Choosability of the square of planar subcubic graphs with large girth

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Abstract

We show that the choice number of the square of a subcubic graph with maximum average degree less than 18/7 is at most 6. As a corollary, we get that the choice number of the square of a subcubic planar graph with girth at least 9 is at most 6. We then show that the choice number of the square of a subcubic planar graph with girth at least 13 is at most 5.

1 Introduction

Let G be a (simple) graph. The *neighbourhood* of a vertex v of G, denoted $N_G(v)$, is the set of its *neighbours*, i.e. is the set of vertices y such that xy is an edge. The *degree* of a vertex v in G, denoted $d_G(v)$, is its number of neighbours. Often, when the graph G is clearly understood from the context, we omit the subscript G. A graph is *subcubic* if every vertex has degree at most 3.

Let $p : V(G) \rightarrow \mathbb{N}$. A *p-list-assignment* is a list-assignment L such that $|L(v)| = p(v)$ for any $v \in V(G)$. G is *p-choosable* if it is L-colourable for any p-list-assignment. By extension, if k is an integer, we say that G is *k-choosable* if it is p-choosable when p is the constant function with value k (i. e. $p(v) = k$ for all $v \in V$). The *choice number* of G, denoted $ch(G)$, is the smallest integer k such that G is k-choosable. Clearly the choice number of G is at least as large as $\chi(G)$, the *chromatic number* of G.

The *square* of G is the graph G^2 with vertex set $V(G)$ such that two vertices are linked by an edge of G^2 if and only if x and y are at distance at most 2 in G. A graph is called *planar* if it can be embedded in the plane. Wegner [9] proved that the square of a subcubic planar graph is 8-colourable. He also conjectured it is 7-colourable. Recently, this conjecture was proved by Thomassen [8].

Theorem 1 (Thomassen [8]) *Let G be a subcubic planar graph. Then $\chi(G^2) \leq 7$.*

Kostochka and Woodall [6] conjectured that, for every square of a graph, the chromatic number equals the choice number.

Conjecture 2 (Kostochka and Woodall [6]) *For all G, $ch(G^2) = \chi(G^2)$.*

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If true, this conjecture together with Theorem 1 implies that every subcubic planar graph is 7-choosable. Very recently, Cranston and Kim [2] showed that the square of every subcubic graph (non necessarily planar) other than the Petersen graph is 8-choosable.

The *average degree* of G, denoted $Ad(G)$ is $\frac{\sum_{v\in V(G)} d(v)}{|V(G)|} = \frac{2|E(G)|}{|V(G)|}$. The *maximum average degree* of G, denoted $Mad(G)$, is $\max\{Ad(H), H \text{ subgraph of } G\}$. In [3], Dvořák, Škrekovski and Tancer proved that the choice number of the square of a subcubic graph G is at most 4 if $Mad(G) < 24/11$ and G has no 5-cycle, at most 5 if $Mad(G) < 7/3$ and at most 6 if $Mad(G) < 5/2$.

The *girth* of a graph is the smallest length of a cycle in G. Planar graphs with prescribed girth have bounded maximum average degree:

Proposition 3 *Every planar graph with girth at least g has maximum average degree less than $2 + \frac{4}{g-2}$.*

Hence the results of Dvořák, Škrekovski and Tancer imply that the choice number of the square of a planar graph with girth g is at most 6 if $g \geq 10$, at most 5 if $g \geq 14$ and at most 4 if $g \geq 24$. The two latter results had been previously proved by Montassier and Raspaud [7].

In this paper, we improve some of these results. We first show (Theorem 4) that the choice number of the square of a subcubic graph with maximum average degree less than 18/7 is at most 6. As a corollary, we get that the choice number of the square of a subcubic planar graph with girth at least 9 is at most 6. Note that this corollary has been proved later and independently by Cranston and Kim [2]. We then show (Theorem 9) that the choice number of the square of a subcubic planar graph with girth at least 13 is at most 5.

2 The main results

The general frame of the proofs is classical. We consider a *k-minimal graph*, that is a subcubic graph such that its square is not k-choosable but the square of every proper subgraph is k-choosable. We prove that some *configurations* (i.e. induced subgraphs) are forbidden in such a graph and then deduce a contradiction. To do so, we will need the following definitions:

An *i-vertex* is a vertex of degree i. We denote by V_i the set of i-vertices of G and by v_i its cardinality. Let v be a vertex. An *i-neighbour* of v is a neighbour of v with degree i. The *i-neighbourhood* of v is $N_i(v) = N(v) \cap V_i$ and its *i-degree* is $d_i(v) = |N_i(v)|$.

Some properties of 6- and 5-minimal graphs have already been proved in [3]. The easy first one is that $V_0 \cup V_1 = \emptyset$, so G has minimum degree 2. This will allow us to use the following definitions for 6- and 5-minimal graphs.

Let G be a subcubic graph with minimum degree 2. A *thread* of G is a path whose endvertices are 3-vertices and whose internal vertices are 2-vertices. The *kernel* of G is the weighted graph K_G such that $V(K_G) = V_3(G)$ and xy is an edge in K_G with weight l if and only if x and y are connected by a thread of length l in G. An edge of weight l is also called *l-edge*. Let x be a 3-vertex of G. The *type* of x is the triple (l_1, l_2, l_3) such that $l_1 \leq l_2 \leq l_3$ and the three edges (a loop being counted twice) incident to x have weight l_1, l_2 and l_3 in K_G. We denote by Y_{l_1,l_2,l_3} the set of 3-vertices of type (l_1, l_2, l_3) and y_{l_1,l_2,l_3} its cardinality. Moreover, for every integer i, we define $Z_i := \bigcup_{l_1+l_2+l_3=i} Y_{l_1,l_2,l_3}$ and $z_i = |Z_i|$. The number of vertices and edges and thus the average degree of G may be easily expressed in terms of the z_i:

$$|V(G)| = \sum_{i\geq 3} \frac{i-1}{2} z_i$$
$$2|E(G)| = \sum_{i\geq 3} i.z_i$$

$$Ad(G) \quad = \quad \frac{\sum_{i\geq 3} i.z_i}{\sum_{i\geq 3} \frac{i-1}{2} z_i} \tag{1}$$

2.1 6-choosability

The aim of this subsection is to prove the following result.

Theorem 4 *Let G be a subcubic graph of maximum average degree $d < 18/7$. Then G^2 is 6-choosable.*

Remark 5 Theorem 4 is tight. Indeed, the graph J_7 depicted in Figure 1 has average degree 18/7 and its square is the complete graph on seven vertices K_7 which is not 6-choosable (nor 6-colourable).

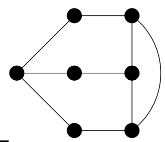

Figure 1: The graph J_7

Theorem 4 and Proposition 3 yield that the square of a subcubic planar graph with girth at least 9 is 6-choosable.

Corollary 6 *The square of a subcubic planar graph with girth at least* 9 *is* 6*-choosable.*

In order to prove Theorem 4, we need to establish some properties of 6-minimal graphs. Some of them have been proved in [3].

Lemma 7 (Dvořák, Škrekovski and Tancer [3]) *Let G be a* 6*-minimal graph. Then the following hold:*

(i) all the edges of K_G have weight at most 2*;*

(ii) every 3*-cycle of G has its vertices in V_3;*

(iii) every 4*-cycle of G has at least three vertices in V_3;*

(iv) a vertex of $Y_{2,2,2}$ is not adjacent in K_G to a vertex of $Y_{1,2,2} \cup Y_{2,2,2}$.

We will prove in Subsection 3.2 some new properties.

Lemma 8 *Let G be a* 6*-minimal graph. Then the following hold:*

(i) if $(v_1, v_2, v_3, v_4, v_1)$ is a 4*-cycle with $v_2 \in V_2$ then v_1 or v_3 is not in $Y_{1,2,2}$;*

(ii) a vertex of $Y_{1,2,2}$ is adjacent in K_G to at most one vertex of $Y_{1,2,2}$ by 2*-edges.*

Proof of Theorem 4. Let G be a 6-minimal planar graph. G has minimum degree 2, so its kernel K_G is defined. Moreover by Lemma 7 (i), Z_i is empty for $i \geq 7$ and $Z_6 = Y_{2,2,2}$ and $Z_5 = Y_{1,2,2}$.

Let us consider a vertex of $Z_4 = Y_{1,1,2}$. Its neighbour in K_G via the 2-edge is in $Z_4 \cup Z_5 \cup Z_6$ because a vertex of $Z_3 = Y_{1,1,1}$ is incident to no edge of weight 2. For $i = 4, 5, 6$, let Z_4^i be the set of vertices of

Z_4 which are adjacent to a vertex of Z_i by their unique 2-edge and z_4^i its cardinality. (Z_4^4, Z_4^5, Z_4^6) is a partition of Z_4 so $z_4 = z_4^4 + z_4^5 + z_4^6$. Hence Equation (1) becomes

$$Ad(G) = \frac{6z_6 + 5z_5 + 4z_4^6 + 4z_4^5 + 4z_4^4 + 3z_3}{\frac{5}{2}z_6 + 2z_5 + \frac{3}{2}z_4^6 + \frac{3}{2}z_4^5 + \frac{3}{2}z_4^4 + z_3}.$$

By Lemma 7 (iv), the three neighbours in K_G of a vertex of Z_6 are not in $Z_6 \cup Z_5$. So they must be in Z_4^6. It follows that $3z_6 = z_4^6$. So

$$Ad(G) = \frac{5z_5 + 6z_4^6 + 4z_4^5 + 4z_4^4 + 3z_3}{2z_5 + \frac{7}{3}z_4^6 + \frac{3}{2}z_4^5 + \frac{3}{2}z_4^4 + z_3}.$$

By Lemma 8 (ii), a vertex of Z_5 is adjacent to at least one vertex of Z_4^5. Thus $z_5 \leq z_4^5$. But $Ad(G)$ is decreasing as a function of z_5 since z_4^6, z_4^5, z_4^4 and z_3 are non-negative. It follows that

$$Ad(G) \geq \frac{6z_4^6 + 9z_4^5 + 4z_4^4 + 3z_3}{\frac{7}{3}z_4^6 + \frac{7}{2}z_4^5 + \frac{3}{2}z_4^4 + z_3} \geq \frac{18}{7}.$$

□

2.2 5-choosability

Dvořák, Škrekovski and Tancer [3] proved that the square of a subcubic graph G with maximum average degree less than 7/3 is 5-choosable. This result is tight since the graph J_6 depicted in Figure 2 has average degree 7/3 and its square is the complete graph on six vertices K_6 which is not 5-choosable (nor 5-colourable). However, we will prove that the square of a subcubic planar graph with girth at least 13

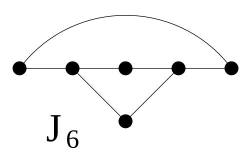

Figure 2: The graph J_6

is 5-choosable, which improves the result of Montassier and Raspaud.

Theorem 9 *The square of a subcubic planar graph with girth at least* 13 *is* 5*-choosable.*

In order to prove this theorem, we need to establish some properties of 5-minimal graphs. Some of them have been proved in [3].

Lemma 10 (Dvořák, Škrekovski and Tancer [3]) *Let G be a* 5*-minimal graph. Then the following hold:*

(i) all the edges of K_G have weight at most 3*;*

(ii) if $i \geq 8$, Z_i is empty.

We will prove in Subsection 3.3 some new properties.

Lemma 11 *Let G be a* 5*-minimal graph of girth at least* 13*. Then in K_G the following hold:*

(i) a vertex of $Y_{2,2,3}$ *and a vertex of* $Y_{1,2,3} \cup Y_{2,2,3}$ *are not linked by a* 2*-edge;*

(ii) a vertex of $Y_{1,3,3}$ *and a vertex of* $Y_{1,2,3} \cup Y_{1,3,3}$ *are not linked by a* 1*-edge;*

(iii) s vertex of $Y_{2,2,2}$ *is not adjacent in* K_G *to three vertices of* $Y_{2,2,3}$ *(by* 2*-edges).*

Proof of Theorem 9. Let G be a 5-minimal planar graph with girth at least 13. G has minimum degree 2, so its kernel K_G is defined. Moreover, by Lemma 10 (i), $Z_7 = Y_{2,2,3} \cup Y_{1,3,3}$, so

$$z_7 = y_{2,2,3} + y_{1,3,3}. \tag{2}$$

Let us count the number e_2 of 2-edges incident to vertices of $Y_{2,2,3}$. Recall that $Z_4 = Y_{1,1,2}$ and $Z_3 = Y_{1,1,1}$. Since 2-edges may not link two vertices of type $(2,2,3)$ according to Lemma 11 (i), we have $e_2 = 2y_{2,2,3}$. Moreover, the ends of such edges which are not in $Y_{2,2,3}$ have to be in $Y_{2,2,2} \cup Y_{1,2,2} \cup Z_4$ by Lemmas 10 and 11 (i). Furthermore, a vertex of $Y_{2,2,2}$ is incident to at most two edges of e_2 according to Lemma 11 (iii) and a vertex of $Y_{1,2,2}$ (resp. Z_4) is incident to at most two (resp. one) 2-edges. Therefore $e_2 \leq 2y_{2,2,2} + 2y_{1,2,2} + z_4$. So,

$$2y_{2,2,3} \leq 2y_{2,2,2} + 2y_{1,2,2} + z_4. \tag{3}$$

Let us now count the number e_1 of 1-edges incident to vertices of $Y_{1,3,3}$. Since 1-edges may not link two vertices of type $(1,3,3)$ according to Lemma 11 (ii), we have $e_1 = y_{1,3,3}$. Moreover, the ends of such edges which are not in $Y_{1,1,3}$ have to be in $Y_{1,2,2} \cup Y_{1,1,3} \cup Z_4 \cup Z_3$ by Lemmas 10 and 11 (ii). Furthermore, vertices of $Y_{1,2,2}$ (resp. $Y_{1,1,3} \cup Z_4$, Z_3) are incident to at most one (resp. two, three) 1-edges. Thus $e_1 \leq y_{1,2,2} + 2y_{1,1,3} + 2z_4 + 3z_3$. So,

$$y_{1,3,3} \leq y_{1,2,2} + 2y_{1,1,3} + 2z_4 + 3z_3. \tag{4}$$

$2 \times (4) + (3)$ yields $2y_{2,2,3} + 2y_{1,3,3} \leq 2y_{2,2,2} + 4y_{1,2,2} + 4y_{1,1,3} + 5z_4 + 6z_3$. Hence, by Equation (2), $2z_7 \leq 2z_6 + 4z_5 + 5z_4 + 6z_3$, so

$$z_7 \leq z_6 + 2z_5 + \frac{5}{2}z_4 + 3z_3.$$

Now by Equation (1) the average degree of G is

$$Ad(G) = \frac{7z_7 + 6z_6 + 5z_5 + 4z_4 + 3z_3}{3z_7 + \frac{5}{2}z_6 + 2z_5 + \frac{3}{2}z_4 + z_3}.$$

As a function of z_7, this is a decreasing function (on $\mathbb{R}^+$); so it is minimum when z_7 is maximum that is equal to $z_6 + 2z_5 + \frac{5}{2}z_4 + 3z_3$. So,

$$Ad(G) \geq \frac{13z_6 + 19z_5 + \frac{43}{2}z_4 + 24z_3}{\frac{11}{2}z_6 + 8z_5 + 9z_4 + 10z_3} \geq \frac{26}{11}.$$

This contradicts the fact that G has girth 13 by Proposition 3. □

Remark 12 It is very likely that using the method below, one can prove that a graph G with maximum average degree less than $\frac{26}{11}$ is 5-choosable unless it contains J_6 as an induced subgraph. However, this will require the tedious study of a large number of configurations.

3 Proofs of Lemmas 8 and 11

In order to prove Lemmas 8 and 11, we need the following lemma proved in [3]. Let S be a set of vertices of a k-minimal graph G. The function $p_S : S \to \mathbb{N}$ is defined by $p_S(v) = k - |N_{G^2}(v) \setminus S|$. Then $p_S(v)$ represents the minimum number of available colours at a vertex $v \in S$ once we have precoloured the square of $G - S$. Hence if $(G-S)^2$ is k-choosable, $(G-S)^2 = G^2 - S$ and $G^2[S]$ is p_S-choosable, one can extend any k-list-colouring of $G - S$ into a k-list-colouring of G, which is a contradiction.

Lemma 13 (Dvořák, Škrekovski and Tancer [3]) *Let S be a set of vertices of a k-minimal graph G. If $(G-S)^2 = G^2 - S$, then $G^2[S]$ is not p_S-choosable.*

In order to use Lemma 13, we need some results on the choosability of some graphs.

3.1 Some choosability tools

Definition 14 Let x and y be two vertices of a graph G. An $(x-y)$-*ordering* of G is an ordering of the vertices such that x is the minimum and y the maximum. An $(x, y-z)$-*ordering* is an ordering of the vertices such that x is minimum, y is the second minimum and z is maximum.

Let $\sigma = (v_1 < v_2 < \ldots < v_n)$ be an ordering of the vertices of G and p a function $V(G) \to \mathbb{N}$. σ is *p-greedy* if, for every i, $|N(v_i) \cap \{v_1, \ldots, v_{i-1}\}| < p(v_i)$. It is *$p$-nice* if, for every i except n, $|N(v_i) \cap \{v_1, \ldots, v_{i-1}\}| < p(v_i)$ and $d(v_n) = p(v_n)$. It is *p-good* if, for every $3 \le i \le n$, $|N(v_i) \cap \{v_1, \ldots, v_{i-1}\}| - \epsilon(v_i) < p(v_i)$ with $\epsilon(v_i) = 1$ if v_i is adjacent to both v_1 and v_2 and $\epsilon(v_i) = 0$ otherwise. By extension, if k is an integer, we say that σ is *k-greedy* (resp. *k-nice*, *k-good*) if it is p-greedy (resp. p-nice, p-good) when p is the constant function with value k (i. e. $p(v_i) = k$ for every $1 \le i \le n$).

The greedy algorithm according to greedy, nice and good orderings yields the following three lemmas.

Lemma 15 *If G has a p-greedy ordering then G is p-choosable.*

Proof. Applying the greedy algorithm according to the p-greedy ordering gives the desired colouring. □

Lemma 16 *Let xy be an edge of graph G and L be a p-list-assignment of G. If $L(x) \not\subset L(y)$ and G has a p-nice $(x-y)$-ordering, then G is L-colourable.*

Proof. Let a be a colour in $L(x) \setminus L(y)$. Assign a to x and proceed the greedy algorithm according to the p-nice $(x-y)$-ordering.. The only vertex which has not more colour in its list than previously coloured neighbours is y for which $|L(y)| = d(y)$. But since $a \notin L(y)$, at most $d(y) - 1$ colours of $L(y)$ are assigned to the neighbours of y. Hence one can colour y. □

Lemma 17 *Let x, y and z be three vertices of a graph $G = (V, E)$ such that $xy \notin E$. If $L(x) \cap L(y) \neq \emptyset$ and G has a p-good $(x, y-z)$-ordering, then G is L-colourable.*

Proof. Let a be a colour in $L(x) \cap L(y)$ and $\sigma = (v_1 < v_2 < \ldots < v_n)$ be a p-good $(x, y-z)$-ordering. (In particular, $v_1 = x$, $v_2 = y$ and $v_n = z$.) Assign a to x and y and proceed the greedy algorithm according to σ. For every $3 \le i \le n$, the number of colours assigned to already coloured neighbours of v_i is at most $|N(v_i) \cap \{v_1, \ldots, v_{i-1}\}| - \epsilon(v_i)$ since v_1 and v_2 are coloured the same. Hence the greedy algorithm gives an L-colouring. □

Remark 18 Note that if $xz, yz \in E$, a p-nice $(x, y-z)$-ordering is also p-good.

Definition 19 The *blocks* of a graph are its maximal 2-connected components. A connected graph is said to be a *Gallai tree* if each of its blocks is either a complete graph or an odd cycle.

The following theorem was proved independently by Borodin [1] and Erdős, Rubin and Taylor [4]:

Theorem 20 (Borodin [1], Erdős, Rubin and Taylor [4]) *Let G be a connected graph and d_G the degree function in G. Then G is d_G-choosable if and only if G is not a Gallai tree.*

Lemma 21 *Let $G = (V, E)$ be a graph and $p : V(G) \to \mathbb{N}$. Let S be a set of vertices such that $p(v) \geq d(v)$ for all $v \in S$. If $G[S]$ is not a Gallai tree and $G - S$ is p-choosable then G is p-choosable.*

Proof. Let L be a p-list-assignment of G. Since $G - S$ is p-choosable, its admits an L-colouring c. Let us now extend it to S. The list $I(v) = L(v) \setminus \{c(w), w \in N(v) \setminus S\}$ of available colours of a vertex $v \in S$ is of size at least $p'(v) = p(v) - |N(v) \setminus S| \geq d_{G[S]}(v)$. Since $G[S]$ is not a Gallai tree, by Theorem 20, $G[S]$ is p'-choosable and thus I-colourable. So, G is L-colourable. □

A 4-regular graph G is *cycle+triangles* if it is the edge union of a Hamiltonian cycle C and a 2-factor consisting of triangles. In other words, the graph induced by the edges of $E(G) \setminus E(C)$ is the disjoint union of 3-cycles.

Theorem 22 (Fleischner and Stiebitz [5]) *Every cycle+triangles graph is 3-choosable.*

3.2 Proof of Lemma 8

Lemma 23 *Let $q \geq 2$ and $C_{4q} = (v_1, \dots, v_{4q}, v_1)$ be the $4q$-cycle and p defined by $p(v_i) = 4$ if i is odd and $p(v_i) = 2$ otherwise. Then C_{4q}^2 is p-choosable.*

Proof. The set S of vertices v for which $p(v) \geq d_{C_{4q}^2}(v)$ is the set of v_i with odd indices. $C_{4q}^2[S]$ is a $2q$-cycle and thus is not a Gallai tree. Moreover $C_{4q}^2 - S$ is also a $2q$-cycle and is 2-choosable. Hence Lemma 21 gives the result. □

Proposition 24 *Let $P_7 = (v_1, \dots, v_7)$ be a path and p the function defined by $p(v_1) = p(v_2) = p(v_6) = p(v_7) = 2$, $p(v_3) = p(v_5) = 4$ and $p(v_4) = 3$. Then P_7^2 is p-choosable.*

Proof. Let L be a p-list-assignment of P_7^2. Since $(v_2 < v_4 < v_6 < v_7 < v_5 < v_3 < v_1)$ is a p-nice ordering of P_7^2, by Lemma 16, we may assume that $L(v_1) = L(v_2)$, and by symmetry of P_7 and p that $L(v_6) = L(v_7)$.

Since $(v_1 < v_4 < v_2 < v_6 < v_7 < v_5 < v_3)$ is p-good, by Lemma 17, we may assume that $L(v_1) \cap L(v_4) = \emptyset$, and by symmetry $L(v_7) \cap L(v_4) = \emptyset$.

Now one can find $c(v_1) \in L(v_1)$, $c(v_2)$ in $L(v_2) \setminus \{c(v_1)\}$, $c(v_6)$ in $L(v_6)$, $c(v_7)$ in $L(v_7) \setminus \{c(v_6)\}$, $c(v_3)$ in $L(v_3) \setminus \{c(v_1), c(v_2)\}$, and $c(v_5)$ in $L(v_5) \setminus \{c(v_3), c(v_6), c(v_7)\}$. Now since $L(v_1) \cap L(v_4) = \emptyset$ and $L(v_1) = L(v_2)$, $c(v_2) \notin L(v_4)$. Analogously, $c(v_6) \notin L(v_4)$. Hence, $L(v_4) \setminus \{c(v_2), c(v_3), c(v_5), c(v_6)\} = L(v_4) \setminus \{c(v_3), c(v_5)\} \neq \emptyset$. So, one can choose $c(v_4)$ in this set to get an L-colouring c of P_7^2. □

Lemma 25 *For $1 \leq i \leq 17$, let F_i be the graphs and p_i be the functions depicted in Figure 3.*

(i) $F_1^2 \cup \{v_5v_6\}$ *is p_1-choosable.*

(ii) $F_2^2 \cup \{v_1v_4\}$ *and* $F_2^2 \cup \{v_4v_7\}$ *are p_2-choosable.*

(iii) $F_3^2 \cup \{v_4v_8\}$ *is p_3-choosable.*

(iv) F_4^2 *is* 6*-choosable.*

(v) $F_5^2 \cup \{v_1v_4, v_1v_6\}$ *is* p_5*-choosable.*

(vi) F_6^2 *is* p_6*-choosable.*

(vii) $F_7^2 \cup \{v_9v_{10}\}$ *is* p_7*-choosable.*

(viii) F_8^2 *is* p_8*-choosable.*

(ix) $F_9^2 \cup \{v_2v_9\}$ *and* $F_9^2 \cup \{v_6v_9\}$ *are* p_9*-choosable.*

(x) $F_{10}^2 \cup \{v_4v_8\}$ *is* p_{10}*-choosable.*

(xi) $F_{11}^2 \cup \{v_4v_8, v_8v_9\}$, $F_{11}^2 \cup \{v_4v_8, v_9v_4\}$ *and* $F_{11}^2 \cup \{v_8v_9, v_9v_4\}$ *are* p_{11}*-choosable and* $F_{11}^2 \cup \{v_4v_8, v_8v_9, v_9v_4\}$ *is* 5*-choosable.*

(xii) $F_{12}^2 \cup \{v_4v_8\}$ *is* p_{12}*-choosable.*

(xiii) F_{13}^2 *is* 6*-choosable.*

Proof.

(i) In $F_1^2 \cup \{v_5v_6\}$, $(v_6 < v_5 < v_4 < v_3 < v_1 < v_2)$ is p_1-greedy. So, by Lemma 15, $F_1^2 \cup \{v_5v_6\}$ is p_1-choosable.

(ii) In $F_2^2 \cup \{v_4v_7\}$, $(v_2 < v_4 < v_7 < v_6 < v_5 < v_3 < v_1)$ is p_2-nice and $p_2(v_2) > p_2(v_1)$. So, by Lemma 16, $F_2^2 \cup \{v_4v_7\}$ is p_2-choosable.

By symmetry, one shows that $F_2^2 \cup \{v_1v_4\}$ is p_2-choosable.

(iii) In $F_3^2 \cup \{v_4v_8\}$, $(v_2 < v_8 < v_4 < v_7 < v_6 < v_5 < v_3 < v_1)$ is p_3-nice and $p_3(v_2) > p_3(v_1)$. So, by Lemma 16, $F_3^2 \cup \{v_4v_8\}$ is p_3-choosable.

(iv) Let L be a 6-list-assignment of F_4^2. Every ordering with maximum v_1 and second maximum v_7 is 6-nice. Thus, by Lemma 16, we may assume that $L(v_j) = L(v_1)$ for $j \in \{2, 3, 4, 5, 6, 8\}$. Analogously, we may assume that $L(v_j) = L(v_7)$ for $j \in \{2, 3, 4, 5, 6, 8\}$. Hence all the lists are the same, say $\{1, 2, 3, 4, 5, 6\}$. Now $c(v_1) = c(v_5) = 1$, $c(v_2) = 2$, $c(v_3) = c(v_7) = 3$, $c(v_4) = 4$, $c(v_6) = 5$ and $c(v_8) = 6$ is an L-colouring of F_4^2.

(v) In $F_5^2 \cup \{v_1v_4, v_1v_6\}$, $(v_7 < v_6 < v_1 < v_4 < v_2 < v_3 < v_5)$ is p_5-nice and $p_5(v_7) > p_5(v_5)$. So, by Lemma 15, $F_5^2 \cup \{v_1v_5\}$ is p_5-choosable.

(vi) In F_6^2, $(v_4 < v_2 < v_8 < v_1 < v_3 < v_5)$ is p_6-nice and $p_6(v_4) > p_6(v_5)$. So, by Lemma 16, F_6^2 is p_6-choosable.

(vii) Let L be a p_7-list-assignment of $F_7^2 \cup \{v_9v_{10}\}$. $(v_2 < v_9 < v_{10} < v_8 < v_6 < v_4 < v_7 < v_5 < v_3 < v_1)$, $(v_2 < v_9 < v_{10} < v_8 < v_6 < v_4 < v_5 < v_7 < v_1 < v_3)$ and $(v_4 < v_9 < v_{10} < v_8 < v_6 < v_2 < v_5 < v_7 < v_1 < v_3)$ are p_7-nice. Thus, by Lemma 16, we may assume that $L(v_2) \subset L(v_1)$, $L(v_2) \subset L(v_3)$ and $L(v_4) \subset L(v_3)$. It follows that $L(v_1) \cap L(v_4) \neq \emptyset$. Because $(v_1 < v_4 < v_{10} < v_9 < v_2 < v_8 < v_6 < v_7 < v_5 < v_3)$ is p_7-good, by Lemma 17, $F_7^2 \cup \{v_9v_{10}\}$ is L-colourable.

(viii) In F_8^2, $(v_6 < v_5 < v_7 < v_9 < v_8 < v_4 < v_3 < v_2 < v_1)$ is p_8-greedy. So, by Lemma 15, F_8^2 is p_8-choosable.

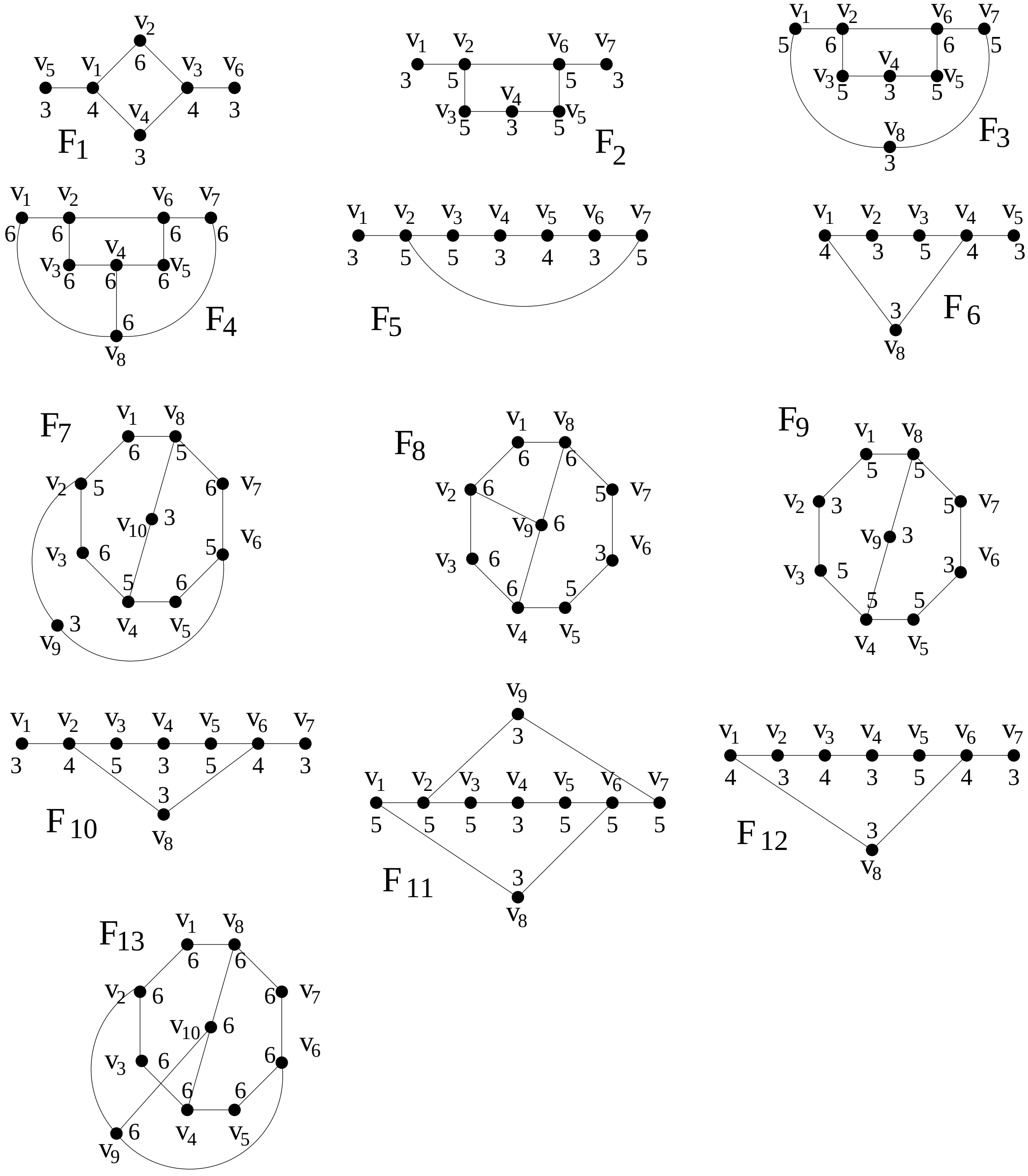

Figure 3: The graphs F_i and functions p_i for $1 \leq i \leq 13$

(ix) Let L be a p_9-list-assignment of $F_9^2 \cup \{v_2v_9\}$. Then $(v_2 < v_9 < v_6 < v_4 < v_8 < v_7 < v_5 < v_3 < v_1)$ and $(v_2 < v_9 < v_6 < v_4 < v_8 < v_7 < v_5 < v_1 < v_3)$ are p_9-nice so by Lemma 16, we may assume that $L(v_2) \subset L(v_3) \cap L(v_1)$. Moreover, $(v_4 < v_2 < v_9 < v_6 < v_8 < v_7 < v_5 < v_1 < v_3)$ is p_9-nice so by Lemma 16, we may assume that $L(v_4) = L(v_3)$. It follows that $L(v_1) \cap L(v_4) \neq \emptyset$. Thus, by Lemma 17, since $(v_1 < v_4 < v_2 < v_9 < v_8 < v_6 < v_7 < v_5 < v_3)$ is p_9-good, $F_9^2 \cup \{v_2v_9\}$ is L-colourable.

By symmetry, one shows that $F_9^2 \cup \{v_6v_9\}$ is p_9-choosable.

(x) In $F_{10}^2 \cup \{v_4v_8\}$, $(v_2 < v_8 < v_4 < v_6 < v_7 < v_5 < v_3 < v_1)$ is p_{10}-nice and $p_{10}(v_2) > p_{10}(v_1)$. So, by Lemma 16, $F_{10}^2 \cup \{v_4v_8\}$ is p_{10}-colourable.

(xi) Let $F \in \left\{F_{11}^2 \cup \{v_4v_8, v_8v_9, v_9v_4\}, F_{11}^2 \cup \{v_4v_8, v_8v_9\}, F_{11}^2 \cup \{v_4v_8, v_9v_4\}, F_{11}^2 \cup \{v_8v_9, v_9v_4\}\right\}$ and L be a 5-list-assignment if $F = F_{11}^2 \cup \{v_4v_8, v_8v_9, v_9v_4\}$ and a p_{11}-list-assignment of F otherwise.

Then $(v_1 < v_8 < v_9 < v_4 < v_2 < v_6 < v_7 < v_5 < v_3)$, $(v_7 < v_9 < v_8 < v_4 < v_6 < v_2 < v_1 < v_5 < v_3)$ and $(v_7 < v_9 < v_8 < v_4 < v_6 < v_2 < v_1 < v_3 < v_5)$ are p-nice in F. So by Lemma 16, we may assume that $L(v_1) = L(v_3) = L(v_5) = L(v_7)$.

If $L(v_8) \not\subset L(v_2)$, let us colour v_8 with $c_8 \in L(v_8) \setminus L(v_2)$, v_4 with $c_4 \in L(v_4) \setminus \{c_8\}$, v_9 with $c_8 \in L(v_8) \setminus \{c_4, c_8\}$, v_1 and v_5 with the same colour $c_1 \in L(v_1) \setminus \{c_4, c_8, c_9\}$, v_3 and v_7 with the same colour $c_3 \in L(v_1) \setminus \{c_1, c_4, c_8, c_9\}$, v_6 with $c_6 \in L(v_6) \setminus \{c_1, c_3, c_8, c_9\}$ and finally v_2 with $c_2 \in L(v_2) \setminus \{c_1, c_3, c_6, c_8, c_9\} = L(v_2) \setminus \{c_1, c_3, c_6, c_9\}$. This gives an L-colouring of F. So we may assume that $L(v_8) \subset L(v_2)$. Exchanging the role of c_4 in c_8 in the preceding argument, we may assume that $L(v_4) \subset L(v_2)$. Moreover by symmetry, we may assume that $L(v_9) \cup L(v_4) \subset L(v_6)$. In particular, this implies that the sets $L(v_8) \cap L(v_9)$, $L(v_8) \cap L(v_4)$, $L(v_9) \cap L(v_4)$ and $L(v_2) \cap L(v_6)$ are non empty.

If $F = F_{11}^2 \cup \{v_4v_8, v_9v_4\}$ then $v_8v_9 \notin F$. Hence $(v_8 < v_9 < v_4 < v_2 < v_6 < v_7 < v_5 < v_3 < v_1)$ is p_{11}-good, so by Lemma 17, F is L-colourable.

If $F = F_{11}^2 \cup \{v_8v_9, v_9v_4\}$ then $v_8v_4 \notin F$. Hence $(v_4 < v_8 < v_9 < v_1 < v_2 < v_6 < v_7 < v_3 < v_5)$ is p_{11}-good, so by Lemma 17, F is L-colourable.

If $F = F_{11}^2 \cup \{v_4v_8, v_8v_9\}$ then $v_9v_4 \notin F$. Hence $(v_4 < v_9 < v_8 < v_1 < v_2 < v_6 < v_7 < v_3 < v_5)$ is p_{11}-good, so by Lemma 17, F is L-colourable.

If $F = F_{11}^2 \cup \{v_4v_8, v_8v_9, v_9v_4\}$, then $(v_2 < v_6 < v_4 < v_8 < v_9 < v_7 < v_5 < v_3 < v_1)$ is 5-good. So, by Lemma 17, F is L-colourable.

(xii) In $F_{12}^2 \cup \{v_4v_8\}$, $(v_6 < v_8 < v_4 < v_2 < v_1 < v_3 < v_5 < v_7)$ is p_{12}-nice and $p_{12}(v_6) > p_{12}(v_7)$. So by Lemma 16, $F_{12}^2 \cup \{v_4v_8\}$ is p_{12}-choosable.

(xiii) Let L be a 6-list-assignment of F_{13}^2. $(v_2 < v_9 < v_{10} < v_8 < v_6 < v_4 < v_7 < v_5 < v_3 < v_1)$, $(v_2 < v_9 < v_{10} < v_8 < v_6 < v_4 < v_5 < v_7 < v_1 < v_3)$ and $(v_4 < v_9 < v_{10} < v_8 < v_6 < v_2 < v_5 < v_7 < v_1 < v_3)$ are 6-nice. Thus, by Lemma 16, we may assume that $L(v_1) = L(v_2) = L(v_3) = L(v_4)$. Because $(v_1 < v_4 < v_{10} < v_9 < v_2 < v_8 < v_6 < v_7 < v_5 < v_3)$ is 6-good, by Lemma 17, F_{13}^2 is L-colourable.

□

Proof of Lemma 8.

To prove this lemma, we will suppose for a contradiction that it does not hold. Then we will find a set X of vertices contradicting Lemma 25. Indeed Lemma 25 will show that $G^2[X]$ is p_X-choosable and for each set X we consider, every vertex of X has at most one neighbour in $G - X$, so $(G - X)^2 = G^2 - X$. Lemma 13 completes the proof.

(i) Suppose for a contradiction that v_1 and v_3 are in $Y_{1,2,2}$. Let v_5 (resp. v_6) be the neighbour of v_1 (resp. v_3) distinct from v_2 and v_4. By Lemma 7 (iv), v_4 is in V_3 and $v_5 \neq v_6$. Set $S = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. Then $G[S] = F_1$, $p_S \geq p_1$ and $G^2[S] \subset F_1^2 \cup \{v_5 v_6\}$. So Lemma 25 contradicts Lemma 13.

(ii) Suppose for a contradiction that, in K_G, a vertex v_4 of $Y_{1,2,2}$ is adjacent to two vertices of $Y_{1,2,2}$ v_2 and v_6 by 2-edges. According to Lemma 7 (iii), $v_2 \neq v_6$. Let v_3 and v_5 be the 2-neighbours of v_4 common with v_2 and v_6 respectively, and v_1 (resp. v_7) be the 2-neighbour of v_2 (resp. v_6) not adjacent to v_4. Set $S = \{v_1, \dots, v_7\}$.

We first claim that $v_1 \neq v_7$. Suppose not. Then $(v_1, v_2, v_3, v_4, v_5, v_6, v_7)$ is a cycle C. It has no chord by Lemma 7 (ii), so $C^2 = G^2[S]$. Moreover, $p_S(v_i) \geq 4$ if i is even and $p_S(v_i) \geq 3$ otherwise. C^2 is a cycle+triangle graph, thus, by Theorem 22, it is 3-choosable and so p_S-choosable. This contradicts Lemma 13.

Let w_1 (resp. w_7) be the neighbour of v_1 (resp. w_7) distinct from v_2 (resp. v_6) and for $i \in \{2, 4, 6\}$, let w_i be the neighbour of v_i not in $\{v_{i-1}, v_{i+1}\}$. Let $W = \{w_1, w_2, w_4, w_6, w_7\}$.

We claim that $W \cap S \neq \emptyset$. Indeed, suppose for a contradiction that $W \cap S \neq \emptyset$. Since G is simple, $w_1 \neq v_2$ and $w_7 \neq v_6$. Moreover by Lemma 7 (i), w_1 and w_7 are in V_3, so $w_1 \neq v_7$ and $w_7 \neq v_1$. Furthermore, by Lemma 7 (ii), $w_2 \neq v_4$ and $w_6 \neq v_4$ and by Lemma 7 (iii), $w_1 \neq v_4$ and $w_7 \neq v_4$. Last, we may not have $w_1 = v_6$ and $w_2 = v_7$ otherwise the 4-cycle $(v_1, v_6, v_7, v_2, v_1)$ would contradict Lemma 7 (iii). Then, by symmetry, we only need to consider the cases $w_2 = v_6$, $w_2 = v_7$.

- Assume that $w_2 = v_6$. Then $G[S] = F_2$, $p_S \geq p_2$ and $G^2[S] \subset F_2^2 \cup \{v_1 v_4, v_4 v_7, v_1 v_7\}$. Thus, by Lemmas 25 and 13, $F_2^2 \cup \{v_1 v_7\} \subset G^2[S]$, so $w_1 = w_7 = v_8$. Let $T = S \cup \{v_8\}$. If $v_8 \neq w_4$, then $G[T] = F_3$ and $p_T \geq p_3$ and $G^2[T] \subset F_3^2 \cup \{v_4 v_8\}$. So Lemma 25 contradicts Lemma 13. If not then $G[T] = G = F_4$, so G is 6-choosable, by Lemma 25. This is a contradiction.

- Suppose that $w_2 = v_7$. Then $G[S] = F_5$, $p_S \geq p_5$ and $G^2[S] \subset F_5^2 \cup \{v_1 v_4, v_1 v_6\}$. Thus Lemma 25 contradicts Lemma 13.

This proves the claim.

Note that by Lemma 7 (ii), $w_1 \neq w_2$ and $w_6 \neq w_7$.

Suppose $w_1 = w_4 = v_8$. Then let $R = \{v_1, v_2, v_3, v_4, v_5, v_8\}$ and w_8 the neighbour of v_8. Then $(G[R], p_R) = (F_6, p_6)$ and $G^2[R] = F_6^2$. Thus Lemma 25 contradicts Lemma 13. Therefore, $w_1 \neq w_4$ and, by symmetry, $w_4 \neq w_7$.

Suppose $w_1 = w_7 = v_8$. Let $T = S \cup \{v_8\}$. Then $G[T]$ is the cycle C_8 and p_T is greater or equal to the function p defined in Lemma 23. So, by Lemmas 23 and 13, $G^2[T] \neq C_8^2$. It follows that either $w_2 = w_6$ or $w_4 = w_8$ with w_8 be the neighbour of v_8 not in S.

- Suppose $w_2 = w_6 = v_9$, and $w_4 = w_8 = v_{10}$. Set $W = \{v_1, \dots, v_{10}\}$. If $v_9 v_{10} \notin E(G)$ then $G[W] = F_7$, $p_W \geq p_7$) and $G^2[W] \subset F_7^2 \cup \{v_9 v_{10}\}$; so Lemma 25 contradicts Lemma 13. If not, $G = G[W] = F_{13}$, so G^2 is 6-choosable, according to Lemma 25, a contradiction.

- Suppose $w_2 = w_4 = w_6 = v_9$. Setting $U = \{v_1, \dots, v_9\}$, we have $(G[U], p_U) = (F_8, p_8)$ and $G^2[U] = F_8^2$. Hence Lemma 25 contradicts Lemma 13.

  By symmetry, we get a contradiction if $w_2 = w_6 = w_8$, $w_2 = w_4 = w_8$ or $w_4 = w_6 = w_8$.

- Suppose $w_4 = w_8 = v_9$, $w_2 \neq v_9$, $w_6 \neq v_9$ and $w_2 \neq w_6$. Setting $U = \{v_1, \dots, v_9\}$, we have $G[U] = F_9$, $p_U \geq p_9$ and $G^2[U] \subset F_9^2 \cup \{v_2 v_9\}$ or $G^2[U] \subset F_9^2 \cup \{v_6 v_9\}$. Hence Lemma 25 contradicts Lemma 13.

  By symmetry, we get a contradiction if $w_2 = w_6 = v_9$, $w_4 \neq v_9$, $w_8 \neq v_9$ and $w_4 \neq w_8$.

Therefore, $w_1 \neq w_7$.

Suppose that $w_2 = w_6 = v_8$. Let $T = S \cup \{v_8\}$. Then $G[T] = F_{10}$, $p_T \geq p_{10}$, and $G^2[T] \subset F_{10}^2 \cup \{v_4v_8\}$, since w_1, w_4 and w_7 are distinct vertices. Hence Lemma 25 contradicts Lemma 13.
Therefore, $w_2 \neq w_6$.

Suppose that $w_1 = w_6 = v_8$ and $w_2 = w_7 = v_9$. Let $U = S \cup \{v_8, v_9\}$. Then $G[U] = F_{11}$ and $G^2[U]$ is a subgraph of $F_{11}^2 \cup \{v_4v_8, v_8v_9, v_9v_4\}$. Moreover $p_U \geq p_{11}$ and, if $G^2[U] = F_{11}^2 \cup \{v_4v_8, v_8v_9, v_9v_4\}$, $p_U(v_i) = 5$ for $1 \leq i \leq 9$. . Hence Lemma 25 contradicts Lemma 13.

Therefore, $w_1 \neq w_6$ or $w_2 \neq w_7$. By symmetry, $w_2 \neq w_7$.

Suppose $w_1 = w_6 = v_8$. Let $T = S \cup \{v_8\}$ and let w_8 be the neighbour of v_8 not in S. Then $G[T] = F_{12}$, $p_T \geq p_{12}$ and $G^2[T] \subset F_{12}^2 \cup \{v_4v_8\}$. Hence Lemma 25 contradicts Lemma 13.
Therefore, $w_1 \neq w_6$.

Hence all the w_i are distinct, so $G[S]^2 = G^2[S]$. Thus Proposition 24 contradicts Lemma 13.

□

Remark 26 The proof of Lemma 8 in the case of planar graphs of girth at least 9 is simpler and shorter because all the configurations considered in the above proof (except the path P_7) have girth less than 9. Thus Corollary 6 has a short direct proof which requires only Proposition 24.

3.3 Proof of Lemma 11

Definition 27 For $1 \leq j \leq 4$, let I_j and q_j be the graphs and functions depicted in Figure 4.

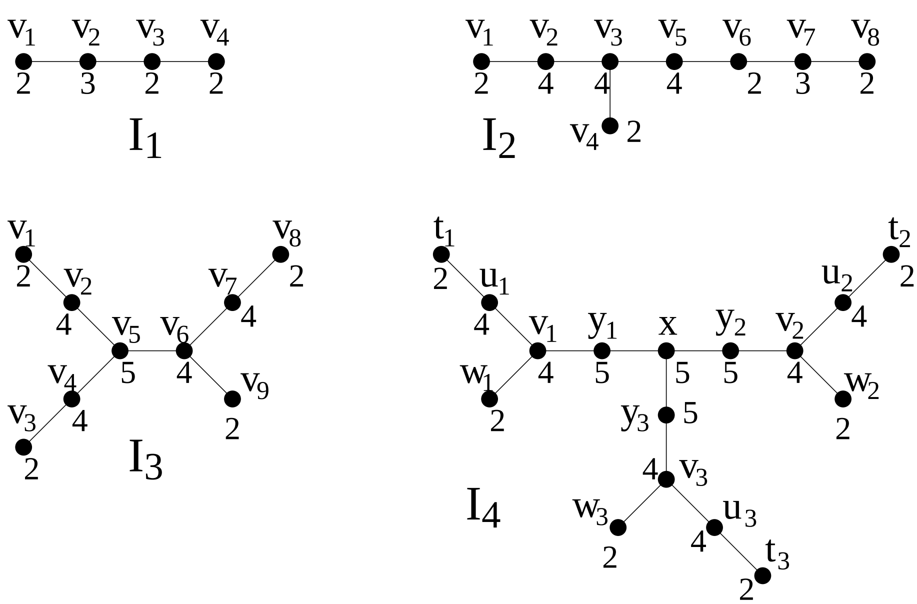

Figure 4: The graphs I_j and functions q_j, $1 \leq j \leq 4$

Lemma 28 *For $1 \leq j \leq 4$, I_j^2 is q_j-choosable.*

Proof.

- Let L be a q_1-list-assignment of I_1^2. The orderings $(v_4 < v_3 < v_1 < v_2)$ and $(v_1 < v_3 < v_4 < v_2)$ are q_1-nice. So, by Lemma 16, we may assume that $L(v_1) \cup L(v_4) \subset L(v_2)$. Hence $L(v_1) \cap L(v_4) \neq \emptyset$. But $(v_4 < v_1 < v_3 < v_2)$ is q_1-good. Thus, by Lemma 17, I_1^2 is L-colourable.

- Let L be a q_2-list-assignment of I_2^2.

  Suppose first that $L(v_3) \not\subset L(v_1) \cup L(v_6)$. Then choose $c(v_3)$ in $L(v_3) \setminus (L(v_1) \cup L(v_6))$ and $c(v_4) \in L(v_4) \setminus \{c(v_3)\}$. Since I_1^2 is q_1-choosable, one can extend c to $\{v_5, v_6, v_7, v_8\}$. Then one can find $c(v_2) \in L(v_2) \setminus \{c(v_3), c(v_4), c(v_5)\}$ and $c(v_1) \in L(v_1) \setminus \{c(v_2), c(v_3)\} = L(v_1) \setminus \{c(v_2)\}$. Hence we may assume that $L(v_3) \subset L(v_1) \cup L(v_6)$, so $L(v_3) = L(v_1) \cup L(v_6)$ and $L(v_1) \cap L(v_6) = \emptyset$.

  Suppose now that $L(v_4) \cap L(v_6) \neq \emptyset$. Then colour v_4 and v_6 with the same colour $c(v_4) = c(v_6) \in L(v_4) \cap L(v_6)$. Choose $c(v_8) \in L(v_8) \setminus \{c(v_6)\}$ and $c(v_7) \in L(v_7) \setminus \{c(v_6), c(v_8)\}$. Now since I_1^2 is q_1-choosable, one can extend c into an L-colouring of I_2^2. So we may assume that $L(v_4) \cap L(v_6) = \emptyset$. Now $(v_4 < v_1 < v_6 < v_8 < v_7 < v_5 < v_3 < v_2)$ is q_2-good so, by Lemma 17, we may assume that $L(v_4) \cap L(v_1) = \emptyset$. It follows that $L(v_4) \cap L(v_3) = \emptyset$ since $L(v_3) = L(v_1) \cup L(v_6)$.

  The ordering $(v_4 < v_8 < v_6 < v_7 < v_5 < v_3 < v_1 < v_2)$ is q_2-nice so, by Lemma 16, we may assume that $L(v_4) \subset L(v_2)$. Then one may assign $c(v_4) \in L(v_4)$ and $c(v_2) \in L(v_4) \setminus \{c(v_4)\}$ to the vertices v_4 and v_2. Now, because $L(v_4) \cap L(v_3) = \emptyset$, one can extend c into an L-colouring of I_2^2 by colouring greedily according to the ordering $(v_1 < v_8 < v_6 < v_7 < v_5 < v_3)$.

- Let L be a q_3-list-assignment of I_3^2. Assign to v_5 a colour c_5 in $L(v_5) \setminus (L(v_1) \cup L(v_9))$ and to v_6 a colour in $L(v_6) \setminus (L(v_8) \cup \{c_5\})$. Then colour the remaining vertices greedily according to $(v_3 < v_4 < v_2 < v_1 < v_9 < v_7 < v_8)$ to get an L-colouring of I_3^2.

- Let L be q_4-list-assignment of I_4^2. Pick $c(y_1)$ in $L(y_1) \setminus L(w_1)$, $c(y_2)$ in $L(y_2) \setminus (L(w_2) \cup \{c(y_1)\})$, $c(y_3)$ in $L(y_3) \setminus (L(w_3) \cup \{c(y_1), c(y_2)\})$ and $c(x)$ in $L(x) \setminus \{c(y_1), c(y_2), c(y_3)\}$. Since I_1^2 is q_1-choosable, one can extend c to a colouring of I_4^2.

□

Proof of Lemma 11.

(i) Suppose that a vertex v_3 of $Y_{2,2,3}$ and v_6 of $Y_{1,2,3} \cup Y_{2,2,3}$ are adjacent via a 2-edge in K_G. Then the subgraph of G induced by v_3, v_6 and the 2-vertices of their incident threads contains I_2 as an induced subgraph. (It is I_2 if v_6 is in $Y_{1,2,3}$ and has one extra vertex otherwise.) Since G has girth at least 13, then $G^2[V(I_2)] = I_2^2$, $(G - V(I_2))^2 = G^2 - V(I_2)$ and $p_{V[I_2]} = q_2$, so Lemma 28 contradicts Lemma 13.

(ii) Suppose that a vertex v_5 of $Y_{1,3,3}$ and v_6 of $Y_{1,2,3} \cup Y_{1,3,3}$ are adjacent via a 1-edge in K_G. Then the subgraph of G induced by v_5, v_6 and the 2-vertices of their incident threads contains I_3. So Lemma 28 contradicts Lemma 13.

(iii) Suppose that a vertex x of $Y_{2,2,2}$ is adjacent to three vertices v_1, v_2 and v_3 of $Y_{2,2,3}$ in K_G. Then the subgraph of G induced by x, v_1, v_2, v_3 and the 2-vertices of their incident threads is I_4. So Lemma 28 contradicts Lemma 13.

□

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