unit interval graph $I_{k,l}$ with maximum degree and clique number equal to $l(k+1)$ which is not k-improperly l-colourable.*

Proof. To construct $I_{k,l}$, just start with a $(l(k+1))$-clique $K = K_{l(k+1)}$ and add a vertex u linked to exactly $(l-1)(k+1)+1$ vertices of K. Suppose $I_{k,l}$ has a k-improper l-colouring: K must have exactly $(k+1)$ vertices of each colour. Thus any vertex of K has impropriety k in K. As u has $(l-1)(k+1)+1$ neighbours in K it must have at least one neighbour of each colour and hence cannot be coloured, a contradiction. $I_{k,l}$ is clearly a unit interval graph. □

This proposition raises the question of the complexity of k-improperly l-colouring unit interval graphs for fixed non-negative integers k and l. We prove now that this problem is polynomial time for general interval graphs, and we provide a dynamic programming algorithm.

Theorem 65 (Havet, Kang and Sereni [68]). *The k-improper l-colourability problem restricted to interval graphs is in P for any fixed non-negative integers k and l.*

This result does not fully answer the complexity question for improper colouring of unit interval graphs: it is unknown whether, for $k > 0$ fixed, there is a polynomial-time algorithm to find $\chi_k(G)$ given a unit interval graph G. The following result, however, shows that only two values are possible: the lower bound given by Proposition 49, or this number plus one.

Theorem 66 (Havet, Kang and Sereni [68]). *For any fixed non-negative integer k, there is a linear-time algorithm that, given a unit interval graph G, finds a k-improper colouring of G using at most $\hat{l} = \left\lceil \frac{\omega(G)}{k+1} \right\rceil + 1$ colours.*

Proof. Let $v_1, \ldots, v_n$ be a unit interval representation for G. Under this ordering, our colouring procedure proceeds by assigning colour 1 to the first $k+1$ vertices, colour 2 to the next $k+1$, and so on until colour $\hat{l}$ has been assigned whereupon it begins assigning colour 1 again. If we now have an invalid colouring, we can suppose without loss of generality that v_{k+1} and $v_{(k+1)\hat{l}+1}$ (both coloured 1) are adjacent. But, because G is a unit interval graph, this implies that $\left\{ v_{k+1}, \ldots, v_{(k+1)\hat{l}+1} \right\}$ induces a clique in G and this contradicts the choice of $\omega(G)$. □

When only k is fixed, one can think of applying the algorithm of Theorem 65 with $l = \lceil \frac{\omega}{k+1} \rceil$. However, this may be polynomial neither in space nor in time, since space and time complexity both are $O(lk)^{l(k+1)}$ and ω can be linear in the number of vertices. In light of this, we pose the following problem:

Problem 67 (Havet, Kang and Sereni [68]). *Let G be a unit interval graph, and k a positive integer. The preceding result states that $\chi_k(G) \in \left\{ \left\lceil \frac{\omega(G)}{k+1} \right\rceil, \left\lceil \frac{\omega(G)}{k+1} \right\rceil + 1 \right\}$. Is there a polynomial-time algorithm to decide which value is correct?*

3.2.5 Improper colouring of weighted hexagonal graphs

For any $k \geq 6$, the k-improper chromatic number of a hexagonal graph is its maximum weight because it has maximum degree 6.

McDiarmid and Reed [117] showed that it is NP-complete to decide whether the chromatic number of a weighted hexagonal graph is 3 or 4. Hence there is no polynomial time algorithm for finding the weighted chromatic number of hexagonal graphs (unless P= NP). Hence one have to find approximate algorithms. An algorithm that gives a colouring with at most $c_1 \times opt + c_2$ colours for some constants c_1 and c_2, where opt is optimal number of colours of such a colouring is said to be c_1*-approximate* or to have *approximation ratio* c_1. For example, Theorem 21 give a 1-approximate algorithm for edge-colouring a graph while Shannon's result give a $3/2$-approximate algorithm for edge-colouring a multigraph.

The better known so far has approximation ratio $4/3$ and is based on the following result.

Theorem 68 (McDiarmid and Reed [117]). *Let G be a weigthed hexagonal graph. For any weight function p then*

$$\chi(G,p) \leq \frac{4}{3}\omega(G,p).$$

A distributed algorithm which guarantees the $\frac{4}{3}\omega(G,p)$ bound is reported by Narayanan and Schende [126, 127]. However, one expects approximate algorithm with ratio better than $4/3$. In particular, McDiarmid and Reed conjecture that for big weight the ratio may be decreased to almost $9/8$.

Conjecture 69 (McDiarmid ans Reed [117]). *There exists a constant c such that for any weigthed hexagonal graph G and weight function p,*

$$\chi(G,p) \leq \frac{9}{8}\omega(G,p) + c.$$

Note that the ratio $9/8$ in the above conjecture is best possible. Indeed consider a 9-cycle C_9 with constant weight k. A colour maybe assigned to at most 4 colours, so $\chi(C_9,\mathbf{k}) \geq \frac{9k}{4}$. Clearly, $\omega(C_9,\mathbf{k}) = 2k$. So $\chi(C_9,\mathbf{k}) \geq \frac{9}{8}\omega(C_9,\mathbf{k})$. In [66], we give an evidence for this conjecture by proving a $7/6$ ratio for hexagonal graphs with girth at least 4. See also [150] for an alternative proof.

Theorem 70 (Havet [66]). *Let G be a hexagonal graph with girth at least four. Then for any weight function p, $\chi(G,p) \leq \frac{7}{6}\omega(G,p) + 5$.*

We also provide [75] a distributed algorithm for $[p]$colouring such hexagonal graph G with $\frac{5}{4}\omega(G,p) + 3$ colours.

Regarding improper colouring, we generalised [68] the above mentionned NP-completeness result of McDiarmid and Reed:

Theorem 71 (Havet, Kang and Sereni [68]). *For $0 \le k \le 5$, the following problem is NP-complete:*
Instance: *a weighted hexagonal graph (G,p).*
Question: *is (G,p) k-improper 3-colourable?*

Hence one cannot expect polynomial-time algorithm for finding the k-improper chromatic number of weighted hexagonal graphs. Therefore, our aim is to find approximate algorithm with approximation ratio as small as possible.

A natural method to finding a k-improper colouring of (G,p) consists in finding first a k-improper colouring of $(G,\mathbf{q})$ with r colours, ideally $\chi_k(G,\mathbf{q})$. We then divide each weight into sets of size q; using r colours for each of these sets, we obtain a k-improper $r \times \left\lceil \frac{p_{\max}}{q} \right\rceil$-colouring of (G,p). As $\chi_k(G,p) \ge p_{\max}$, we obtain an (r/q)-approximate algorithm.

In [16], we improve this approximation ratio.

To do so, instead of considering only $p_{\max}$, we consider the number of colours that a vertex and its neighbours may require. As shown by the following, this number may be larger than $p_{\max}$. The graph $K_{1,k+1}$ is the graph with $k+2$ vertices and $k+1$ edges linking one v ertex, called the *centre* to the $k+1$ others, called *spikes*.

Proposition 72. *For every weight function p, $\chi_k(K_{1,k+1},p) \ge \dfrac{1}{k+1} \displaystyle\sum_{v \in V(K_{1,k+1})} p(v)$.*

Proof. Let u be the centre of $K_{1,k+1}$ and $v_1,\dots,v_{k+1}$ its spikes. Consider a k-improper colouring C of $K_{1,k+1}$. For $1 \le i \le k+1$, set $q(v_i) = |C(v_i) \setminus C(u)|$. The colouring C uses at least $M = \max\{q(v_i) + p(u) \mid 1 \le i \le k+1\} \ge p(u) + \frac{1}{k+1}\sum_{i=1}^{k+1} q(v_i)$. But a colour in $C(u)$ is assigned to at most k of the spikes because the colouring is k-improper. Thus $\sum_{i=1}^{k+1} q(v_i) \ge \sum_{i=1}^{k+1} p(v_i) - kp(u)$. It follows $M \ge \frac{1}{k+1}(p(u) + \sum_{i=1}^{k+1} p(v_i))$. □

We denote by $\theta_k(G,p) = \max\{p(H)/(k+1) \mid H \text{ star of } G\}$ and $\omega_k(G,p) = \max\{p_{\max}, \theta_k(G,p)\}$. According to Proposition 72, $\omega_k(G,p) \le \chi_k(G,p)$.

Our aim is to show algorihtms that k-improper $[p]$colours a graph G with at most $\alpha_k \times \omega_k(G,p) + \beta_k$ colours, so proving an α_k-approximate algorithm.

Theorem 73 (Bermond, Havet, Huc Linhares [16]). *Let $\alpha_k(r,q) = \frac{(k+1)r^2}{(k+2)rq-q^2}$ and $\beta_k(r,q) = \max\{(k+2)r^2 - rq, (k+1)r^2 + krq\}$.*

Given a k-improper colouring C of $(G,\mathbf{q})$ with r colours, there exists a polynomial algorithm that gives a k-improper colouring (G,p) with at most $\alpha_k(r,q) \times \omega_k(G,p) + \beta_k(r,q)$ colours.

In particular, if $\chi_k(G,\mathbf{q}) \le r$, then $\chi_k(G,p) \le \alpha_k(r,q) \times \omega_k(G,p) + \beta_k(r,q)$.

For 1-improper colouring or when the maximum degree Δ of the graph G is $k+1$, one can get algorithms with better approximation ratio.

Theorem 74 (Bermond, Havet, Huc Linhares [16]). *Let r and q be two integers. Set $a = 2r - 2q$ if $r \ge 2q$ and $a = r$ if $r \le 2q$, and $\alpha'_1(r,q) = \frac{ar+rq}{aq+rq}$ and $\beta'_1(r,q) = 4r^2$.*

Given a 1-improper colouring C of $(G,\mathbf{q})$ with r colours, there exists a polynomial algorithm that gives a 1-improper colouring of (G,p) with at most $\alpha'_1(r,q) \times \omega_k(G,p) + \beta'_1(r,q)$ colours.

In particular, if $\chi_1(G,\mathbf{q}) \le r$, then $\chi_1(G,p) \le \alpha'_1(r,q) \times \omega_1(G,p) + \beta'_1(r,q)$.

Theorem 75 (Bermond, Havet, Huc Linhares [16]). *Let r and q be two integers. Set $a=(k+1)r-2$ if $r \geq 2q$ and $a=kr$ if $r \leq 2q$, and $\alpha''_k(r,q)=\frac{ar+rq}{aq+rq}$ and $\beta''_k(r,q)=ar+r^2$.*

Let G be a graph with maximum degree $k+1$. Given a k-improper colouring C of $(G,\mathbf{q})$ with r colours, there exists a polynomial algorithm that gives a k-improper colouring of (G,p) with at most $\alpha''_k(r,q)\times\omega_k(G,p)+\beta''_k(r,q)$ colours.

In particular, if $\chi_k(G,\mathbf{q})\leq r$, then $\chi_k(G,p)\leq \alpha''_k(r,q)\times\omega_k(G,p)+\beta''_k(r,q)$.

For the triangular lattice R, we show [16] that $\chi_1(R,\mathbf{q})=\left\lceil\frac{5q}{2}\right\rceil$, $\chi_2(R,\mathbf{q})=2q$, $\chi_3(R,\mathbf{q})=\left\lceil\frac{3q}{2}\right\rceil$, $\chi_4(R,\mathbf{q})=\left\lceil\frac{4q}{3}\right\rceil$ and $\chi_5(R,\mathbf{q})=\left\lceil\frac{7q}{6}\right\rceil$. Hence according to Theorems 74, 73 and 75, for $1\leq k\leq 5$, we obtain α_k-approximate algorithms for finding a k-improper colouring of a weighted hexagonal graph, where $\alpha_1=\frac{20}{11}$, $\alpha_2=\frac{12}{7}$, $\alpha_3=\frac{18}{13}$, $\alpha_4=\frac{80}{63}$, and $\alpha_5=\frac{41}{36}$.

4 Colouring the arcs of a digraph

An *edge-colouring* of a graph is a mapping from the edge-set $E(G)$ into a set of colours such that two edges get different colours if they are adjacent that i.e. have an end-vertex in common. If we consider digraphs, one may imagine several arc-colourings since there are different types of adjacency. Indeed two adjacent arcs e and f may intersect into three different ways:

- either $head(e)=head(f)$ in which case e and f are said *coheaded*;
- or $tail(e)=tail(f)$, in which case e and f are said *cotailed*;
- or $tail(e)=head(f)$ or $head(f)=tail(e)$, in which case e and f are said *consecutives*.

Let D be a digraph. Suppose we would like to colour the arcs of D. We will get different colourings depending on which types of adjacency impose two arcs to have different colours.

If arcs which are either coheaded, cotailed or consecutives must have different colours, then we are looking for a usual edge-colouring of the underlying digraphs and the colour classes are matchings. See Subsection 1.3.

If only coheaded arcs must get different colours then the colour classes are outforests and exactly $\Delta^-(D)$ colours are needed and suffice. Indeed at a vertex v of indegree $\Delta^-(D)$, all the arcs with head v must get different colours. Conversely, if we assign at each vertex different colours arbitrarily to incoming arcs, we obtain the desired colouring.

If coheaded and cotailed arcs must get different colours but consecutives ones may be coloured the same, then each colour class is a union of disjoint directed paths or circuits. The problem is then equivalent to finding an (optimal) edge-colouring of the bipartite graph with vertex-set $\bigcup_{v\in V(D)}\{v^-,v^+\}$ and edge-set $\{u^-v^+ \mid uv\in A(D)\}$. Hence, $\max\{\Delta^+(D),\Delta^-(D)\}$ colours are needed and suffice.

If only consecutives arcs must get different colours then the colour classes are *cuts*. A *cut* is a set of arcs with tail in a fixed subset S of $V(G)$ and head in $V(G)\setminus S$. Such colourings, introduced by Poljak and Rödl [131], are called *arc-colourings* and are in one-to-one correspondence with the colouring of the *line-digraph* of D. This digraph $L(D)$ has vertex set $V(L(D))=E(D)$ and an arc $a\in E(D)$ dominates an arc $b\in E(D)$ in $L(D)$ if and only if $head(a)=tail(b)$. The *arc-chromatic number* of D, denoted $\chi_a(D)$, is the minimum number of colours of an arc-colouring of D. Clearly $\chi_a(D)=\chi(L(D))$, where χ denotes the chromatic number. In the first section, show how some problems in function theory may be modelled as an arc-colouring problem of digraphs in which each vertex has bounded outdegree or bounded indegree. We then give the upper and lower bounds, shown in [17], on the number of colours needed for such a colouring

If coheaded and consecutives arcs must get different colours then the colour classes are *outstar forests*. An *outtree* is a digraph in which every vertex has indegree 1 except one, called the *root* which has indegree 0; an *outforest* is the disjoint union of outtrees; an *outstar* is an outtree in which all the vertices are dominated by the root and an *outstar forest* or *galaxy* is an outforest of outstars. The minimum number of colours of such an arc colouring has been introduced by Guiduli [62] as the *directed star arboricity*. It is an analog of the *star arboricity* defined in [2]. In [6], we show that finding the directed star arboricity of a digraph is a NP-hard problem. More precisely, we show that determining if the directed star arboricity of a digraph with out- and indegree at most 2 is NP-complete. In Subsection 4.2, we show how some wavelength assignment problems in optical networks may be modelled by directed star arboricity. We then expose some upper bounds on the directed star arboricity of digraphs with bounded outdegree and/or bounded indegree proved in [6].

4.1 Arc-colouring of digraphs

Definition 76. We denote by $\overline{H}_k$ the complementary of the hypercube of dimension k, that is the digraph with vertex-set all the subsets of $\{1,\dots,k\}$ and with arc-set $\{xy \mid x \not\subset y\}$.

A *homomorphism* $h : D \rightarrow D'$ is a mapping $h : V(D) \rightarrow V(D')$ such that for every arc xy of D, $h(x)h(y)$ is an arc of D'.

Let c be an arc-colouring of a digraph D into a set of colours S. For any vertex x of D, we denote by $Col^+_c(x)$ or simply $Col^+(x)$ the set of colours assigned to the arcs with tail x. We define $Col^-(x) = S \setminus Col^+(x)$. Note that $Col^-(x)$ contains (but may be bigger than) the set of colours assigned to the arcs with head x. The cardinality of $Col^+(x)$ (resp. $Col^-(x)$) is denoted by $col^+(x)$ (resp. $col^-(x)$).

Lemma 77. *For every digraph D, $\chi_a(D) = \min\{k : D \rightarrow \overline{H}_k\}$.*

Proof. Assume that D admits an arc-colouring with $\{1,\dots,k\}$. It is easy to check that Col^+ is a homomorphism from D to $\overline{H}_k$.
Conversely, suppose that there exists a homomorphism h from D to $\overline{H}_k$. Assign to each arc xy an element of $h(y) \setminus h(x)$, which is not empty. This provides an arc-colouring of D. □

Definition 78. The *complete digraph of order n*, denoted $\vec{K}_n$, is the digraph with vertex set $\{v_1, v_2, \dots, v_n\}$ and arc set $\{v_iv_j \mid i \neq j\}$.

The *transitive tournament of order n*, denoted TT_n, is the digraph with vertex set $\{v_1, v_2, \dots, v_n\}$ and arc set $\{v_iv_j \mid i < j\}$.

θ is the function defined by $\theta(k) = \min\{s : \binom{s}{\lceil \frac{s}{2} \rceil} \geq k\}$.

The following corollary of Lemma 77 provides bounds on the arc-chromatic number of a digraph according to its chromatic number.

Lemma 79 (Poljak and Rödl [131]). *For every digraph D,*

$$\lceil log(\chi(D)) \rceil \leq \chi_a(D) \leq \theta(\chi(D)).$$

Proof. By the definition of the chromatic number, $D \to \vec{K}_{\chi(D)}$. As the subsets of $\{1,\dots,k\}$ with cardinality $\lceil \frac{k}{2} \rceil$ induce a complete digraph on $\binom{k}{\lceil \frac{k}{2} \rceil}$ vertices in $\overline{H}_k$, we obtain a homomorphism from D to $\overline{H}_{\theta(\chi(D))}$. So $\chi_a(D) \leq \theta(\chi(D))$.

By Lemma 77, we have $D \to \overline{H}_{\chi_a(D)}$. As $\chi(\overline{H}_{\chi_a(D)}) = 2^{\chi_a(D)}$, we obtain $D \to \vec{K}_{2^{\chi_a(D)}}$. □

These bounds are tight since the lower one is achieved by transitive tournaments and the upper one by complete digraphs by Sperner's Lemma (see [149]).

4.1.1 Function theory and arc-colouring

Let f and g be two maps from a finite set A into a set B. Suppose that f and g are *nowhere coinciding*, that is for all $a \in A$, $f(a) \neq g(a)$. A subset A' of A is (f,g)-*independent* if $f(A') \cap g(A') = \emptyset$. We are interested in finding the largest (f,g)-independent subset of A and the minimum number of (f,g)-independent subsets to partition A. As shown by El-Sahili [45], this can be translated into an arc-colouring problem.

Let $D_{f,g}$ and $H_{f,g}$ be the digraphs defined as follows :

- $V(D_{f,g}) = B$ and $(b,b') \in E(D_{f,g})$ if there exists an element a in A such that $g(a) = b$ and $f(a) = b'$. Note that if for all a, $f(a) \neq g(a)$, then $D_{f,g}$ has no loop.
- $V(H_{f,g}) = A$ and $(a,a') \in E(H_{f,g})$ if $f(a) = g(a')$.

We associate to each arc (b,b') in $D_{f,g}$ the vertex a of A such that $g(a) = b$ and $f(a) = b'$. Then (a,a') is an arc in $H_{f,g}$ if, and only if, $head(a) = tail(a')$ (as arcs in $D_{f,g}$). Thus $H_{f,g} = L(D_{f,g})$. Note that for every digraph D, there exists maps f and g such that $D = D_{f,g}$.

It is easy to see that an (f,g)-independent subset of A is an independent set in $H_{f,g}$ and thus a cut of $D_{f,g}$. In [44] El-Sahili proved the following :

Theorem 80 (El-Sahili [44]). *Let f and g be two nowhere coinciding maps from a finite set A into a set B. Then there exists an (f,g)-independent subset A' of cardinality at least $|A|/4$.*

As noted in [17], this theorem may be easily shown using the digraphs defined above. Indeed consider a partition (V_1,V_2) be a partition of $V(D_{f,g})$ that maximizes the number $a(V_1,V_2)$ of arcs with tail in V_1 and head in V_2. It is well-known that $a(V_1,V_2) \geq |E(D_{f,g})|/4$.

Let f and g be two nowhere coinciding maps from a finite set A into B. We define $\phi(f,g)$ as the minimum number of (f,g)-independent sets to partition A. Then $\phi(f,g) = \chi(H_{f,g}) = \chi_a(D_{f,g})$.

Let $\Phi(k)$ (resp. $\Phi^\vee(k,l)$) be the maximum value of $\phi(f,g)$ for two nowhere coinciding maps f and g from A into B such that for every z in B, $g^{-1}(z) \leq k$ (resp. either $g^{-1}(z) \leq k$ or $f^{-1}(z) \leq l$). The condition $f^{-1}(z)$ (resp. $g^{-1}(z)$) has at most k elements means that each vertex has indegree (resp. outdegree) at most k in $D_{f,g}$. Therefore we are interested in the arc-chromatic number of digraphs with maximal outdegree k called *k-digraph*, and digraphs in which every vertex has either outdegree at most k or indegree at most l called $(k \vee l)$-digraphs. Let $\Phi(k)$ (resp. $\Phi^\vee(k,l)$) is the maximum value of $\chi_a(D)$ for D a k-digraph (resp. $(k \vee l)$-digraph).

The two functions Φ and $\Phi^\vee$ are closely related as

Proposition 81 (Bessy, Birmelé and Havet [17]).

$$\Phi(k) \leq \Phi^\vee(k,0) \leq \cdots \leq \Phi^\vee(k,k) \leq \Phi(k) + 2$$

Proof. The sole inequality that does not immediately follow the definitions is $\Phi^\vee(k,k) \leq \Phi(k)+2$. Let us prove it.
Let D be a $(k \vee k)$-digraph. Let V^+ be the set of the vertices of D with outdegree at most k and $V^- = V(D) \setminus V^+$. One can colour the arcs in $D[V^+] \cup D[V^-]$ with $\Phi(k)$ colours. It remains to colour the arcs with tail in V^- and head in V^+ with one new colour and the arcs with tail in V^+ and head in V^- with a second new colour. □

In [17], we conjecture that $\Phi^\vee(k,k)$ is never equal to $\Phi(k)+2$.

Conjecture 82 (Bessy, Birmelé and Havet [17]).

$$\Phi^\vee(k,k) \leq \Phi(k)+1$$

In view of Proposition 81, getting good bounds on Φ will also give good bounds on $\Phi^\vee$. Therefore, we mainly focus on Φ. It is easy to check that $\Phi(1)=3$.

Theorem 83 (Bessy, Birmelé and Havet [17]). *If* $k \geq 2$,

$$\max\{\log(2k+3), \theta(k+1)\} \leq \Phi(k) \leq \theta(2k+1).$$

Proof. It is easy to see that the multigraph underlying a k-digraph is $2k$-degenerate and so $(2k+1)$-colourable, by Proposition 6. Then Theorem 79 gives the upper bound on $\Phi(k)$.

The lower bounds are obtained by considering a regular tournament on $2k+1$ vertices and the complete digraph $\vec{K}_{k+1}$. □

Analogously, we showed [17] the following bound on $\Phi^\vee$.

Theorem 84 (Bessy, Birmelé and Havet [17]).

$$\max\{\log(2k+2l+4), \theta(k+1), \theta(l+1)\} \leq \Phi^\vee(k,l) \leq \theta(2k+2l+2)$$

We obtained [17] slightly better bounds on Φ and $\Phi^\vee$. Moreover, we provide the exact values of $\Phi(k)$ and $\Phi^\vee(k,l)$ for $l \leq k \leq 3$. They are summarized in the following table :

$\Phi^\vee(0,0)=1$	$\Phi^\vee(1,0)=\Phi(1)=3$	$\Phi^\vee(2,0)=\Phi(2)=4$	$\Phi^\vee(3,0)=\Phi(3)=4$
	$\Phi^\vee(1,1)=3$	$\Phi^\vee(2,1)=4$	$\Phi^\vee(3,1)=4$
		$\Phi^\vee(2,2)=4$	$\Phi^\vee(3,2)=5$
			$\Phi^\vee(3,3)=5$

In view of all these results, we made the following conjecture:

Conjecture 85 (Bessy, Birmelé and Havet [17]). *Let l be a positive integer. There exists an integer k_l such that if $k \geq k_l$ then $\Phi^\vee(k,l) = \Phi(k)$.*

As an evidence, we showed that $\Phi^\vee(k,0) = \Phi(k)$. Moreover, if $l=1$, then for every k, at least one of Conjecture 82 and Conjecture 85 holds : $\Phi^\vee(k,k) \leq \Phi(k)+1$ or $\Phi^\vee(k,1) = \Phi(k)$.

4.2 Directed star arboricity

4.2.1 Motivation: WDM in star network

We are given a star network in which a center node is connected by an optical fiber to a set of nodes V. Each node v of V sends a set of multicasts $M_1(v),\dots,M_{s(v)}(v)$ to the sets of nodes $S_1(v),\dots,S_{s(v)}(v)$. Using WDM (*wavelength-division multiplexing*), different signals may be sent at the same time through the same fiber but on different wavelengths. The central node is an all-optical transmitter: hence, it may redirect a signal arriving from a node on a particular wavelength to some of the others nodes on the same wavelength. Therefore for each multicast $M_i(v)$, the node v should send the message to the central node on a set of wavelengths so that the central node redirect it to each node of $S_i(v)$ using one of these wavelengths. The aim is to minimize the total number of used wavelengths.

In the very fundamental case when the fiber is unique and each vertex v sends a unique multicast $M(v)$ to the set $S(v)$ of nodes. Let D be the digraph with vertex set V such that the outneighbourhood of a vertex v is $S(v)$. Note that this is a digraph and not a multidigraph (there is no multiple arcs) as $S(v)$ is a set. Then the problem is to find the directed star arboricity of D.

For a vertex v, its indegree $d^-(v)$ corresponds to the number of multicasts it receives. A sensible assumption is that a node receives a bounded number of multicasts. Hence, we studied the directed star arboricity of a digraph D with maximum indegree Δ^-.

4.2.2 Directed star arboricity of digraph with bounded (in)degree

Brandt and Gonzalez [33] showed that $dst(D) \leq \lceil 5\Delta^-/2 \rceil$. This upper bound is tight if $\Delta^- = 1$ because odd circuits have directed star arboricity 3. However it can be improved for larger value of $\Delta^- = 1$. We conjecture that if $\Delta^- \geq 2$, then $dst(D) \leq 2\Delta^-$.

Conjecture 86 (Amini et al. [6]). Every digraph D with maximum indegree $k \geq 2$ satisfies $dst(D) \leq 2k$.

This conjecture would be tight as Brandt [32] showed that for every k, there is an acyclic digraph D_k such that $\Delta^-(D_k) = k$ and $dst(D_k) = 2k$. Note that to prove this conjecture, it is sufficient to prove it for $k = 2$ and $k = 3$. Indeed a digraph with maximum indegree $k \geq 2$ has an arc-partition into $k/2$ digraphs with maximum indegree 2 if k is even and into $(k-1)/2$ digraphs with maximum indegree 2 and one with maximum indegree 3.

It is easy to see that an outforest has directed star arboricity 2. Hence, an idea to prove Conjecture 86 would be to show that every digraph has an arc-partition into Δ^- outforests. However this statement is false. Indeed A. Frank [51] (see also [142], p.908) characterized digraphs having an arc-partition into k outforests. Let $D = (V,A)$. For any $U \subset V$, the digraph induced by the vertices of U is denoted $D[U]$.

Theorem 87 (A. Frank). *A digraph $D = (V,A)$ has an arc-partition into k outforests if and only if $\Delta^-(D) \leq k$ and for every $U \subset V$, the digraph $D[U]$, has at most $k(|U|-1)$ arcs.*

However, Theorem 87 implies that every digraph D has an arc-partition into $\Delta^- + 1$ outforests. Indeed for any $U \subset V$, $\Delta^-(D[U]) \leq \min\{\Delta^-, |U|-1\}$, so $D[U]$ has at most $\min\{\Delta^-, |U|-1\} \times |U| \leq (\Delta^- + 1)(|U|-1)$ arcs. Hence, every digraph has directed star arboricity at most $2\Delta^- + 2$.

Corollary 88. *Every digraph D satisfies $dst(D) \leq 2\Delta^- + 2$.*

In [6], we lessen this upper bound by one so showing Conjecture 86 up to 1.

Theorem 89 (Amini et al. [6]).

$$dst(D) \leq 2\Delta^- + 1$$

Proof. The idea to prove this theorem is to show that every digraph has an arc-partition into Δ^- outforests and a galaxy G. To do so, we shall prove a stronger result by induction.

A *sink* is a vertex with outdegree 0. A *source* is a vertex with indegree 0. A multidigraph is *k-nice* if $\Delta^- \leq k$ and if the tails of parallel arcs, if any, are sources. A *k-decomposition* of a digraph D is an arc-partition into k outforests and a galaxy G such that every source of D is isolated in G. Let u be a vertex of D. A k-decomposition of D is *u-suitable* if no arc of G has head u.

Let u be a vertex of a k-nice multidigraph D. Then D has a u-suitable k-decomposition.

We proceed by induction on $n+k$. We now discuss the connectivity of D.

- If D is not connected, we apply induction on every component.

- If D is strongly connected, every vertex has indegree at least one. Remember also that there is no parallel arcs. Let v be an outneighbour of u. There exists a spanning arborescence T with root v which contains all the arcs with tail v. Let D' be the digraph obtained from D by removing the arcs of T and v. Observe that D' is $(k-1)$-nice. By induction, it has a u-suitable $(k-1)$-decomposition $(F_1,\dots,F_{k-1},G)$. Note that F_i, T and G contain all the arcs of D except those with head v. By construction, $G' = G \cup uv$ is a galaxy since no arc of G has head u. Let $u_1,\dots,u_{l-1}$ be the inneighbours of v distinct from u, where $l \leq k$. Let $F_i' = F_i \cup u_i v$, for all $1 \leq i \leq l-1$. Then each F_i' is an outforest, so $(F_1,\dots,F_{k-1},T,G')$ is a u-suitable k-decomposition of D.

- If D is connected but not strongly connected, we consider a strongly connected terminal component D_1. Set $D_2 = D \setminus D_1$. Let u_1 and u_2 be two vertices of D_1 and D_2, respectively, such that u is one of them.

  If D_2 has a unique vertex v (thus $u_2 = v$), since D is connected, there exists a spanning arborescence T of D with root v. Now $D' = D \setminus A(T)$ is a $(k-1)$-nice multidigraph, so by induction it has a u_1-suitable $(k-1)$-decomposition. Adding T to this decomposition, we obtain a u_1-suitable k-decomposition, which is also u_2-suitable since u_2 is a source. Since $u = u_1$ or $u = u_2$, we have our conclusion.

  If D_2 has more than one vertex then , by induction, it admits a u_2-suitable k-decomposition $(F_1^2,\dots,F_k^2,G^2)$. Moreover the digraph D_1' obtained by contracting D_2 to a single vertex v has a u_1-suitable k-decomposition $(F_1^1,\dots,F_k^1,G^1)$. Moreover, since v is a source, it is isolated in G^1. Hence $G = G^1 \cup G^2$ is a galaxy. We now let F_i be the union of F_i^1 and F_i^2 by replacing the arcs of F_i^1 with tail v by the corresponding arcs in D. Then $(F_1,\dots,F_k,G)$ is a k-decomposition of D which is suitable for both u_1 and u_2.

□

Moreover, we settle Conjecture 86 for acyclic digraphs.

Remark 90. Note that we restrict ourselves to digraphs, i.e. circuits of length two are permitted, but not multiple arcs. When multiple arcs are allowed, all the bounds above do not hold. Indeed the multidigraph T_k with three vertices u, v and w and k parallel arcs uv, vw and wu satisfies $dst(T_k) = 3k$. One can easily show (see [6]) that for multidigraphs, the bound $dst(D) \leq 3\Delta^-$ is sharp.

We then study the directed star arboricity of a digraph bounded with maximum degree. By Theorem 23, one can colour the edges of a multigraph with $\Delta(G)+\mu(G)$ colours so that incident edges have different colours. Since the multigraph underlying a digraph has maximum multiplicity at most two, for any digraph D, $dst(D) \leq \Delta+2$. We conjecture the following:

Conjecture 91 (Amini et al. [6]). *Let D be a digraph with maximum degree $\Delta \geq 3$. Then $dst(D) \leq \Delta$.*

This conjecture would be tight since every digraph with $\Delta = \Delta^-$ has directed star arboricity at least Δ. In [6], we prove that Conjecture 91 holds when $\Delta = 3$.

Theorem 92 (Amini et al. [6]). *Let D be a digraph. If $\Delta(D) \leq 3$ then $dst(D) \leq 3$.*

A first step towards Conjectures 86 and 91 would be to prove the following statement which is weaker than these two conjectures.

Conjecture 93 (Amini et al. [6]). *Let $k \geq 2$ and D be a digraph. If $\max(\Delta^-,\Delta^+) \leq k$ then $dst(D) \leq 2k$.*

This conjecture holds and is far from being tight for large k. Indeed Guiduli [62] showed that if $\max(\Delta^-,\Delta^+)$ then $dst(D) \leq k+20\log k+84$. Since $\max(\Delta^-,\Delta^+) \leq \Delta$, every digraph D satisfies $dst(D) \leq \Delta+20\log\Delta+84$. Guiduli's proof is based on the fact that, when both out and indegree are bounded, the colour of an arc depends of the colour of few other arcs. This bounded dependency allows the use of the Lovász Local Lemma. This idea was first used by Algor and Alon [2], for the star arboricity of undirected graphs. Note also that Guiduli's result is (almost) tight since there are digraphs D with $\max(\Delta^-,\Delta^+) \leq p$ and $dst(D) \geq p+\Omega(\log p)$. (See [62].) Note also that similarly as for Conjecture 86, it is sufficient to prove Conjecture 93 for $k=2$ and k=3. In [6], we prove that Conjecture 93 holds for $k=2$. By the above remark, it implies that Conjecture 93 holds for all even k.

4.2.3 Several fibers and several multicasts

Next, we study the more general (and more realistic) problem in which the center is connected to the nodes of V with n optical fibers. Morover each node may sent several multicasts. We model it as a labelled digraph problem: We consider a digraph D on vertex set V. For each multicast $(v,S_i(v))$ we add the set of arcs $A_i(v)=\{vw, w\in S_i(v)\}$ with label i. The label of an arc a is denoted by $l(a)$. Thus for every couple (u,v) of vertices and label i there is at most one arc uv labelled by i. If each vertex sends at most m multicasts, there are at most m labels on the arcs. Such a digraph is said to be *m-labelled*. One wants to find a *n-fiber wavelength assignment* of D, that is a mapping $\Phi : A(D) \rightarrow \Lambda\times\{1,\dots,n\}\times\{1,\dots n\}$ in which every arc uv is associated a triple $(\lambda(uv), f^+(uv), f^-(uv))$ such that :

(i) $(\lambda(uv), f^-(uv)) \neq (\lambda(vw), f^+(vw))$;

(ii) $(\lambda(uv), f^-(uv)) \neq (\lambda(u'v), f^-(u'w))$;

(iii) if $l(vw) \neq l(vw')$ then $(\lambda(vw), f^+(vw)) \neq (\lambda(vw'), f^+(vw'))$.

$\lambda(uv)$ corresponds to the wavelength of uv, and $f^+(uv)$ and $f^-(uv)$ the fiber used in u and v respectively. Hence the condition (i) corresponds to the fact that an arc entering v and an arc leaving v have either different wavelength or different fibers; the condition (ii) corresponds to the fact that two arcs entering v have either different wavelength or different fibers; the condition (iii) corresponds to the fact that two arcs leaving v with different labels have either different wavelengths or different fibers. The problem is then to find the minimum cardinality $\lambda_n(D)$ of Λ such that there exists an *n-fiber wavelength assignment* of D.

The crucial thing in an n-fiber wavelength assignment is the function λ which assigns colours (wavelengths) to the arcs. It must be an *n-fiber colouring*, that is a function $\phi : A(D) \rightarrow \Lambda$, such that at each vertex v, for each colour in $\lambda \in \Lambda$, $in(v,\lambda) + out(v,\lambda) \leq n$ with $in(v,\lambda)$ the number of arcs coloured λ entering v and $out(v,\lambda)$ the number of labels l such that there exists an arc leaving v coloured λ. Once we have an n-fiber colouring, one can easily find a suitable wavelength assignment by assigning for every vertex v and every colour λ a different fiber to each arc entering v with colour λ and each set of arcs leaving v coloured λ and labelled the same. Hence $\lambda_n(D)$ is the minimum number of colours such that there exists an n-fiber colouring.

We are particularly interested in $\lambda_n(m,k) = \max\{\lambda_n(D) \mid D \text{ is } m\text{-labelled and } \Delta^-(D) \leq k\}$ that is the maximum number of wavelengths that may be necessary if there are n-fibers and each node sends at most m and receives at most k multicasts. In particular, $\lambda_1(1,k) = \max\{dst(D) \mid \Delta^-(D) \leq k\}$. So our above mentionned results show that $2k \leq \lambda_1(1,k) \leq 2k+1$. Brandt and Gonzalez showed that for $n \geq 2$ then $\lambda_n(1,k) \leq \lceil \frac{k}{n-1} \rceil$. In [6], we study the case when $n \geq 2$ and $m \geq 2$. We show that if $m \geq n$ then

$$\left\lceil \frac{m}{n} \left\lceil \frac{k}{n} \right\rceil + \frac{k}{n} \right\rceil \leq \lambda_n(m,k) \leq \left\lceil \frac{m}{n} \left\lceil \frac{k}{n} \right\rceil + \frac{k}{n} \right\rceil + C\frac{m^2 \log k}{n} \quad \text{for some constant } C.$$

We also show that if $m < n$ then

$$\left\lceil \frac{m}{n} \left\lceil \frac{k}{n} \right\rceil + \frac{k}{n} \right\rceil \leq \lambda_n(m,k) \leq \left\lceil \frac{k}{n-m} \right\rceil.$$

The lower bound generalizes Brandt and Gonzalez [33] results which established this inequality in the particular cases when $k \leq 2$, $m \leq 2$ and $k = m$. The digraphs used to show this lower bound are all acyclic. We show that if $m \geq n$ then this lower bound is tight for acyclic digraphs. Moreover the above mentionned digraphs have large outdegree. Generalizing the result of Guiduli [62], we show that for an m-labelled digraph D with both in- and outdegree bounded by k then few colours are needed:

$$\lambda_n(D) \leq \frac{k}{n} + C'\frac{m^2 \log k}{n} \quad \text{for some constant } C'.$$

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