Complexity of $(p, 1)$-total labelling

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Abstract

A $(p, 1)$-*total labelling* of a graph $G = (V, E)$ is a total coloring L from $V \cup E$ into $\{0, \ldots, l\}$ such that $|L(v) - L(e)| \geq p$ whenever an edge e is incident to a vertex v. The minimum l for which G admits a $(p, 1)$-total labelling is denoted by $\lambda_p(G)$. The case $p = 1$ corresponds to the usual notion of total colouring, which is NP-hard to calculate even for cubic bipartite graphs [3]. We assume $p \geq 2$ in this paper. It is easy to show that $\lambda_p(G) \geq \Delta + p - 1$, where Δ is the maximum degree of G. Moreover, when G is bipartite, $\Delta + p$ is an upper bound for $\lambda_p(G)$, leaving only two possible values. In this paper, we completely settle the computational complexity of deciding whether $\lambda_p(G)$ is equal to $\Delta + p - 1$ or to $\Delta + p$ when G is bipartite. This is trivial when $\Delta \leq p$, polynomial when $\Delta = 3$ and $p = 2$, and NP-complete in the remaining cases.

Key words: Total labelling, total colouring, distance constrained colouring.
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1 Introduction

Let $G = (V, E)$ be a graph and p be a positive integer. A $(p, 1)$-*total labelling* of G is a mapping L from $V \cup E$ into $\{0, \ldots, l\}$, for some integer l, such that:

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- if x and y are adjacent vertices then $L(x) \neq L(y)$;
- if e and f are adjacent edges then $L(e) \neq L(f)$;
- if an edge e is incident to a vertex x then $|L(x) - L(e)| \geq p$.

A $(1,1)$-total labelling is the usual notion of *total colouring*. Clearly every graph admits a $(p,1)$-total labelling, if l is chosen large enough. The minimum l for which G has a $(p,1)$-total labelling into $\{0,\dots,l\}$ is denoted by $\lambda_p(G)$. The notion of $(p,1)$-total labelling has been introduced by Havet and Yu in [2], where are proved the following easy bounds (here χ stands for the chromatic number and χ' the chromatic index):

Proposition 1 (Havet and Yu, [2]) *Let $G=(V,E)$ be a graph with maximum degree $\Delta > 0$.*

(i) $\lambda_p(G) \geq \Delta + p - 1$.
(ii) If G is regular and $p \geq 2$ then $\lambda_p(G) \geq \Delta + p$.
(iii) If $p \geq \Delta$, then $\lambda_p(G) \geq \Delta + p$.

Proposition 2 (Havet and Yu, [2]) *Let $G=(V,E)$ be a graph with maximum degree $\Delta > 0$.*

(i) $\lambda_p(G) \leq \chi(G) + \chi'(G) + p - 2$.
(ii) $\lambda_p(G) \leq 2\Delta(G) + p - 1$.

In this paper, we are interested in the complexity of calculating $\lambda_p(G)$. In the case of total colouring, Sánchez-Arroyo [4] first proved that it is NP-hard to determine the total chromatic number of graphs. Furthermore, McDiarmid and Sánchez-Arroyo [3] showed that it is still NP-hard when restricted to k-regular bipartite graphs (if $k \geq 3$).

Here we study the problem when $p \geq 2$. Contrary to total colouring, determining the $(p,1)$-total labelling number of a regular bipartite graph is easy since it is always $\Delta + p$ by Propositions 1 and 2 (since $\chi(G) = 2$ and $\chi'(G) = \Delta(G)$ by König's theorem). Hence we will study the problem restricted either to the class of bipartite graphs. If G is bipartite, Propositions 1 and 2 yield $\lambda_p(G) \in \{\Delta(G)+p-1, \Delta(G)+p\}$. Hence we investigate the complexity of the following problem:

$\mathbf{\Delta}$-Bipartite $\mathbf{(p,1)}$-Total Labelling Problem:
INSTANCE: Bipartite graph G with maximum degree Δ.
QUESTION: Does $\lambda_p(G) = \Delta + p - 1$?

Note that Proposition 1 (iii) implies that this problem is trivial when $\Delta \leq p$ since it is always answered in the negative.

The aim of this paper is to prove the NP-completeness of the Δ-Bipartite $(p, 1)$-Total Labelling Problem for any $\Delta \geq p+1$ except if $\Delta = 3$ and $p = 2$ in which case we give a polynomial algorithm to solve it. We first give in Section 3 a polynomial algorithm that decides if $\lambda_2(G) = 4$ or $\lambda_2(G) = 5$ for a bipartite graph with maximum degree 3. This algorithm is based on induced matching in bipartite graphs. We also show that the same decision problem for graphs (non necessarily bipartite) with maximum degree 3 is NP-complete. In Section 4, we prove the NP-completeness of the Δ-Bipartite $(p, 1)$-Total Labelling Problem in all other cases. To achieve so, we need to distinguish three cases: $\Delta \geq 2p$ (Section 4.1), $2p - 1 \geq \Delta \geq p + 2$ (Section 4.2) and $\Delta = p + 1$ (Section 4.3).

2 Preliminaries

Let $G = (V, E)$ be a graph. The degree of a vertex v is denoted by $d_G(v)$ or simply $d(v)$, when G is clearly understood. A *path* is a non-empty graph P of the form

$$V(D) = \{v_0, v_1, \ldots, v_k\} \qquad E = \{v_0v_1, v_1v_2, \ldots, v_{k-1}v_k\},$$

where the v_i are all distinct. The vertices v_0 and v_k are called the *ends* of P. We often refer to a path by the natural sequence of its vertices, writing $P = v_0v_1 \ldots v_k$. For any pair of vertices x and y, an *xy-path* is a path with ends x and y.

Given two sets of vertices X and Y of G, the *distance* from X to Y denoted $dist(X, Y)$ is the length of a shortest xy-path with $x \in X$ and $y \in Y$. By extension, the distance between two edges uv and xy is defined by $dist(uv, xy) = dist(\{u, v\}, \{x, y\})$.

We will often make use of the following (easy) facts:

Proposition 3 *Let $p \geq 2$ and $k \geq p + 1$ be an integer. Let G be a graph admitting a $(p, 1)$-total labelling L into $\{0, \ldots, k + p - 1\}$.*

(i) If $d(v) = k$, then either $L(v) = 0$ and its incident edges are labelled by $\{p, \ldots, k + p - 1\}$ or $L(v) = k + p - 1$ and its incident edges are labelled by $\{0, \ldots, k - 1\}$.
(ii) If two vertices v and w of degree k are adjacent then $L(vw) \in \{p, \ldots, k-1\}$.
(iii) If $p \geq 3$ and $d(v) = k - 1$, then $L(v) \in \{0, 1, k + p - 2, k + p - 1\}$.

Proof:

(i) Suppose that $L(v) \notin \{0, k + p - 1\}$. Then $|\{L(v) - p + 1, \ldots, L(v) + p - 1\} \cap \{0, \ldots, k + p - 1\}| \geq p + 1$. Hence at most $k - 1$ labels are available to colour the edges adjacent to v. So $d(v) \leq k-1$.

(ii) It follows directly from (i).

(iii) Suppose that $L(v) \notin \{0, 1, k+p-2, k+p-1\}$. Then $|\{L(v)-p+1, \dots, L(v)+p-1\} \cap \{0, \dots, 2p\}| \geq p+2$. Hence at most $k-2$ labels are available to colour the edges adjacent to v. So $d(v) \leq k-2$. (Note that this inequality does not hold if $p = 2$ since $|\{L(v)-1, L(v), L(v)+1\}| = 3$.)

□

Observe that all properties of Proposition 3 do not hold if $p = 1$. The graph I is the graph drawn in Figure 1.

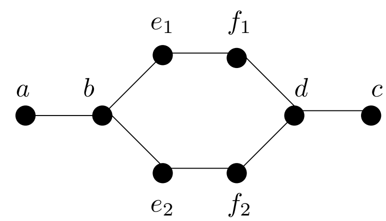

Fig. 1. The graph I

Proposition 4 *Let $p \geq 2$ be an integer. Let G be a graph admitting a $(p,1)$-total labelling L into $\{0, \dots, 2p\}$.*

(i) An edge labelled by p has its two endvertices labelled by 0 and $2p$.

(ii) Two edges labelled p are at distance at least two.

(iii) If two vertices x and y of degree $p+1$ have a common neighbour u and different labels, say $L(x) < L(y)$, then $L(x) = 0$, $L(xu) = 2p$, $L(u) = p$, $L(uy) = 0$ and $L(y) = 2p$.

(iv) If two vertices x and y of degree $p+1$ have two common neighbours then $L(x) = L(y)$.

(v) If three vertices x, y and z of degree $p+1$ have a common neighbour, then $L(x) = L(y) = L(z)$.

(vi) If $p \geq 3$ and I is a subgraph of G with $d_G(a) = d_G(b) = d_G(c) = d_G(d) = p+1$ and $d_G(f_1) = d_G(f_2) = p$ then $L(a) = L(c)$ and $L(b) = L(d)$.

Proof:

(i) Trivial.

(ii) Assume for contradiction that there are two edges xy and uv, both labelled p, at distance one (distance zero is impossible by definition of $(p,1)$-total labelling).Without loss of generality, we may assume that yu is an edge. Then y is labelled 0 and u is labelled $2p$. Thus the unique label allowed by its ends for the edge yu is p, which is a contradiction.

(iii) By Proposition 3 (i), $L(x) = 0$ and $L(y) = 2p$. Moreover the edge xu is labelled in $\{p, \dots, 2p\}$, so $L(u) \leq p$ and the edge uy is labelled in $\{0, \dots, p\}$, thus $L(u) \geq p$. Hence $L(u) = p$, so $L(xu) = 2p$ and $L(uy) = 0$.

(iv) It follows directly from (iii).

(v) It follows also easily from (iii).

(vi) Suppose for a contradiction that it is not true. By (i), the edges ab and cd are both labelled p and $L(a) = L(d)$ and $L(b) = L(c)$. Without loss of generality, we may assume that $L(b) = 0$ and $L(d) = 2p$. The vertex f_1 of I has degree p so by Proposition 3 (iii), $L(f_1) \in \{0, 1, 2p-1, 2p\}$. Moreover $L(d) = 2p$ and $L(f_1 d) \leq p - 1$, so $L(f_1) = 2p - 1$. Hence $L(e_1 f_1) \leq p - 1$ so $L(e_1) \geq p$. Now $L(be_1) \geq p$, so $L(e_1) \leq p$. Thus $L(e_1) = p$ and $L(be_1) = 2p$. Analogously $L(be_2) = 2p$ which is a contradiction.

□

3 The case $\mathbf{\Delta = 3}$ and $\mathbf{p = 2}$

3.1 A polynomial algorithm for bipartite graphs

Let G be a bipartite graph with maximum degree three. Our aim is to show a polynomial time algorithm which decides if $\lambda_2(G)$ is equal to 4 or 5.

An *induced matching* is a matching M of G such that any two distinct edges of M are at distance at least two. A *good matching* is an induced matching M such that every vertex of maximum degree is incident to an edge of M. Observe that from the definition of a good matching, an edge which is incident to two vertices of maximum degree is necessarily in every good matching. Conversely, an edge which is in a path of length 2 joining two vertices of maximum degree is never in a good matching.

Theorem 1 *Let G be a bipartite graph with maximum degree at most 3. The graph G has a good matching if and only if $\lambda_2(G) = 4$.*

Proof: If $\lambda_2(G) = 4$, we consider the set M of edges labelled 2 in a $(2,1)$-total labelling of G in $\{0, \ldots, 4\}$. Then by Proposition 3 (i) every vertex of degree 3 is incident to an edge of M and by Proposition 4 (ii), M is a good matching.

Suppose now that there is a good matching M in G. Let us find a $(2,1)$-total labelling L of G into $\{0, \ldots, 4\}$. Let (A, B) be the bipartition of G. Label the edges of M with 2 and the vertices adjacent to the edges of M with 0 if they are in A and 4 if they are in B.

Because every vertex of degree 3 is incident to an edge of M, the graph $G \setminus M$ has maximum degree 2. So it is the union of disjoint (even) cycles and paths. Let D be an orientation of $G \setminus M$ such that every cycle is a directed cycle and every path is a directed path (i.e. an orientation such that $|d^+(x) - d^-(x)| \leq 1$ for every vertex x). If a cycle or a path of $G \setminus M$ is not incident to any edge of M (and thus forms a connected component of G), we simply label its vertices by an alternating 0,1 sequence and its edges by an alternating 3,4 sequence. So we assume now that every component of D contains a vertex of $V(M)$. Let $\mathcal{P}$ be the set of maximal oriented paths of D whose internal vertices are not incident to an edge of M (such a path can have the same endvertices when it comes from a cycle of D which is incident to exactly one edge of M). Observe that every arc of D belongs to a unique path of $\mathcal{P}$.

We label the vertices and the arcs of each path $P = (x_0, x_1, \ldots, x_l)$ of $\mathcal{P}$ as follows:

- Suppose that x_0 and x_l are both incident to an edge of M. Then since M is a good matching, we have $l \geq 2$.
  - If l is even, then $L(x_0) = L(x_l)$.
    If $L(x_0) = 0$ then for $0 \leq i \leq l-1$, if i is even, set $L(x_i) = 0$ and $L(x_i x_{i+1}) = 3$, and, if i is odd, set $L(x_i) = 1$ and $L(x_i x_{i+1}) = 4$.
    If $L(x_0) = 4$ then for $0 \leq i \leq l-1$, if i is even, set $L(x_i) = 4$ and $L(x_i x_{i+1}) = 1$, and, if i is odd, set $L(x_i) = 3$ and $L(x_i x_{i+1}) = 0$.
  - If l is odd, then $L(x_0) \neq L(x_l)$.
    If $L(x_0) = 0$ then set $L(x_0 x_1) = 3$, $L(x_1) = 1$, $L(x_1 x_2) = 4$, $L(x_2) = 2$ and $L(x_2 x_3) = 0$. Furthermore, for $3 \leq i \leq l-1$, if i is odd, set $L(x_i) = 4$ and $L(x_i x_{i+1}) = 1$, and if i is even, set $L(x_i) = 3$ and $L(x_i x_{i+1}) = 0$.
    If $L(x_0) = 4$ then set $L(x_0 x_1) = 1$, $L(x_1) = 3$, $L(x_1 x_2) = 0$, $L(x_2) = 2$ and $L(x_2 x_3) = 4$. Moreover, for $3 \leq i \leq l-1$, if i is odd, set $L(x_i) = 0$ and $L(x_i x_{i+1}) = 3$, and, if i is even, set $L(x_i) = 1$ and $L(x_i x_{i+1}) = 4$.
- If x_0 is incident to an edge of M, and x_l is not, we suppose without loss of generality that $L(x_0) = 0$. We colour $L(x_i) = 0$ and $L(x_i x_{i+1}) = 3$, if i is even, and $L(x_i) = 1$ and $L(x_i x_{i+1}) = 4$ if i is odd.
  The case x_l incident to an edge of M is treated similarly.

To see that L is a $(2,1)$-total labelling of G, observe that a vertex $x \in V(M)$ is the origin of at most one path P of $\mathcal{P}$ and the end of at most one path Q of $\mathcal{P}$. In addition, the first edge of P is coloured 3 (resp. 1) and the last edge of Q is coloured 4 (resp. 0) if $L(x) = 0$ (resp. 4). □

Clearly, a graph has a good matching if and only if it has a *restricted good matching* that is a good matching such that each edge is adjacent to a vertex of maximal degree. From now on, by good matching, we understand restricted good matching.

Theorem 2 *The following problem is polynomially solvable:*
INSTANCE: Graph G with maximum degree 3.
QUESTION: Does G have a good matching?

Proof: Given a graph G with maximal degree at most 3, the following algorithm finds a good matching of G if it exists or answers "G has no good matching" otherwise.

For any edge e, we denote $B_2(e)$ the union of the set of edges and vertices at distance strictly less than two from e. If F is a set of edges, then $B_2(F) = \bigcup_{e \in F} B_2(e)$. Note that if e is an edge of a good matching M then $B_2(e) \cap M = \{e\}$.

Good Matching(G)
Step 0: Initialize H to G, S to the set of vertices of degree 3 and M to the set of edges with both endvertices in S.
Step 1: If M is an induced matching, then remove $B_2(M)$ from H and the endvertices of each edge of M from S. Otherwise return "G has no good matching".
Step 2: Remove the edges of every path of length 2 joining two vertices of S.
Step 3: Repeat until no vertex u of S satisfies one of the following cases:
Case 1: If u has degree 0 in H then return "G has no good matching".
Case 2: If u has a unique neighbour v or a neighbour v that has degree one in H, then add uv to M, remove $B_2(uv)$ from H and u from S.
Case 3: If there is a path uvw in H such that w is is not adjacent to some vertex of S then add uv to M, remove $B_2(uv)$ from H and u from S.
Step 4: Repeat until $S = \emptyset$: Pick a vertex u of S with minimum degree in H. Take a path $uvwx$ starting at u (observe that $x \in S$). Add uv to M, remove uvw from H and remove u from S.
Step 5: Return M.

Along the algorithm M is the set of edges that are selected to be in the desired good matching and S denotes the set of vertices that must be incident to an edge of a good matching and that are not yet incident to an edge of M. Finally, H is the subgraph of G where the remaining edges of the good matching can be.

At Step 0, S is initialized to the set of vertices of degree 3.

By Proposition 3 (ii), any good matching must contain the edges joining two vertices of degree 3. So M is initialized to this set. At Step 1, we check that M is an induced matching which is a necessary condition for G to have a good matching.

From Step 2, M is an induced matching. Indeed each time, we will add an edge e to M, we remove $B_2(e)$ from the graph G. Hence, all the edges of the remaining graph are at distance at least 2 from e in particular those edges that will be added to M after e. Therefore once S will be reduced to the emptyset, M will be a good matching.

At Step 2, we remove all the paths of length 2 between vertices of S since their edges are in no good matching.

Let us prove that at each iteration of the loop of Step 3, the following "correctness statement" holds : if there is a good matching M_1 then there is a good matching M_2 containing M.
Case 1: There is no more edges to be incident to u. Thus G has no good matching containing M, so by the correctness statement G has no good matching.
Case 2: Suppose that there is a good matching M_1 containing M. Let e_u be the edge incident to u. Let us prove that $M_2 = (M_1 - e_u) \cup \{uv\}$ is also a good matching. Let e be an edge of $M_2 \setminus \{uv\}$ that is the closest to uv and let P be a smallest path connecting e to uv in (the initial) G. If v is an endvertex of P, then the two first edges of P are not in H and thus not in M. If not $dist(e_u, e) \leq dist(uv, e)$. In both cases, $dist(uv, e) \geq 2$. Thus M_2 is an induced matching and then a good matching.
Analogously one can prove the correctness statement if we are in Case 3.

At the end of Step 3, H has a nice structure: a path joining to vertices of S with no interval vertices in S has length exactly 3, and each vertex of S is adjacent to at least one such path. In particular this implies that G has a good matching. Then Step 4 extends the matching M in a good matching. □

Theorems 1 and 2 immediatly imply:

Corollary 1 *The* 3*-Bipartite* $(2,1)$*-Total Labelling Problem is polynomially solvable.*

3.2 NP-completeness for general graphs

Theorem 3 *The following problem is NP-complete:*
INSTANCE: Graph G with maximum degree 3*.*
QUESTION: Is $\lambda_2(G) = 4$?

Proof: We reduce the problem to **Not-All-Equal 3-SAT Problem**. We need the following construction in order to emulate variables, clauses and negation.

Let $\mathcal{C} = \{C_1, \ldots, C_n\}$ be a collection of clauses over a set U of variables. We will construct a graph $G(\mathcal{C}, U)$. For every variable $u \in U$, create a *variable subgraph* P_u defined as follows:

$$V(P_u) = \bigcup_{i=1}^{n} \{a_i(u), b_i(u), s_i(u)\}$$

$$E(P_u) = \bigcup_{i=1}^{n} \{a_i(u)b_i(u), a_i(u)s_i(u), a_i s_{i-1}(u)\}$$

with $s_0(u) = s_n(u)$.

For every clause $C_i = x \vee y \vee z$, create a subgraph D_i defined as follows:

$$V(D_i) = \{a_i(x), b_i(x), a_i(y), b_i(y), a_i(z), b_i(z), c_i, d_i, t_i(x), t_i(y), t_i^1(z), t_i^2(z)\}$$

$$E(D_i) = \{a_i(x)b_i(x), b_i(x)t_i(x), a_i(y)b_i(y), b_i(y)t_i(y), a_i(z)b_i(z), b_i(z)t_i^1(z), b_i(z)t_i^2(z), c_i d_i, c_i t_i(x), c_i t_i(y), d_i t_i^1(z), d_i t_i^2(z)\}$$

If x is a non-negated literal u identify the vertices $a_i(u)$ and $b_i(u)$ of P_u with the vertices $a_i(x)$ and $b_i(x)$ of D_i.

If x is a negated literal $\bar{u}$ create two new vertices $q_i(u)$ and $r_i(u)$ and join them both to the vertices $b_i(u)$ of P_u and $a_i(x)$ of D_i.

Let us prove now that $G(\mathcal{C}, U)$ has a $(2,1)$-total labelling in $\{0, \dots, 4\}$ if and only if there is a truth assignment such that each clause in $\mathcal{C}$ has at least one true literal and at least one false literal.

Suppose first that there exists a $(2,1)$-total labelling L of $G(\mathcal{C}, U)$ in $\{0, \dots, 4\}$.

$A = \{a_i(u), b_i(u) \mid 1 \leq i \leq n, u \in U\} \cup \{a_i(x), b_i(x), a_i(y), b_i(y), a_i(z), b_i(z), c_i, d_i \mid C_i = x \vee y \vee z \text{ clause}\}$ is the set of vertices of degree 3 in $G(\mathcal{C}, U)$. By construction, every vertex of A has exactly one neighbour in A. Hence by Proposition 3, every vertex of A is labelled 0 or 4 and an edge with its two ends in A is labelled 2.

Let us show that for every $u \in U$, all the $a_i(u)$ are labelled the same (0 or 4). By Proposition 3 (i), for all $1 \leq i \leq n$, the $a_i(u)$ are labelled 0 or 4 and $L(a_i(u)b_i(u)) = 2$. Suppose they are not all labelled the same there exists $i_0 < i_1$ such that $L(a_{i_0}(u)) = 0 = L(a_{i_1+1}(u))$ and $L(a_i(u)) = 4$ if $i_0 < i \leq i_1$. Then by Proposition 4 (i), $L(a_{i_0(u)} s_{i_0}(u)) = 4$, $L(s_{i_0}(u)) = 2$ and $L(s_{i_0}(u)a_{i_0+1}(u)) = 0$, then $L(a_{i_0+1}(u)s_{i_0+1}(u))$ is necessary 1. So $L(s_{i_0+1}(u)) = 3$ and $L(s_{i_0+1}(u)a_{i_0+2}(u)) = 0$. And so on by induction, if $i_0 < i \leq i_1$, $L(s_{i-1}(u)a_i(u)) = 0$, $L(a_i(u)s_i(u)) = 1$. But by Proposition 3 (i), $L(a_{i_1}(u)s_{i_1}(u)) = 0$ which is a contradiction.

Hence we may define the truth assignment ϕ by $\phi(u) = \mathit{true}$ if $L(a_i(u)) = 2p$

and $\phi(u) = false$ if $L(a_i(u)) = 0$. Let us prove that each clause in $\mathcal{C}$ has at least one true literal and at least one false literal under ϕ.

Let $C_i = x \vee y \vee z$ be a clause. Let t be one of its literals. If t is a non-negated literal u, then $L(a_i(t)) = L(a_i(u))$ since $a_i(t) = a_i(u)$. If t is a negated literal $\bar{u}$ then, according to Proposition 4 (iv), $L(a_i(t)) = L(b_i(u)) \neq L(a_i(u))$ since $a_i(x)$ and $b_i(u)$ have two common neighbours $q_i(u)$ and $r_i(u)$. Hence to prove the result it suffices to prove that $L(a_i(x))$, $L(a_i(y))$ and $L(a_i(z))$ are not all equal.

Suppose (reductio ad absurdum) that they are all equal. Without loss of generality, we may suppose they are 0. Then since $a_i(x)b_i(x)$, $a_i(y)b_i(y)$ and $a_i(z)b_i(z)$ are edges labelled p, then $b_i(x)$, $b_i(y)$ and $b_i(z)$ are labelled 4. Now c_id_i is also labelled p and, because they have two common neighbours, d_i and $b_i(z)$ are labelled the same by Proposition 3 (iv). Thus d_i is labelled 4 and so c_i is labelled 0. Now c_i and $b_i(x)$ have a common neighbour $t_i(x)$ so $L(t_i(x)c_i) = 4$ according to Proposition 3 (iii). Analogously, $L(t_i(y)c_i) = 4$ which is a contradiction.

Let us now suppose that there is a truth assignment ϕ such that each clause in $\mathcal{C}$ has at least one true literal and at least one false literal. For every variable $u \in U$, we do the following

- if $\phi(u) = true$ then, for $1 \leq i \leq n$, set $L(a_i(u)) = 4$, $L(b_i(u)) = 0$, $L(a_i(u)b_i(u)) = 2$, and label the neighbours of $a_i(u)$ different from $b_i(u)$ with 3. We then label $a_i(u)s_i(u)$ with 0 and $a_i(u)s_{i-1}(u)$ with 1.
- if $\phi(u) = false$ then, for $1 \leq i \leq n$, set $L(a_i(u)) = 0$, $L(b_i(u)) = 4$, $L(a_i(u)b_i(u)) = 2$,$L(s_i(u)) = 1$, $L(a_i(u)s_i(u)) = 3$ and $L(a_i(u)s_{i-1}(u)) = 4$.

For every literal x of clause C_i, set $L(a_i(x)) = 4$, $L(b_i(x)) = 0$, $L(a_i(x)b_i(x)) = 2$ if $\phi(x) = true$ and set $L(a_i(x)) = 0$, $L(b_i(x)) = 4$, $L(a_i(x)b_i(x)) = 2$ if $\phi(x) = false$. Note that if x is a non-negated literal u then the vertices $a_i(x) = a_i(u)$, $b_i(x) = b_i(u)$ and the edge $a_i(x)b_i(x) = a_i(u)b_i(u)$ get the same label with the labelling of the clause and the labelling of the variable.

If x is the negated literal $\bar{u}$, then $a_i(x)$ and $b_i(u)$ are labelled the same. Hence set $L(q_i(u)) = L(r_i(u)) = 1$, $L(b_i(u)q_i(u)) = L(r_i(u)a_i(x)) = 3$ and $L(b_i(u)r_i(u)) = L(q_i(u)a_i(x)) = 4$ if they are labelled 0 and $L(q_i(u)) = L(r_i(u)) = 3$, $L(b_i(u)q_i(u)) = L(r_i(u)a_i(x)) = 1$ and $L(b_i(u)r_i(u)) = L(q_i(u)a_i(x)) = 0$ if they are labelled 4.

Let us now extend the labelling to each clause graph D_i. Since C_i has one true literal and one false literal then $\{b_i(x), b_i(y), b_i(z)\}$ has one vertex labelled 0 and one is labelled 4.

- If $L(b_i(x)) = L(b_i(y)) = 0$ and $L(b_i(z)) = 4$, set $L(c_i) = 0$, $L(d_i) = 4$, $L(c_i d_i) = 2$, $L(t_i(x)) = L(t_i(y)) = 1$ and $L(t_i^1(z)) = L(t_i^2(z)) = 3$.
- If $L(b_i(x)) = L(b_i(z)) = 0$ and $L(b_i(y)) = 4$, set $L(c_i) = 2p$, $L(d_i) = 0$, $L(c_i d_i) = 2$, $L(t_i(x)) = 2$ and $(t_i(y)) = 3$, $L(t_i^1(z)) = L(t_i^2(z)) = 1$, $L(ct_i(x) = 0$, $L(t_i(x)b_i(x)) = 4$.

In other cases, we proceed analogously, since x and y are equivalent and by symmetry of the labelling $l \to 2p - l$. □

4 NP-completeness of the bipartite $(p, 1)$-Total Labelling Problem

4.1 The case $\mathbf{\Delta \geq 2p}$

Theorem 4 *If $\Delta \geq 2p \geq 4$, the Δ-Bipartite $(p, 1)$-Total Labelling Problem is NP-complete.*

Proof: We reduce the problem to the following NP-complete problem [5] (L03 in the book of Garey and Johnson [1]):

Not-All-Equal $(\mathbf{p+1})$-SAT Problem:
INSTANCE: Set U of variables, collection $\mathcal{C}$ of clauses over U such that each clause $C \in \mathcal{C}$ has $p + 1$ literals.
QUESTION: Is there a truth assignment such that each clause in $\mathcal{C}$ has at least one true literal and at least one false literal?

Let $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$, we construct a graph $G(\mathcal{C}, U)$ as follows: For each variable u, create the *variable subgraph* $P(u)$ from the path $b_0(u)a_1(u)b_1(u)a_2(u)b_2(u) \ldots a_n(u)b_n(u)$ by blowing up each $a_i(u)$, $1 \leq i \leq n$ into a stable set $A_i(u)$ of cardinality p and each $b_i(u)$, $0 \leq i \leq n$, into a stable set $B_i(u)$ of cardinality $\left\lceil \frac{\Delta - p}{2} \right\rceil$ if i is odd and $\left\lfloor \frac{\Delta - p}{2} \right\rfloor$ if i is even.

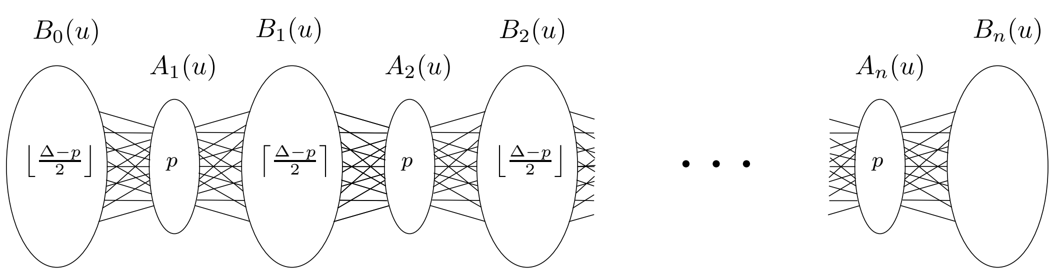

Fig. 2. The variable subgraph $P(u)$

Let C_i be a clause and $u \in U$ a variable. Let $s_i(u)$ be a vertex in $A_i(u)$. This vertex will correspond to the non-negated literal u in the clause C_i. Let us create a *negation subgraph* $N_i(u)$ containing a vertex $s_i(\bar{u})$ corresponding to the negated literals $\bar{u}$ in the clause C_i. The vertex set $V(N_i(u))$ is $A_i(u) \cup \{p_i(u), q_i(u), s_i(\bar{u})\} \cup R_i(u)$, with $R_i(u)$ a set of $\Delta - p$ new vertices and $E(N_i(u)) = \{ap_i(u) \mid a \in A_i(u)\} \cup \{rq_i(u) \mid r \in R_i(u)\} \cup \{rs_i(\bar{u}) \mid r \in R_i(u)\} \cup \{p_i(u)q_i(u)\}$ (see Fig. 3).

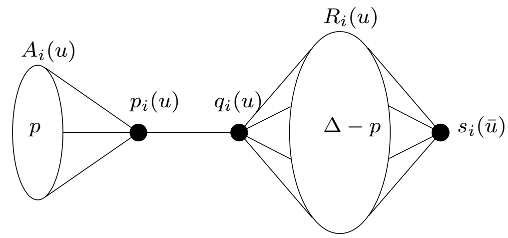

Fig. 3. The negation subgraph $N_i(u)$

For each clause C_i create a vertex v_i. Connect v_i to $s_i(l)$ for every literal l in C_i.

Finally, add as many as necessary extra vertices of degree 1 adjacent to the vertices of $S = \bigcup_{u \in U} V(P(u)) \cup \bigcup_{u \in U, 1 \leq i \leq n} [V(N_i(u) \setminus \{p_i(u)\}]$ in such a way that all these vertices get degree Δ.

By construction, $G(\mathcal{C}, U)$ is bipartite with maximum degree Δ. Let us prove that $\lambda_p^T(G(\mathcal{C}, U)) = \Delta + p - 1$ if and only if there is a truth assignment such that each clause in C has at least one true literal and at least one false literal.

If there is a truth assignment ϕ, we do the following for each variable u:

- Label the edges of $P(u)$ with labels of $\{p, \dots, \Delta - 1\}$. This is possible by König's theorem since $P(u)$ is bipartite of maximal degree $\Delta - p$.
- If $\phi(u) = true$ then label each $a \in A_i(u)$ with $\Delta + p - 1$ and each $b \in B_i(u)$ with 0. Otherwise label each $a \in A_i(u)$ with 0 and each $b \in B_i(u)$ with $\Delta + p - 1$.
- Label the edges of $\{rq_i(u) \mid r \in R_i(u)\} \cup \{rs_i(\bar{u}) \mid r \in R_i(u)\}$ with labels of $\{p, \dots, \Delta - 1\}$.
- If $\phi(u) = true$ then label each $r \in R_i(u)$ with $\Delta + p - 1$, and $q_i(u)$ and $s_i(\bar{u})$ with 0. Otherwise label each $r \in R_i(u) \cup \{r_i(u)\}$ with 0, and $q_i(u)$ and $s_i(\bar{u})$ with $\Delta + p - 1$.
- If $\phi(u) = true$ then label the edges of $\{ap_i(u) \mid a \in A_i(u)\}$ with $\{0, \dots, p - 1\}$, $p_i(u)q_i(u)$ with $\Delta + p - 1$ and $p_i(u)$ with $2p - 1$. Otherwise label the edges of $\{ap_i(u) \mid a \in A_i(u)\}$ with $\{\Delta, \dots, \Delta + p - 1\}$, $p_i(u)q_i(u)$ with 0 and $p_i(u)$ with $\Delta - p$. This is valid since $\Delta \geq 2p$.

Now each vertex v_i is adjacent to the $p+1$ vertices $s_i(l)$ for l literal of C_i. These vertices are labelled in $\{0, \Delta+p-1\}$ with at least one labelled 0 and at least one labelled $\Delta+p-1$. Let us denote by $t_1, t_2, \dots, t_j$ the neighbours of v_i labelled 0 and $t_{j+1}, \dots, t_{p+1}$ the neighbours of v_i labelled $\Delta+p-1$. For $1 \leq l \leq j$, label $v_i t_j$ with $\Delta+p-l$ and for $j+1 \leq l \leq p+1$, label $v_i t_j$ with $l-j+1$. Now label v_i with $2p-j$. This is possible because $\Delta \geq 2p$.

This labelling may trivially be extended to the extra vertices and their incident edges to get a $(p,1)$-total labelling of $G(\mathcal{C}, U)$.

Suppose now that there is a $(p,1)$-total labelling L of $G(\mathcal{C}, U)$ in $\{0, \dots, \Delta+p-1\}$. By Proposition 3 (i), for any $u \in U$, all the vertices in $\bigcup_{i=1}^{n} A_i(u)$ have the same label $L_u \in \{0, \Delta+p-1\}$ and all the vertices in $\bigcup_{i=0}^{n} B_i(u)$ are labelled with the integer $\bar{L}_u$ of $\{0, \Delta+p-1\} \setminus L_u$. Moreover the edges of $P(u)$ are labelled in $\{p, \dots, \Delta-1\}$ by Proposition 3 (ii). Now, since every vertex a of $a_i(u)$ has degree $\Delta-p$ in $P(u)$, each label of $\{p, \dots, \Delta-1\}$ is assigned to an edge incident to a in $P(u)$.

Let us show that $L(s_i(\bar{u})) = \bar{L}_u$. Without loss of generality, we may assume that $L_u = \Delta+p-1$.

Suppose for a contradiction that $L(s_i(\bar{u})) \neq 0$. By Proposition 3 (i), $L(s_i(\bar{u})) = \Delta+p-1$. Furthermore by Proposition 3 (ii), each vertex in $R_i(u)$ is labelled 0, $L(q_i(u)) = \Delta+p-1$, and the $\Delta-p$ edges of $\{q_i(u)r_i(u)\} \cup \{q_i(u)r \mid r \in R_i(u)\}$ are labelled with the $\Delta-p$ integers of $\{p, \dots, \Delta-1\}$. Hence the $p+1$ edges adjacent to $p_i(u)$ are labelled in $\{0, \dots, p-1\}$. This is a contradiction.

Let ϕ be the truth assignment defined by $\phi(u) = true$ if $L_u = \Delta+p-1$ and $\phi(u) = false$ if $L_u = 0$.

Let us prove that each clause in $\mathcal{C}$ has at least one true literal and at least one false literal. The vertex v_i is adjacent to $p+1$ vertices, namely the $s_i(l)$ for all the literal l of C_i. If C_i has all its literals true (resp. false) then all the neighbours of v_i are labelled 0 (resp. $\Delta+p-1$). Moreover they are incident to edges labelled $p, \dots, \Delta-1$ in $P(u)$ or $N_i(u)$. Hence the $p+1$ edges incident to v_i cannot be labelled since they are only p labels available, those of $\{0, \dots, p-1\}$ (resp. $\{\Delta, \dots, \Delta+p-1\}$). □

4.2 The case $\mathbf{p+2 \leq \Delta \leq 2p-1}$

Theorem 5 *If $2p-1 \geq \Delta \geq p+2 \geq 4$, the Δ-Bipartite $(p,1)$-Total Labelling Problem is NP-complete.*

Proof: We reduce the problem to **Not-All-Equal 3-SAT Problem**.

Let $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$, we construct a graph $G(\mathcal{C}, U)$ as follows: For each variable u, create the *variable subgraph* $P(u)$ from the path $b_0(u)s_1(u)b_1(u)s_2(u)b_2(u)\ldots s_n(u)b_n(u)$ by blowing up each $b_i(u)$ into a stable set $B_i(u)$ of cardinality $\left\lceil \frac{\Delta-p}{2} \right\rceil$ if i is odd and $\left\lfloor \frac{\Delta-p}{2} \right\rfloor$ if i is even. The vertex $s_i(u)$ will correspond to the non-negated literal u in C_i.

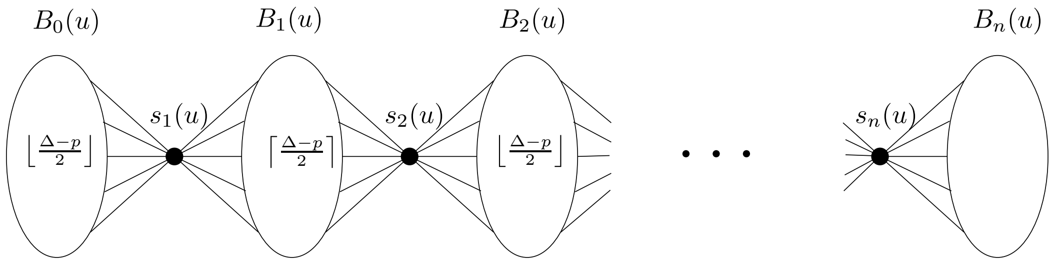

Fig. 4. The variable subgraph $P(u)$

Let us now create a *negation subgraph* $N_i(u)$ containing a vertex $s_i(\bar{u})$ corresponding to the negated literal $\bar{u}$ in the clause C_i. The vertex set $V(N_i(u))$ is $\{s_i(u), p_i^1(u), p_i^2(u), r_i(u), s_i(\bar{u})\} \cup Q_i(u) \cup T_i(u) \cup V_i(u)$ where $Q_i(u)$ is a set of $p-1$ vertices and $T_i(u)$ and $V_i(u)$ are two sets of $\Delta-p-1$ vertices. The edge set $E(N_i(u))$ is $\{s_i(u)p_i^1(u), s_i(u)p_i^2(u), r_i(u)s_i(\bar{u})\} \cup \{pq \mid p \in \{p_i^1(u), p_i^2(u)\}, q \in Q_i(u)\} \cup \{xr_i(u), \mid x \in Q_i(u) \cup T_i(u)\} \cup \{vs_i(\bar{u}) \mid v \in V_i(u)\}$.

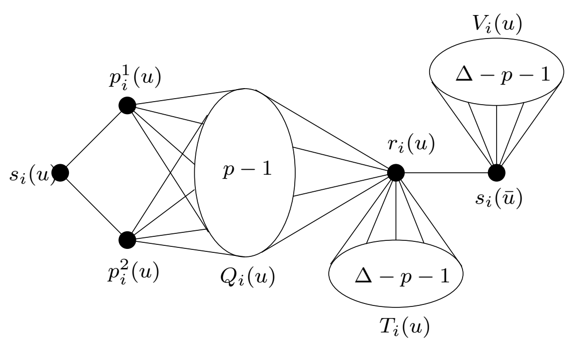

Fig. 5. The negation subgraph $N_i(u)$

Now for each clause $C_i = x \vee y \vee z$ create a *clause subgraph* $C(i)$ that connects the three vertices $s_i(x)$, $s_i(y)$ and $s_i(z)$. The vertex set $V(C(i))$ is $\{s_i(x), s_i(y), s_i(z), p_i^1, p_i^2, r_i, w_i\} \cup D_i^1 \cup D_i^2 \cup Q_i \cup T_i$ with D_i^1, D_i^2, Q_i and T_i four sets of cardinality respectively $\left\lceil \frac{\Delta-p}{2} \right\rceil$, $\left\lfloor \frac{\Delta-p}{2} \right\rfloor$, $p-1$ and $\Delta - p - 1$. The edge set $E(C(i))$ is $\{s_i(z)p_i^1, s_i(z)p_i^2, r_i w_i\} \cup \{pq \mid p \in \{p_i^1, p_i^2\}, q \in Q_i\} \cup \{xr_i, x \in Q_i \cup T_i\} \cup \{w_i d \mid d \in D_i^1 \cup D_i^2\} \cup \{s_i(x)d \mid d \in D_i^1\} \cup \{s_i(y)d \mid d \in D_i^2\}$.

Finally, add as many as necessary extra vertices of degree 1 adjacent to the vertices of $S = \bigcup_{u \in U} V(P(u)) \cup \bigcup_{1 \le i \le n} [\{r_i, w_i\} \cup T_i] \cup \bigcup_{u \in U, 1 \le i \le n} [\{r_i(u), s_i(\bar{u})\} \cup$

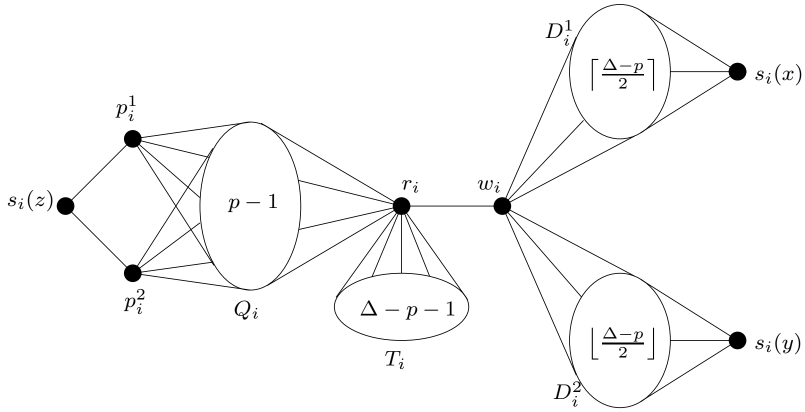

Fig. 6. The clause subgraph $C(i)$

$T_i(u) \cup V_i(u)]$ and $S' = \bigcup_{u\in U, 1\le i\le n} Q_i(u) \cup \bigcup_{1\le i\le n} Q_i \cup D_i^1 \cup D_i^2$ in such a way that the vertices of S get degree Δ and those of S' degree $\Delta-1$.

By construction, $G(\mathcal{C}, U)$ is bipartite with maximum degree Δ. Let us prove that $\lambda_p^T(G(\mathcal{C}, U)) = \Delta + p - 1$ if and only if there is a truth assignment such that each clause in C has at least one true literal and at least one false literal.

Suppose first that there exists such a truth assignment ϕ. Let us exhibit a $(p, 1)$-total labelling L of $G(\mathcal{C}, U)$ in $\{0, \dots, \Delta + p - 1\}$. Let u be a variable.

Suppose that $\phi(u) = true$. Then label the vertices and edges of $P(u)$ as follows:

- Label the edges of $P(u)$ with labels of $\{p, \dots, \Delta - 1\}$. This is possible by König's theorem since $P(u)$ is bipartite of maximal degree $\Delta - p$.
- For $1 \le i \le n$, $L(s_i(u)) = \Delta + p - 1$ and $L(b) = 0$ for any $b \in \bigcup_{0\le i\le n} B_i(u)$.

Furthermore, for every $1 \le i \le n$, label the vertices and edges of $N_i(u)$ as follows:

- $L(s_i(\bar{u})) = 0$; $L(r_i(u)) = \Delta + p - 1$; $L(v) = \Delta + p - 1$ for $v \in V_i(u)$; $L(t) = 0$ for $t \in T_i(u)$ $L(p_i^1(u)) = L(p_i^2(u)) = \Delta + p - 3$; $L(q) = \Delta + p - 2$ for $q \in Q_i(u)$.
- $L(r_i(u)s_i(\bar{u}) = p$; label the edges of $\{r_i(u)t \mid t \in T_i(u)\} \cup \{s_i(\bar{u})v \mid v \in V_i(u)\}$ with $\{p+1, \dots, \Delta - 1\}$; $L(s_i(u)p_i^1(u)) = 0$; $L(s_i(u)p_i^2(u)) = 1$; let $q_i(u)$ be vertex of $Q_i(u)$; label $p_i^1(u)q_i(u)$ with 1 and $p_i^2(u)q_i(u)$ with 0; label the edges of $\{pq \mid p \in \{p_i^1(u), p_2^1(u)\}, q \in Q_i(u) \setminus \{q_i(u)\}\} \cup \{r_i(u), q \mid q \in Q_i(u)\}$ with $\{2, \dots, \Delta - 3\}$. This is possible by König's theorem and valid since $\Delta \ge p + 2$.

If $\phi(u)$ is false, we label the vertices and the edges of $P(u)$ and the $N_i(u)$ in the symmetric way, that is a label l when $\phi(u)$ is true is replaced by a label $\Delta + p - 1 - l$ when $\phi(u)$ is false.

Let us now label the edges and vertices of each clause subgraph for each clause C_i. So far, the vertex $s_i(x)$ is label $\Delta + p - 1$ if the literal x is true and 0 if x is false. Hence, since C_i has one true and one false literal with ϕ, one vertex among $s_i(x_i)$, $s_i(y_i)$ and $s_i(z_i)$ is labelled $\Delta + p - 1$ and another 0.

Suppose first that $L(s_i(x_i)) = L(s_i(y_i)) = \Delta + p - 1$ and $L(s_i(z_i)) = 0$. Then label the vertices and edges of $C(i)$ as follows:

- The labelling of vertices and edges in $N(i)$ is labelled in the same way as $N_i(u)$ when $\phi(u) = false$. In such a way $L(w_i) = \Delta+p-1$ and $L(r_i w_i) = p$.
- label the vertices of $D_i^1 \cup D_i^2$ with $\Delta + p - 2$.
- Label the edges of $\{w_i d \mid d \in D_i^1 \cup D_i^2\} \cup \{s(x_i)d \mid d \in D_i^1\} \cup \{s(y_i)d \mid d \in D_i^2\}$ with labels in $\{0, \dots, p-1\}$. This is possible by König's theorem because $\Delta - p \leq p$.

Suppose now that $L(s_i(x_i)) = \Delta + p - 1$ and $L(s_i(y_i)) = L(s_i(z_i)) = 0$. Then label the vertices and edges of $C(i)$ as follows:

- The labelling of vertices and edges in $N(i)$ is labelled in the same way as $N_i(u)$ when $\phi(u) = false$. In such a way $L(w_i) = \Delta+p-1$ and $L(r_i w_i) = p$.
- label the vertices of D_i^1 with $\Delta + p - 2$ and those of D_i^1 with 1.
- Label the edges of $\{s(x_i)d \mid d \in D_i^1\} \cup \{w_i d \mid d \in D_i^1\}$ with labels in $\{0, \dots, p-1\}$. Label the edges of $\{w_i d \mid d \in D_i^2\}$ with labels in $\{\Delta, \dots, \Delta + p - 1\}$.
- Finally label the edges of $\{s(y_i)d \mid d \in D_i^2\}$ with labels in $\{p+1, \dots, \Delta-1\}$. This is possible by König's theorem because $\Delta - p - 1 \geq \frac{\Delta-p}{2}$.

All the other cases are obtained from these two by symmetry of the graph and labels. This labelling may trivially be extended to the extra vertices and their incident edges to get a $(p,1)$-total labelling of $G(\mathcal{C}, U)$.

Suppose now that there exists a $(p,1)$-total labelling L of $G(\mathcal{C}, U)$ in $\{0, \dots, \Delta + p - 1\}$.

By Proposition 3, for any $u \in U$, all the vertices $s_i(u)$, $1 \leq i \leq n$, have the same label $L_u \in \{0, \Delta + p - 1\}$ and all the vertices of $\bigcup_{i=0}^n B_i(u)$ are labelled with the integer $\bar{L}_u$ of $\{0, \Delta + p - 1\} \setminus L_u$. Moreover the edges of $P(u)$ are labelled in $\{p, \dots, \Delta - 1\}$. Since every vertex $s_i(u)$ has degree $\Delta - p$ in $P(u)$, each label of $\{p, \dots, \Delta - 1\}$ is assigned to an edge of $P(u)$ incident to $s_i(u)$.

Let us now show that $s_i(\bar{u})$ is assigned $\bar{L}_u$. Without loss of generality, we may assume that $L_u = \Delta + p - 1$.

Suppose for a contradiction that $L(s_i(\bar{u})) \neq 0$. By Proposition 3, $L(s_i(\bar{u})) = \Delta + p - 1$ and so $L(r_i(u)) = 0$ and $L(t) = \Delta + p - 1$ for any $t \in T_i(u)$. Furthermore, the $\Delta - p$ edges joining $r_i(u)$ to $T_i(u) \cup s_i(\bar{u})$ are labelled in $\{p, \ldots, \Delta-1\}$. So each integer of this set label one of those edges. It follows that the edges of $\{r_i(u), q \mid q \in Q_i(u)\}$ are labelled in $\{\Delta, \ldots, \Delta+p-1\}$. Now each vertex $q \in Q_i(u)$ is labelled in $\{0, 1, \Delta+p-1, \Delta+p-2\}$ by Proposition 3 (iii). So $L(q) = 1$ (0 is forbidden because of $r_i(u)$ and $\Delta+p-1$ and $\Delta+p-2$ by the edges $qr_i(u)$). It follows that the edges of $\{p_i^1(u)q \mid q \in Q_i(u)\}$ are labelled in $\Gamma = \{p+1, \ldots, \Delta+p-1\} \setminus \{L(p_i^1(u))-p+1, \ldots, L(p_i^1(u))+p-1\}$. Hence $L(p_i^1(u)) \leq p$ otherwise $|\Gamma| \leq \Delta-p-1 \leq p-2$ which is a contradiction. But $L(s_i(u)p_i^1(u)) \in \{0, \ldots, p-1\}$ because $s_i(u)$ is labelled $\Delta+p-1$ and adjacent to an edge labelled l in $P(u)$ for any $l \in \{p, \ldots, \Delta-1\}$. Thus $L(s_i(u)p_i^1(u)) = 0$ and $L(p_i^1(u)) = p$. Analogously, we have $L(s_i(u)p_i^2(u)) = 0$ which is a contradiction.

Hence $s_i(\bar{u})$ is labelled $\bar{L}_u$. Moreover, by Proposition 3, each label of $\{p, \ldots, \Delta-1\}$ is assigned to an edge of $\{vs_i(u) \mid v \in \{r_i(u)\} \cup V_i(u)\}$.

Let us define the truth assignment ϕ by $\phi(u) = true$ if $L_u = \Delta + p - 1$ and $\phi(u) = false$ if $L_u = 0$. Let us show that each clause C_i, $1 \leq i \leq n$, has at least one true literal and at least one false literal.
Suppose for a contradiction that the clause $C_i = x_i \vee y_i \vee z_i$ has all its literals *true*. Then $s_i(x_i) = s_i(y_i) = s_i(z_i) = \Delta + p - 1$. In the same way as we proved that $L(s_i(\bar{u}))$ is labelled $\bar{L}_u$, we can prove that $L(w_i) = 0$. Now each edge of $\{s_i(x_i)d \mid d \in D_i^1\}$ is labelled in $\{\Delta, \ldots, \Delta+p-1\}$ since $s_i(x_i)$ is adjacent to an edge labelled l for all $l \in \{p, \ldots, \Delta - 1\}$, either in $P(x_i)$ if x_i is a non-negated literal or in $N_i(u)$ if x_i is the negated literal $\bar{u}$. Moreover, by Proposition 3 (iii), every vertex of D_i^1 is labelled in $\{0, 1, \Delta+p-2, \Delta+p-1\}$. It follows that every vertex of D_i^1 is labelled $\Delta+p-2$. Analogously, we show that every vertex of D_i^2 is labelled $\Delta+p-2$. Hence the edges of $F = \{w_i d \mid d \in D_i^1 \cup D_i^2\}$ are assigned distinct labels in $\Gamma' = \{p, \ldots, \Delta - 2\}$. But $|F| = 2p - 2 > |\Gamma'| = \Delta - p - 1$ which is a contradiction. □

4.3 The case $\mathbf{\Delta = p + 1}$ *and* $\mathbf{p \geq 3}$

Theorem 6 *Let $p \geq 3$. The $(p + 1)$-Bipartite $(p, 1)$-Total Labelling Problem is NP-complete.*

Proof: We reduce the problem to **Not-All-Equal 3-SAT Problem**. We need the following construction in order to emulate variables, clauses and negation.

Let $\mathcal{C} = \{C_1, \ldots, C_n\}$ be a collection of clauses over a set U of variables. We will construct a graph $G(\mathcal{C}, U)$. For every variable $u \in U$, create a *variable subgraph* P_u defined as follows:

$$V(P_u) = \{a_i(u) \mid 1 \le i \le n\} \cup \{b_i(u) \mid 1 \le i \le n\} \cup \{s_j(u) \mid 1 \le j \le n/2\}$$
$$E(P_u) = \{a_i(u)b_i(u) \mid 1 \le i \le n\} \cup \{a_{2j-1}(u)s_j(u) \mid 1 \le j \le n/2\} \cup \{a_{2j}s_j(u) \mid 1 \le j \le n/2\} \cup \{a_{2j+1}(u)s_j(u) \mid 1 \le j \le n/2\}$$

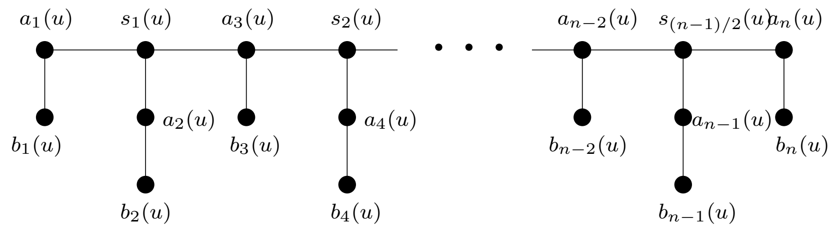

Fig. 7. The variable subgraph P_u

For every clause $C_i = x \vee y \vee z$, create a *clause subgraph* D_i defined as follows:

$$V(D_i) = \{a_i(x), b_i(x), a_i(y), b_i(y), a_i(z), b_i(z), c_i, d_i, t_i(x), t_i(y), v_i^1, v_i^2, w_i^1, w_i^2\}$$

$$E(D_i) = \{a_i(x)b_i(x), b_i(x)t_i(x), a_i(y)b_i(y), b_i(y)t_i(y), a_i(z)b_i(z), b_i(z)v_i^1, b_i(z)v_i^2, c_id_i, c_it_i(x), c_it_i(y), d_iw_i^1, d_iw_i^2, v_i^1w_i^1, v_i^2w_i^2\}$$

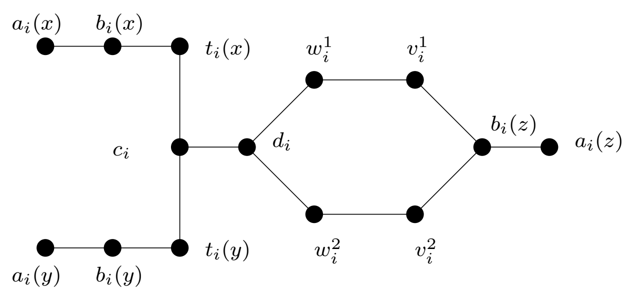

Fig. 8. The clause subgraph D_i

If x is a non-negated literal u identify the vertices $a_i(u)$ and $b_i(u)$ of P_u with the vertices $a_i(x)$ and $b_i(x)$ of D_i.

If x is a negated literal $\bar{u}$ create four new vertices $q_i^1(u)$, $q_i^2(u)$, $r_i^1(u)$, and $r_i^2(u)$ and add the edges $b_i(u)q_i^1(u)$, $b_i(u)q_i^2(u)$, $q_i^1(u)r_i^1(u)$, $q_i^2(u)r_i^2(u)$, $r_i^1(u)a_i(x)$ and $r_i^2(u)a_i(x)$.

Finally, add as many as necessary vertices of degree 1 adjacent to the vertices of $A = \{a_i(u), b_i(u), \mid 1 \le i \le n, u \in U\} \cup \{a_i(x), b_i(x), a_i(y), b_i(y), a_i(z), b_i(z), c_i, d_i, \mid C_i = x \vee y \vee z \text{ clause}\}$ so that they have degree $p+1$ and to the vertices of $B = \{w_i^1, w_i^2, \mid C_i \text{ clause}\} \cup \{r_i^1(u), r_i^2(u) \mid \bar{u} \text{ a literal of } C_i\}$ so that they have degree p. This is possible since $p \ge 3$.

It is simple matter to check that $G(\mathcal{C}, U)$ is bipartite. One set of the partition contains the a_i, d_i, t_i, v_i, and q_i, and the other the b_i, s_i, c_i, w_i and r_i.

Let us prove now that $G(\mathcal{C}, U)$ has a $(p, 1)$-total labelling in $\{0, \ldots, 2p\}$ if and only if there is a truth assignment such that each clause in $\mathcal{C}$ has at least one true literal and at least one false literal.

Suppose first that there exists a $(p, 1)$-total labelling L of $G(\mathcal{C}, U)$ in $\{0, \ldots, 2p\}$.

By construction, every vertex of A has exactly one neighbour in A. Hence by Proposition 3, every vertex of A is labelled 0 or $2p$ and an edge with its two ends in A is labelled p. Furthermore, by Proposition 4 (v), for any variable u, the vertices $a_i(u)$, $1 \le i \le n$, are labelled the same (either 0 or $2p$) since the vertices $a_{2j-1}(u)$, $a_{2j}(u)$ and $a_{2j+1}(u)$ have s_j as common neighbour. Hence we may define the truth assignment ϕ by $\phi(u) = true$ if $L(a_i(u)) = 2p$ and $\phi(u) = false$ if $L(a_i(u)) = 0$. Let us prove that each clause in $\mathcal{C}$ has at least one true literal and at least one false literal under ϕ.

Let $C_i = x \vee y \vee z$ be a clause. Let t be one of its literals. If t is a non-negated literal u, then $L(a_i(t)) = L(a_i(u))$ since $a_i(t) = a_i(u)$. If t is a negated literal $\bar{u}$ then, according to Proposition 4 (vi), $L(a_i(t)) = L(b_i(u)) \neq L(a_i(u))$. Hence to prove the result it suffices to prove that $L(a_i(x))$, $L(a_i(y))$ and $L(a_i(z))$ are not all equal.

Suppose (reductio ad absurdum) that they are all equal. Without loss of generality, we may suppose they are 0. Then since $a_i(x)b_i(x)$, $a_i(y)b_i(y)$ and $a_i(z)b_i(z)$ are edges labelled p, then $b_i(x)$, $b_i(y)$ and $b_i(z)$ are labelled $2p$. Now $c_i d_i$ is also labelled p. By Proposition 4 (vi), d_i and $b_i(z)$ are labelled the same. Thus d_i is labelled $2p$ and so c_i is labelled 0. Now c_i and $b_i(x)$ have a common neighbour $t_i(x)$ so $L(t_i(x)c_i) = 2p$ according to Proposition 3 (iii). Analogously, $L(t_i(y)c_i) = 2p$ which is a contradiction.

Let us now suppose that there is a truth assignment ϕ such that each clause in $\mathcal{C}$ has at least one true literal and at least one false literal. For every variable $u \in U$, we do the following

- if $\phi(u) = true$ then, for $1 \le i \le n$, set $L(a_i(u)) = 2p$, $L(b_i(u)) = 0$, $L(a_i(u)b_i(u)) = p$, and for $1 \le i \le n$, $L(s_j(u)) = 2p - 1$, $L(a_{2j-1}s_j(u)) = 0$, $L(a_{2j}s_j(u)) = 1$ and $L(a_{2j+1}s_j(u)) = 2$.
- if $\phi(u) = false$ then, for $1 \le i \le n$, set $L(a_i(u)) = 0$, $L(b_i(u)) = 2p$, $L(a_i(u)b_i(u)) = p$, andand for $1 \le i \le n$, $L(s_j(u)) = 1$, $L(a_{2j-1}s_j(u)) = 2p$, $L(a_{2j}s_j(u)) = 2p - 1$ and $L(a_{2j+1}s_j(u)) = 2p - 2$.

For every literal x of clause C_i, set $L(a_i(x)) = 2p$, $L(b_i(x)) = 0$, $L(a_i(x)b_i(x)) =$

p if $\phi(x) = true$ and set $L(a_i(x)) = 0$, $L(b_i(x)) = 2p$, $L(a_i(x)b_i(x)) = p$ if $\phi(x) = false$. Note that if x is a non-negated literal u then the vertices $a_i(x) = a_i(u)$, $b_i(x) = b_i(u)$ and the edge $a_i(x)b_i(x) = a_i(u)b_i(u)$ get the same label with the labelling of the clause and the labelling of the variable.

If x is the negated literal $\bar{u}$, then $a_i(x)$ and $b_i(u)$ are labelled the same. Hence if they are labelled 0, set $L(q_i^1(u)) = L(q_i^2(u)) = 2$, $L(r_i^1(u)) = L(r_i^2(u)) = 1$, $L(b_i(u)q_i^1(u)) = L(r_i^1(u)a_i(x)) = L(q_i^2(u)r_i^2(u)) = 2p$ and $L(b_i(u)q_i^2(u)) = L(r_i^2(u)a_i(x)) = L(q_i^1(u)r_i^1(u)) = 2p-1$, and if they are labelled $2p$, set $L(q_i^1(u)) = L(q_i^2(u)) = 2p-2$, $L(r_i^1(u)) = L(r_i^2(u)) = 2p-1$, $L(b_i(u)q_i^1(u)) = L(r_i^1(u)a_i(x)) = L(q_i^2(u)r_i^2(u)) = 0$ and $L(b_i(u)q_i^2(u)) = L(r_i^2(u)a_i(x)) = L(q_i^1(u)r_i^1(u)) = 1$.

Let us now extend the labelling to the clause graph D_i. Since C_i has one true literal and one false literal then $\{b_i(x), b_i(y), b_i(z)\}$ has one vertex labelled 0 and one is labelled $2p$.

- If $L(b_i(x)) = L(b_i(y)) = 0$ and $L(b_i(z)) = 2p$, set $L(c_i) = 0$, $L(d_i) = 2p$, $L(c_id_i) = p$, $L(t_i(x)) = L(t_i(y)) = 1$, $L(v_i^1) = L(v_i^2) = 2p-2$, $L(w_i^1) = L(w_i^2) = 2p-1$, $L(b_i(x)t_i(x)) = L(t_i(y)c_i) = 2p$, $L(b_i(y)t_i(y)) = L(t_i(x)c_i) = 2p-1$, $L(b_i(z)v_i^1) = L(w_i^1d_i) = L(v_i^2w_i^2) = 0$ and $L(b_i(z)v_i^2) = L(w_i^2d_i) = L(v_i^1w_i^1) = 1$.
- If $L(b_i(x)) = L(b_i(z)) = 0$ and $L(b_i(y)) = 2p$, set $L(c_i) = 2p$, $L(d_i) = 0$, $L(c_id_i) = p$, $L(t_i(x)) = p$, $L(t_i(y)) = 2p-1$, $L(v_i^1) = L(v_i^2) = 2$, $L(w_i^1) = L(w_i^2) = 1$, $L(c_it_i(x) = 0$, $L(t_i(x)b_i(x) = 2p$, $L(c_it_i(y) = 1$, $L(t_i(y)b_i(y) = 0$, $L(b_i(z)v_i^1) = L(w_i^1d_i) = L(v_i^2w_i^2) = 2p$ and $L(b_i(z)v_i^2) = L(w_i^2d_i) = L(v_i^1w_i^1) = 2p-1$.

In other cases, we proceed analogously, since x and y are equivalent and by symmetry of the labelling $l \to 2p - l$.

Trivially, this labelling may be extended to the degree 1 vertices (added to ensure that elements of A and B have degree $p+1$ and p) and their incident edges. □

5 Conclusion

In this paper, we completely characterize the complexity of finding the $(p,1)$-total labelling number when the graph is bipartite. It would be interesting to do the same for k-regular graphs

k-Regular $(\mathbf{p}, \mathbf{1})$-Total Labelling Problem:
INSTANCE: k-regular graph G.

QUESTION: What is $\lambda_p(G)$?

When $p = 1$ McDiarmid and Sanchez-Arroyo [3] showed it to be NP-hard if $k \geq 3$ and polynomial otherwise.

When $p \geq 2$, remains unclear even if we expect some dichotomy NP-hard/polynomial.

Havet and Yu [2] showed that every 2-regular graph has $(2,1)$-total labelling number 4. Moreover, they show that for $p \geq 3$, the $(p,1)$-total number of a 2-regular graph is $p+3$ if and only one of its component is an odd cycle and $p+2$ otherwise. So for any p, one can polynomially find the $(p,1)$-total number of a 2-regular graph.

If G is a connected 3-regular graph, by Proposition 1 (ii), $\lambda_2(G) \geq 5$. Moreover, Havet and Yu [2] conjecture that $\lambda_2(G) = 5$ unless $G = K_4$. This would trivially imply that the 3-Regular (2,1)-Total Labelling Problem is polynmial time solvable.

Moreover one can determine polynomially the $(3,1)$-total number of a 3-regular graph. Indeed if G is 3-regular then $\lambda_3(G) \geq 6$, by Proposition 1 (ii), $\lambda_3(G) \leq 7$ as proved by Havet and Yu [2], and $\lambda_3(G) = 6$ if and only if G is bipartite below.

Theorem 7 *Let $p \geq k \geq 3$ be integers. Let G be a k-regular graph. Then $\lambda_p(G) = p + k$ if and only if G is bipartite.*

Proof: If G is bipartite, then by Proposition 2 (i), $\lambda_p(G) \leq p + k$.

Suppose now that G has a $(p,1)$-total labelling L of G in $\{0, \ldots, p+k\}$. Then one can easily see that every vertex msut be coloured in $\{0, 1, p+k-1, p+k\}$. Let A, (resp. B) be the set of vertices of H labelled with 0 or $p+k-1$, (resp. 1 or $p+k-1$). Then A and B are stable sets since the endvertices of an edge may not be labelled with 0 and $p+k-1$ or $p+k$ and 1. So (A, B) is a bipartition of G. $\square$

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