Finding a five bicolouring of a triangle-free subgraph of the triangular lattice.

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July 8, 1999

Abstract

A basic problem in the design of mobile telephone networks is to assign sets of radio frequency bands (colours) to transmitters (vertices) to avoid interference. Often the transmitters are laid out like vertices of a triangular lattice in the plane. We investigate the corresponding colouring problem of assigning sets of colours of size $p(v)$ to each vertex of the triangular lattice so that the sets of colours assigned to adjacent vertices are disjoint. A n-$[p]$*colouring* of a graph G is a mapping c from $V(G)$ into the set of the subsets of $\{1, 2, \ldots, n\}$ such that $|c(v)| = p(v)$ and for any adjacent vertices u and v, $c(u) \cap c(v) = \emptyset$. We give here an alternative proof of the fact that every triangular-free induced subgraph of the triangular lattice is 5-[2]colourable. This proof yields a constant time distributed algorithm that finds a 5-[2]colouring of such a graph. We then give a distributed algorithm that finds a $[p]$colouring of a triangle-free induced subgraph of the triangular lattice with at most $\frac{5\omega_p(G)}{4} + 3$ colours.

1 Introduction.

A basic problem in the design of mobile telephone networks is to assign sets of radio frequency bands (colours) to transmitters (vertices) to avoid interference. The number $p(v)$ of bands demanded at transmitter v may vary between transmitters. We assume that the transmitters are located like vertices in a triangular lattice in the plane: this pattern is often used as it gives a good coverage. We assume also that adjacent vertices must not be assigned the same band, so as to avoid interference. There are more refi ned versions of this 'channel assignment problem', see for example [3, 4], in which we insist on a minimum separation between channels assigned to two transmitters (where this minimum separation depends on the proximity of the transmitters). But we consider only the most basic case here.

The channel assignment problem described above is a 'weighted colouring' problem on the triangular lattice. A *weight function* on a graph G is a function from $V(G)$, the set of vertices of G, into the set of non-negative integers. Let us denote the set $\{1, 2, \ldots, n\}$ by $[1, n]$ and let p be a weight function on a graph G. A n-$[p]$*colouring* of G is a mapping c from $V(G)$ into the set of the subsets of $[1, n]$ such that $|c(v)| = p(v)$ and for any adjacent vertices u and v, $c(u) \cap c(v) = \emptyset$. The graph G is n-$[p]$*colourable* if it admits an n-$[p]$colouring. The $[p]$*chromatic number* of a graph G, denoted by $\chi_p(G)$, is the smallest integer n such that G is n-$[p]$colourable. If $\chi_p(G) = n$, we say that G is n-$[p]$*chromatic*. Let k be a positive integer. For convenience, we denote by k the weight function such that $k(v) = k$ for each vertex v of G. The [1]colouring is the usual colouring: to each vertex, we associate a colour in such a way that two adjacent vertices become different colours. In this paper, we call [2]colourings *bicolourings*, and [3]colourings *tricolourings*.

There is a natural graph G_p associated with a pair (G, p) as above, obtained by blowing up each vertex v by a clique on $p(v)$ vertices. Weighted $[p]$colourings of G correspond to usual colourings of the graph G_p, so $\chi_p(G) = \chi(G_p)$. More generally, n-$[pq]$colouring of the graph G is equivalent to n-$[p]$colouring of the graph G_q. So $\chi_{pq}(G) = \chi_p(G_q)$. Moreover $\chi_{p+q}(G) \leq \chi_p(G) + \chi_q(G)$ $(\star)$.

The $[p]$*weighted clique number* of G, denoted by ω_p, is $\max\{\sum_{v \in C} p(v), C \text{ clique of } G\}$. Clearly, $\omega_p(G) = \omega(G_p)$. Thus $\chi_p \geq \omega_p$.

We are interested in the $[p]$chromaticity of an induced subgraph of the triangular lattice, as this corresponds precisely to the basic channel assignment problem described above. This lattice graph may be described as follows. The vertices are all integer linear combinations $a\mathbf{p} + b\mathbf{q}$ of the two vectors $\mathbf{p} = (1, 0)$ and $\mathbf{q} = (\frac{1}{2}, \frac{\sqrt{3}}{2})$: thus we may identify the vertices with the pairs (a, b) of integers. Two vertices are adjacent when the Euclidean distance between them is 1. Thus each vertex $x = (a, b)$ has the six neighbours: its *left neighbour* $(a - 1, b)$, its *right neighbour* $(a + 1, b)$, its *leftup neighbour* $(a - 1, b + 1)$, its *rightup neighbour* $(a, b + 1)$, its *leftdown neighbour* $(a, b - 1)$ and its *rightdown neighbour* $(a + 1, b - 1)$.

There is an obvious 3-colouring of the infinite triangular lattice which gives rise to the partition of the vertex set of any triangular lattice graph into three independent sets, *Red*, *Blue* and *Green* such that if x is in Red (resp. Blue or Green) then its right neighbour is in Blue (resp. Green or Red). According to this partition each vertex is called *red*, *blue* or *green*.

McDiarmid and Reed [5] proved that for every induced subgraph of the triangular lattice and every weight function p:

$$\chi_p \leq \frac{4\omega_p + 1}{3}$$

A distributed algorithm which guarantees the $4\omega_p/3$ bound is reported by Narayanan and Schende [7]. McDiarmid and Reed conjectured that this ratio $4/3$ can be improved:

Conjecture 1 (McDiarmid and Reed) *There is a constant c such that for every induced subgraph of the triangular lattice and any weight function p*

$$\chi_p(G) \leq \frac{9\omega_p(G)}{8} + c.$$

This ratio $9/8$ is best possible: the cycle on 9 vertices, C_9 is an induced subgraph of the triangular lattice and $\chi_k(C_9) = \lceil 9k/4 \rceil = \lceil 9\omega_k/8 \rceil$.

We first investigate this conjecture when the weight function is a constant k. The triangular lattice is $3k$-$[k]$chromatic. Therefore, any of its subgraphs is $3k$-$[k]$colourable. If it contains a triangle, then it is $3k$-$[k]$chromatic and $\omega_k = 3k$. So $\chi_k = \omega_k$.

So we just need to study the $[k]$chromaticity of triangle-free induced subgraphs of the triangular lattice. Let $f(k)$ be the maximum $[k]$-chromatic number of such a graph. The above conjecture of Reed and McDiarmid restricted to [k]colouring is the following:

Conjecture 2 (McDiarmid, Reed) *There is a constant c such that $f(k) = \frac{9k}{4} + c$.*

Havet [2] proved that every triangle-free induced subgraph of the triangular lattice is 7-tricolourable. This implies that $f(k) \leq \lceil \frac{7k}{3} \rceil$. But his proof did not yield any distributed algorithm to find such a $[k]$colouring. In the next section, we expose an alternative proof of Theorem 1 of [2] that states that every triangle-free induced subgraph of the triangular lattice is 5-bicolourable. This proof gives a distributed algorithm which finds a 5-bicolouring (and then a $\lceil \frac{5k}{2} \rceil - [k]$colouring) of such a graph.

In the third section, using this algorithm, we exhibit a distributed algorithm which finds a $(\frac{5\omega_p}{4} + 3)$-[p]colouring of a triangle-free induced subgraphs of the triangular lattice.

2 5-bicolouring of the triangle-free induced subgraphs of the triangular lattice.

Lemma 1 (Havet, [2]) *Let $P = (x_0, x_1, x_2, \dots, x_m)$ be a path of length m and c_0 and c_m two 2-subsets of $[1,5]$. There exists a 5-bicolouring of P such that $c(x_0) = c_0$ and $c(x_m) = c_m$ if and only if:*

$m = 1$ and $d(c_0, c_1) = 2$, $m = 2$ and $d(c_0, c_2) \leq 1$, $m = 3$ and $d(c_0, c_2) \geq 1$, or $m \geq 4$

For the sake of completeness we give a sketch of proof here. Note that the graph with vertices being 2-subsets of the set $[1,5]$ in which two sets are adjacent if and only if they have empty intersection is isomorphic to the famous Petersen graph. As it is well-known that in the Petersen graph any pair of vertices lies on a 5-cycle, the assertion of the above lemma follows.

Theorem 1 *Let G be a triangle-free induced subgraph of the triangular lattice. Then G is 5-bicolorable.*

Proof. A vertex of G is said to be *suitable* if its left, rightup and rightdown neighbours in the lattice do not belong to G. It is easy to see that two suitable vertices are not adjacent in G. Let S be the set of suitable vertices of G. Let L be the bicolouring of the vertices of S defined as follows:

if v is a red vertex, $L(v) := \{1, 2\}$,

if v is a blue vertex, $L(v) := \{2,3\}$,
if v is a green vertex, $L(v) := \{1,5\}$.

We will prove that this bicolouring of S may be extended to a 5-bicolouring of G.

A path $(x_1, x_2, \ldots, x_n)$ is *left* (resp. *rightup, rightdown*) if x_{i+1} is the left (resp. rightup, rightdown) neighbour of x_i in the triangular lattice. A *tristar* is the union of one left, one rightup and one rightdown path emerging from a common origin.

Let G' be the graph induced by G on the vertices of $V(G) \setminus S$. It is easy to see that the components of G' are tristars and only the terminus of the left (resp. rightup, rightdown) path may have a left (resp. rightup, rightdown) neighbour in S.

Let T be a tristar of G'. Let $(x, u_1, u_2, \ldots, u_k)$ be the left path of T, $(x, v_1, v_2, \ldots, v_l)$ the rightup path of T and the $(x, w_1, w_2, \ldots, w_m)$ rightdown path of T. And let u be the left neighbour of u_k, v the rightup neighbour of v_l and w the rightdown neighbour of w_m. Let G_T be the subgraph induced by G on $V(T) \cup \{u, v, w\}$. We shall prove that there exists a 5-bicolouring c of G_T that coincides with L on $\{u, v, w\}$.

Suppose first that two paths of the tristar T have the same length. By symmetry, we may suppose that $l = m$. Then v and w have the same label (red, blue or green) and so $L(v) = L(w)$. If $k + l \geq 2$, then in G_T, the path $P = (u, u_k, u_{k-1}, \ldots u_1, x, v_1, \ldots, v_l, v)$ has length at least four. So, by Lemma 1, there exists a 5-bicolouring c of P such that $c(u) = L(u)$ and $c(v) = L(v)$. Thus setting $c(w_i) = c(v_i)$, the 5-bicolouring c of G_T coincides with L on $\{u, v, w\}$. If $k + l = 1$, then in G_T, $P = (u, \ldots, x, \ldots, v)$ is a path of length three. Furthermore, u and v have different labels, and hence $L(u) \neq L(v)$. Thus, by Lemma 1, there exists a 5-bicolouring c of P such that $c(u) = L(u)$ and $c(v) = L(v)$. Setting $c(w_i) = c(v_i)$, the 5-bicolouring c of G_T coincides with L on $\{u, v, w\}$. If $k + l = 0$, then $k = l = m = 0$. So T is a single vertex x with three (possible) neighbours in G, u, v and w. Now, u, v and w have the same label so $L(u) = L(v) = L(w)$. Setting $c(x)$ be a 2-subset of $[1,5]$ disjoint from $L(u)$, we have the result.

Suppose now that the three paths of T have distinct lengths. By symmetry, we may suppose that $k > l > m$. We will prove the result by induction on k.

If $k = 2$ then $l = 1$ and $m = 0$. If u is red, then v is blue and w is green. The colouring c defined as follows is suitable: $c(u_2) = \{3,4\}$, $c(u_1) = c(v_1) = \{1,5\}$, $c(x) = \{2,3\}$. If u is blue, then v is green and w is red. The colouring c defined as follows is suitable: $c(u_2) = \{1,5\}$, $c(u_1) = c(v_1) = \{2,4\}$, $c(x) = \{3,5\}$. If u is green, then v is red and w is blue. The colouring c defined as follows is suitable: $c(u_2) = \{2,4\}$, $c(u_1) = c(v_1) = \{3,5\}$, $c(x) = \{4,1\}$.

If $k \geq 3$, by induction hypothesis, there exists a 5-bicolouring c of $T - \{u_k, u_{k-1}, u_{k-2}\}$ such that $c(v) = L(v)$ and $c(w) = L(w)$. $(u_{k-3}, u_{k-2}, u_{k-1}, u_k, u)$ is a path of length four. So, by Lemma 1, one can extend c in a 5-bicolouring of T that coincides with L on $\{u, v, w\}$. $\square$

The proof of the above theorem gives the following distributed algorithm for finding a bicolouring of an induced subgraph of the triangular lattice which runs in constant time. We may suppose that each vertex knows its label. And that each fourth vertex on a line is *special*. These two assumptions make sense because they are fixed by the triangular lattice and do not depend on the triangle-free graph we want to 5-bicolour.

Step 1: Colour every red (resp. blue, green) suitable vertex by $\{1,2\}$ (resp. $\{2,3\}$, $\{1,5\}$). This can be done in one communication unit of time, because a vertex only need to know its neighbours to decide whether it is suitable or not.

Step 2: Bicolour arbitrarily every special vertex that is at distance at least four of all suitable vertices and all centers of tristars. This can be done in four communication units of time. Every suitable vertex and every center of tristar send a message that is transmitted from neighbour to neighbour. Each special vertex that has received no message, four units of time after is given an arbitrary 2-set of colours.

Step 3: Extend the bicolouring to the whole graph. This is possible according to Lemma 1 and the proof of Theorem 1. Indeed, this can be done in at most 16 units of communication time : the graph induced by the vertices which are not bicoloured after Step 2 is the union of paths of length at most two and tristars which are unions of paths of length at most seven. So it requires at most 16 units of time for a non-coloured vertex to completely know its component and the bicouloring of the neighbours of the endvertices of its component.

3 [p]colouring of the triangle-free induced subgraphs of the triangular lattice.

We now give a distributed algorithm that finds a $\frac{5\omega_p(G)}{4}+3$ [p]colouring of a triangle-free induced subgraph of the triangular lattice. We suppose that each vertex of the graph is already labelled by Red, Blue or Green according to the 3-colouring of the triangular lattice.

Step 0: Initialize $k := 1$, $H := G$ and $q := p$.

Step 1: Assign to each red (resp. blue, green) vertex v of H such that $q(v) = 1$, the colour Red, (resp. Blue, Green).

Step 2: Let H' be the graph induced by H on the set of vertices v for which $q(v) \geq 2$. Assign to each isolated vertex v of H' the colours $(k,1)$, $(k,2)$, $k,3)$, $(k,4)$ and $(k,5)$ (or $(k,1),\dots(k,q(v))$ if $q(v) \leq 4$. Let $q'(v) = q(v) - 5$ (or $q'(v) = 0$ if $q(v) \leq 4$).

Step 3: Let H'' be the graph induced by the vertices of H' which are not isolated. Colour each of these vertex by two colours in $\{(k,1)$, $(k,2)$, $k,3)$, $(k,4)$,$(k,5)\}$ by the previous algorithm. For a vertex of H'', let $q'(v) = q(v) - 2$.

Step 4: Let H''' be the graph induced by the vertices v of H' such that $q'(v) > 0$. If H''' is not empty, do $H := H'''$, $q := q'$, $k := k + 1$, and go to Step 1.

It is easy to see that this algorithm yields a $\frac{5\omega_p(G)}{4} + 3$ [p]colouring of G. Indeed it is easy to see that for each loop from Step 1 to Step 4, $\omega_{q'}(H''') \leq \omega_q(H) - 4$. Thus the algorithm makes at most $\frac{\omega_p(G)}{4}$ loops. And the algorithm uses 3 special colours Red, Blue and Green and 5 colours at each loop. So the algorithm uses at most $\frac{5\omega_p(G)}{4} + 3$ colours. Note that this algorithm uses that many colours only if two vertices of an edge have weight $\omega_p(G)/2$. Otherwise this algorithm will use fewer colours. In particular, if the vertices of a stable set of G have huge weight and the other vertices have small weight then this algorithm will use roughly $\omega_p(G)$ colours.

Each loop of the above algorithm can be done simultaneously. This yields the following algorithm:

Step 1: Assign to each red (resp. blue, green) vertex of G with $p(v) = 1 (\mathrm{mod}\ 2)$ the colour Red (resp. Blue, Green). Let $q(v) = 2\lfloor p(v)/2 \rfloor$.

Step 2: Find a 5-bicolouring c of G.

Step 3: Let v be a vertex of G, $\{a, b\} = c(v)$ and $q^*(v) = \max\{q(w), w$ neighbour of $v\}$. Set $m = q(v)/2$ and $k = q^*(v)/2$.

If $q(v) \leq q^*(v)$, assign to v the colours (i, a) and (i, b) for $1 \leq i \leq m$.

If $q(v) > q^*(v)$ assign to v the colours (i, a) and (i, b) for $1 \leq i \leq k$. Let $q - q^* = 5\alpha + \beta$. Assign to v the colours $(i, 1)$ $(i, 2)$, $(i, 3)$, $(i, 4)$ and $(i, 5)$ for $k \leq i \leq k + \alpha$ and the colours $(k + \alpha + 1, 1) \ldots (k + \alpha + 1, \beta)$.

Since the algorithm for finding a 5-bicoloration is distributed and runs in constant time, so does the above algorithm. Hence,

Theorem 2 *Let G be a triangle-free subgraph of triangular lattice and p an arbitrary weigh function. Then there is a constant time distributed algorithm which finds a $[p]$colouring of G using at most $\frac{5\omega_p(G)}{4} + 3$ colours.*

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