SGT 2015 on Topological methods in graph theory
Lecture Notes

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We first compute the chromatic number of Kneser and Schrijver graphs using the Borsuk-Ulam theorem and Gale's Lemma. Then, we consider their local-chromatic number (a coloring parameter bounded above by the chromatic number) using the Zig-Zag Theorem (about colorings of graphs for which topological methods are relevant when estimating their chromatic number) proved by Ky Fan's Theorem.

1 Chromatic number of Kneser and Schrijver graphs

In this section, we define the Kneser and Schrijver graphs and consider their chromatic number. Let $[m]$ denote the set $\{1, \cdots, m\}$ and $\binom{S}{\ell}$ the set of all ℓ-subsets of S.

Definition 1. Let $m, \ell \in \mathbb{N}$ and $m \geq 2\ell$. The *Kneser Graph* $KG(m, \ell)$ has vertex set $\binom{[m]}{\ell}$, i.e., all ℓ-subsets of a set with m elements, and $AB \in E(KG(m, \ell))$ iff $A \cap B = \emptyset$.

If $m = 2\ell$, then $KG(m, \ell)$ is a matching.
If $\ell = 1$, then $KG(m, \ell) \cong K_m$.
$KG(5, 2)$ is isomorphic to the Petersen Graph.

Lemma 2. $\chi(KG(m, \ell)) \leq m - 2\ell + 2$.

Proof. Let $A \in \binom{[m]}{\ell}$. Let $c(A) = \min A$ if $\min A < m - 2\ell + 2$ and $c(A) = m - 2\ell + 2$ otherwise. Let us check that c is a proper coloring of $KG(m, \ell)$. Assume that $c(A) = c(B) = i$. If $i < m - 2\ell + 2$, then $\min A = \min B = i$. Hence A and B intersects, and so $AB \notin E(KG(m, \ell))$. If $i = m - 2\ell + 2$, then $\min A \geq m - 2\ell + 2$ and $\min B \geq m - 2\ell - 2$. Thus A and B are two ℓ-subsets of $\{m - 2\ell + 2, \ldots m\}$, a set of $2\ell - 1$ elements. Therefore A and B must intersect and $AB \notin E(KG(m, \ell))$. □

A more general way for properly coloring $KG(m, \ell)$ with $m - 2\ell + 2$ colors can be defined as follows. Partition $[m]$ into $m - 2\ell + 2$ odd cardinality subsets $T_1, \cdots, T_{m-2\ell+2}$. For any $A \in \binom{[m]}{\ell}$, color it with any i such that $|A \cap T_i| > |T_i|/2$. Such an i exists since otherwise, $|A| = \sum_i |A \cap T_i| \leq \sum_{1 \leq i \leq m-2\ell+2} (|T_i| - 1)/2 = \sum_{1 \leq i \leq m-2\ell+2} |T_i|/2 - (m - 2\ell + 2)/2 = \ell - 1$. Now the coloring is proper because any two sets receiving the same colour i must have at least one element of T_i in common as they both contain more than a half of T_i. Observe that the coloring given in Lemma 2 is obtained for $|T_1| = \cdots = |T_{m-2\ell+1}| = 1$ and $|T_{m-2\ell+1}| = 2\ell - 1$.

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In 1955, Kneser conjectured [7] that $\chi(KG(m,l)) = m - 2\ell + 2$. This was proved in 1978 by Lovász [11]. (Dates refer to publication year.) The same year another proof was given by Bárány.

Theorem 3 (Lovász 1978 [11])**.** $\chi(KG(m,\ell)) = m - 2\ell + 2$.

While Lovász's proof gave a general lower bound on the chromatic number of graphs in terms of topology, Bárány's proof is very specifically tailored for Kneser graphs. On the other hand, it is very simple and cute. Schrijver used Bárány's idea in a more general setting thereby obtaining the result we present below.

Definition 4. Let $m, \ell \in \mathbb{N}$ and $m \geq 2\ell$. The *Shrijver Graph* $SG(m,\ell)$ is the subgraph of $KG(m,\ell)$ induced by the vertices A that are *stable*, i.e., the sets $A \in \binom{[m]}{\ell}$ that do not contain consecutive elements modulo m, i.e., the subsets of $[m]$ that induce a stable (i.e. independent) set on the m-cycle with vertex set $[m]$ in order.

If $m = 2\ell + 1$, $SG(m,\ell)$ is a cycle of size m.

By definition, $SG(m,\ell)$ is a subgraph of $KG(m,\ell)$, so $\chi(SG(m,\ell)) \leq \chi(KG(m,\ell))$. Note that while $KG(m,\ell)$ is vertex-transitive, $SG(m,\ell)$ is not.
Still in 1978, Schrijver [17] generalized Theorem 3:

Theorem 5 (Schrijver [17])**.** $\chi(SG(m,\ell)) = m - 2\ell + 2$.

In some sense, Theorem 5 is best possible. Indeed, Schrijver also proved that $SG(m,\ell)$ is vertex-color-critical. We recall that a graph G is *vertex-color-critical* if $\chi(G - v) < \chi(G)$ for every vertex $v \in V(G)$.

We now first give a proof of Theorem 5, which clearly implies the Lovász-Kneser Theorem (Theorem 3), and then present also a direct proof of Theorem 3.

1.1 Proof of Shrijver's Theorem

Bárány's proof [2] used the following lemma.

Lemma 6 (Gale's lemma)**.** *It is possible to place m points on the sphere $S^{m-2\ell}$ such that every open hemisphere contains at least ℓ points.*

To prove his theorem, Schrijver strengthened this lemma.

Lemma 7 (Strong Gale Lemma (Schrijver))**.** *It is always possible to place m points on the sphere $S^{m-2\ell}$ s.t. every open hemisphere contains at least ℓ points corresponding to a stable ℓ-subset (i.e., a vertex of $SG(m,\ell)$).*

The main ingredient of basically all proofs of the Lovász-Kneser and Shrijver's theorems is the celebrated Borsuk-Ulam theorem. See the book [13] of Matoušek. (We note that by now Matoušek [14] found a purely combinatorial proof, too. However, it uses a combinatorial lemma, called Tucker's Lemma, which is a combinatorial version of the Borsuk-Ulam theorem. Cf. the lectures by Frédéric Meunier.) This theorem has many equivalent versions. One of them is the following.

Theorem 8 (Borsuk-Ulam)**.** *Let $f : S^d \to \mathbb{R}^d$ be a continuous map. There exists $x \in S^d$ such that $f(x) = f(-x)$.*

Another, equivalent form we state as a corollary.
A set $X \subseteq S^d$ is said to be *antipodal-free* if for every $x \in X$ we have $-x \notin X$.

Corollary 9. *If k antipodal-free open (resp. closed) sets cover S^d, then $k \geq d+2$.*

Proof. Assume for a contradiction that $d+1$ antipodal-free closed sets $D_1, \cdots, D_{d+1}$ cover S^d. Let $f: S^d \to \mathbb{R}^d$ be defined by $f(x) = (\mathrm{dist}(x, D_1), \cdots, \mathrm{dist}(x, D_d))$, where dist stands for distance. By the Borsuk-Ulam Theorem (Theorem 8), there exists $x \in S^d$ such that $f(x) = f(-x)$. Observe that if $f_i(x) = 0$ for any $i \leq d$, then $\mathrm{dist}(x, D_i) = 0 = \mathrm{dist}(-x, D_i)$ meaning that $x, -x \in D_i$, a contradiction. So $f_i(x) > 0$ for every i, thus neither of $x, -x$ are in D_i for $i \leq d$. Hence, both x and $-x$ must be in D_{d+1}, a contradiction.

Proving the statement for antipodal-free open sets from here is a routine topological argument: If an open covering as in the statement is given, then each open set involved in it can be shrinked a little to obtain closed sets still satisfying the conditions. So no fewer open sets are enough than closed ones for such a covering. □

Proof of Theorem 5. For any $x \in S^{m-2\ell}$, let $H(x)$ be the open hemisphere "centered" in x.

Place m points on $S^{m-2\ell}$ as in the Strong Gale Lemma. Hence, for any $x \in S^{m-2\ell}$, $H(x)$ contains a stable ℓ-subset. Let us consider a proper coloring of $SG(m, \ell)$ with t colors. We define a covering of $S^{m-2\ell}$ as follows. For any $1 \leq i \leq t$, let A_i be the set of $x \in S^{m-2\ell}$ such that $H(x)$ contains a stable ℓ-subset with color i.

By the placement, $\bigcup_{i \leq t} A_i = S^{m-2\ell}$. Moreover, all A_i, $1 \leq i \leq t$, are open sets.

Assume for a contradiction that both x and $-x$ are in A_i. Then $H(x)$ and $H(-x)$ contain stable ℓ-subsets with color i. But these sets are disjoint, contradicting the assumption that the coloring is proper.

Hence, $A_1, \cdots, A_t$ satisfy the hypothesis of Corollary 9. Hence, $t \geq m - 2\ell + 2$. □

1.2 Proof of the Lovász-Kneser Theorem, by J. Greene

Greene's proof [6] of the Lovász-Kneser Theorem is also based on the Borsuk-Ulam Theorem. It uses a slight variation of Corollary 9.

Corollary 10. *If k antipodal-free sets cover S^d, and $k-1$ of them are open and the remaining one is closed, then $k \geq d+2$.*

Proof of Theorem 3. Place m points on $S^{m-2\ell+1}$ in general position (no $m - 2\ell + 2$ points on a great circle). Fix a coloring of $KG(m, \ell)$ with t colors and let us define $A_1, \cdots, A_t$ as in the previous proof. As before, for any $i \leq t$, A_i is open and antipodal-free.

Let $B = S^{m-2\ell+1} \setminus \bigcup_{i \leq t} A_i$. The set B is closed. For purpose of contradiction, assume that both x and $-x$ are in B. Then at most $\ell - 1$ points are in $H(x)$ (resp., in $H(-x)$). Hence, at least $m - 2\ell + 2$ points are in $S^{m-2\ell+1} \setminus (H(x) \cup H(-x))$ contradicting the general position.

Since $A_1, \cdots, A_t, B$ cover $S^{m-2\ell+1}$, Corollary 10 implies $t \geq m - 2\ell + 2$. □

The advantage of this proof is that it avoids the use of Gale's lemma. On the other hand, it does not seem to generalize for Shrijver graphs.

1.3 Fractional chromatic number

The fractional chromatic number is given by the fractional relaxation of the integer program defining the chromatic number.

Let us be more precise. A *fractional coloring* is a collection $\{S_1, \ldots, S_\ell\}$ of independent sets along with corresponding non-negative real weights $\{w_1, \ldots, w_\ell\}$ such that the sum of the weights of the independent sets containing any vertex is at least 1, i.e., $\forall v \in V, \quad \sum_{\{S_i \mid v \in S_i\}} w_i \geq$

1. Observe that a coloring is simply a fractional coloring in which every weight is 0 or 1. A fractional coloring is a *fractional c-coloring* if the sum of the weights of the independent sets equals c, that is $\sum_i w_i = c$. The *fractional chromatic number* of G, denoted by $\chi_f(G)$, is the infimum of the real numbers c such that G has a fractional c-coloring. Notice that $\chi_f(G) \leq \chi(G)$.

One of the reasons why Kneser (and Schrijver) graphs are interesting is because they are graphs with a large gap (as large as we want) between the fractional chromatic number and the chromatic number. (In some sense Kneser graphs are "the canonical" examples of such graphs. This has to do with the possibility of defining the fractional chromatic number via the existence of graph homomorphisms to certain Kneser graphs.) This gap also somehow explains why it is difficult to bound the chromatic number of Kneser graphs from below, since elementary lower bounds usually also bound the fractional chromatic number from below.

Proposition 11. $\chi_f(KG(m,\ell)) \leq m/\ell.$

Proof. Set $G = KG(m,\ell)$. Because G is vertex-transitive, $\chi_f(G) = |V(G)|/\alpha(G)$ (see [16]), where $\alpha(G)$ is the maximum size of an independent set of G.

The $\binom{m-1}{\ell-1}$ sets of $\binom{[m]}{\ell}$ containing i form an independent set S_i, so $\alpha(G) \geq \binom{m-1}{\ell-1}$. Hence $\chi_f(G) \leq \frac{|V(KG(m,\ell))|}{\binom{m-1}{\ell-1}} = \frac{\binom{m}{\ell}}{\binom{m-1}{\ell-1}} = m/\ell.$ □

We note that $\alpha(KG(m,l)) = \binom{m-1}{\ell-1}$ by the celebrated Erdős-Ko-Rado theorem, and thus there is equality in the above Proposition.

It is well-known that $\chi_f(G) = 2$ if and only if G is bipartite. In contrast, for every $\varepsilon > 0$ and large M, there exists a graph with $\chi_f(G) \leq (2+\varepsilon)$ and $\chi(G) \geq M$. It suffices to consider a Kneser graph $KG(m,\ell)$ with $M - 2 + 2\ell \leq m \leq (2+\varepsilon)\ell$ and such an m exists if $\ell \geq \frac{M-2}{\varepsilon}$.

Remark 12. There are not very many graph families known with the property of having a large gap between chromatic and fractional chromatic number. In several of these cases one can use the topological method to determine the chromatic number. An exceptional family is that of the *shift graphs* H_m defined by $V(H_m) = \{(i,j) \mid i,j \in [m]\ i < j\}$ and $E(H_m) = \{(i,j),(k,l) \mid j = k \text{ or } i = l\}$.

H_m can also be defined as the line graph of the transitive tournament on m points, where the line graph $L(\vec{D})$ of a directed graph $\vec{D}$ is defined on the arcs of $\vec{D}$ as vertices with two of them being connected if the head of one is the tail of the other. It is a nice exercise (see [12] Problem 9.26) to show that $\chi(L(\vec{D})) \geq \log_2 \chi(D)$, where $\chi(D)$ is simply the chromatic number of the underlying undirected graph D of $\vec{D}$. In case of the transitive tournament this gives $\chi(H_m) \geq \log_2 m$, which is actually tight.

On the other hand, it is not very difficult to prove that $\chi_f(H_m) < 4$. Let the complete directed graph on m vertices be the one containing an arc in both directions between any pair of its m vertices. Its line graph S_m is vertex-transitive and contains H_m as an induced subgraph. it is easy to show that S_m has independent sets of size larger than $(m^2-1)/4$, while the number of its vertices is $m(m-1)$. This gives $\chi_f(H_m) \leq \chi_(S_m) = \frac{m(m-1)}{\alpha(S_m)} < 4$.

2 Local Chromatic Number

Definition 13. Let $c : V \to \mathbb{N}$ be a coloring of $G = (V,E)$. For any $A \subseteq V$, let $c(A) = \{c(v) \mid v \in A\}$.

A coloring c is r-*local* if it is proper and $\max_{v\in V} |c(N[v])| \leq r$ where $N[v]$ is the closed neighborhood of v, i.e., $N[v] = N(v) \cup \{v\}$. Observe that for a proper coloring, $|c(N[v])| = |c(N(v))| + 1$.

For any graph $G = (V, E)$, let $\Psi(G) = \min_{c \text{ proper coloring of } G} \max_{v \in V} |c(N[v])|$.

2.1 Around $\chi_f \le \Psi \le \chi$

By definition, we directly have :

Proposition 14. *For any graph G, $\Psi(G) \le \chi(G)$.*

Note that, in any proper coloring with $\chi(G)$ colors, any color class contains a vertex that sees every other color. Hence, any r-local coloring of G with $r < \chi(G)$ should use strictly more than $\chi(G)$ colors. It is somewhat counterintuitive that such a waste of colors globally can pay off locally. But this is actually the case.

Note that if G has an r-local coloring and $u, v \in V$ have the same color and see the same colors, then we can identify them without destroying r-locality. This leads to the following definition.

Definition 15. Let $U(m, r)$ be the graph defined by

$$\begin{aligned} V(U(m,r)) &= \{(x, A) \mid x \in [m], A \subseteq [m], x \notin A, |A| = r - 1\} \\ E(U(m,r)) &= \{\{(x, A), (y, B)\} \mid x \in B, y \in A\}. \end{aligned}$$

In essence, for a vertex (x, A), x represents its color and the set A represents the set of colors of its neighbors. Thus the following proposition is straightforward.

Proposition 16. *G admits an r-local coloring if and only if there is an $m \in \mathbb{N}$ such that a homomorphism $h : G \to U(m, r)$ exists.*

Proposition 17 ([8])**.** *$\chi_f(U(m, r)) = r$.*

Proof. Set $G = U(m, r)$. The graph G is vertex transitive so $\chi_f(G) = |V(G)|/\alpha(G)$. But $|V(G)| = r\binom{m}{r}$ and $\alpha(G) \ge \binom{m}{r}$ because the sets (x, A) such that $x = \min(A \cup \{x\})$ form an independent set in G. This shows $\chi_f(G) \le r$.

The equality comes from the fact that $\chi_f(G) \ge \omega(G) \ge r$. Indeed, the set $(x, R \setminus \{x\})$ for any r-set R of $[m]$ is a clique in G. $\square$

Corollary 18. *For any graph G, $\chi_f(G) \le \Psi(G)$.*

Proof. Let G be a graph with $\Psi(G) = r$. Then there is a homomorphism from G to $U(m, r)$ for some m, and χ_f can only be increased by a homomorphism. $\square$

Proposition 14 and Corollary 18 yield

$$\chi_f(G) \le \psi(G) \le \chi(G) \quad \text{for all graphs } G.$$

It is easy to see that $\psi(G) = 2$ if and only if $\chi(G) = 2$, and so if and only if $\chi_f(G) = 2$. In contrast, for larger values of $\psi(G)$, Erdős et al. [5] proved that $\chi(G)$ can be arbitrarily large.

Theorem 19 (Erdős, Füredi, Hajnal, Komjáth, Rödl, Seress 1986, [5])**.** *For any $k \ge 3$, there exists a graph G s.t. $\Psi(G) = 3$ and $\chi(G) \ge k$.*

Proof. (Sketch.) It can be proved that $lim_{m\to\infty}\chi(U(m, 3)) = +\infty$. Since $\psi(U(m, 3)) = 3$ this implies the statement.

To verify our claim about the above limit we can use what is said in Remark 12. Note that each directed graph $\vec{D}$ naturally induces an orientation on $L(\vec{D})$: the edge of $L(\vec{D})$ between arcs a, b of $\vec{D}$ is oriented from a to b if the head of a is the tail of b. For natural numbers $1 \le x < y < z \le m$ the ordered pair (y, xz) consisting of the number y and the 2-element set $xz = \{x, z\}$ is a vertex of $U(m, 3)$. On the other hand it can easily be identified with an arc of the just described oriented version of the shift graph H_m considered as the line graph of the transitive tournament on m vertices. We can observe that the line graph of this oriented graph $\vec{H}_m$ is isomorphic to the graph induced by $U(m, 3)$ on the above type of vertices. Using the general inequality mentioned in Remark 12 this gives $\chi(U(m,3)) \ge \chi(L(\vec{H}_m)) \ge \log_2 \log_2 m$ that indeed goes to infinity with m. $\square$

2.2 Local chromatic number of Kneser and Schrijver graphs

Since there is an arbitrarily large gap between the fractional chromatic and the chromatic number of both Kneser and Schrijver graphs $\Psi(KG(m,\ell)) =?$, $\Psi(SG(m,\ell)) =?$ are naturally occurring questions

This is partially answered by the following.

Theorem 20 (Simonyi and Tardos [19], Simonyi, Tardos and Vrećica[21])**.** *Let $\chi = m - 2\ell + 2$ be fixed.*

For m, ℓ large enough, $\Psi(SG(m,\ell)) = \lfloor \frac{\chi}{2} \rfloor + 2$.

A proper coloring $c : V \to \mathbb{N}$ is *wide* if, for any $x, y \in V$ with $c(x) = c(y)$, there is no walk of length 5 between x and y.

Lemma 21. *If G admits a wide coloring with t colors, then $\Psi(G) \le \lfloor \frac{t}{2} \rfloor + 2$.*

Proof. Let c be a wide coloring of G with t colors. Let $U = \{v \in V(G) \mid |c(N(v))| > t/2\}$. Let $h, h' \in N(U)$. Let $x, y \in U$ such that $h \in N(x)$ and $h' \in N(y)$. There is a color s such that there are $a \in N(x)$ and $b \in N(y)$ such that $c(a) = c(b) = s$. (This is because both $c(N(x))$ and $c(N(y))$ contain more than half of the colors.) If $\{h, h'\} \in E(G)$, then (a, x, h, h', y, b) is a 5-walk from a to b, two vertices of the same color, a contradiction. Hence, $N(U)$ is an independent set. Therefore we can recolor $N(U)$ with one new color. Doing this, every vertex in U has only 1 color in its neighborhood (the new one), and vertices not in U have at most $\lfloor \frac{t}{2} \rfloor + 1$ colors in their neighborhood, (the at most $\lfloor \frac{t}{2} \rfloor$ original ones by definition of U plus possibly the new one). Hence the new coloring is a ($\lfloor \frac{t}{2} \rfloor + 2$)-local coloring. $\square$

We shall also use another kind of graphs, namely Borsuk graphs, that were introduced by Erdős and Hajnal [4].

Definition 22. Let $0 < \alpha < 2$. The *Borsuk graph* $B(n, \alpha)$ is defined by $V(B(n,\alpha)) = S^{n-1}$ and $E(B(n,\alpha)) = \{xy \mid \mathrm{dist}(x,y) > \alpha\}$. (Nodes are linked with "almost" antipodal ones).

The following statement is equivalent to the Borsuk-Ulam Theorem.

Theorem 23 (Borsuk-Ulam theorem in terms of Borsuk graphs)**.** $\chi(B(n,\alpha)) \ge n + 1$.

For $\alpha < 2$ large enough the above inequality is an equality. Indeed, for large $\alpha < 2$ $B(n,\alpha))$ can be colored with $n+1$ colors: Consider the regular simplex of dimension n inscribed into S^{n-1} and project its $n+1$ facets $F_1, \ldots, F_{n+1}$ onto the sphere from its center. It is easy to see that if every $x \in S^{n-1}$ gets color j for some j satisfying that x is in the image of F_j, then no pair of points of the same color will have a distance close to 2. So this is a proper coloring of $B(n,\alpha)$ for $\alpha < 2$ large enough.

In fact, more is true. For every odd k there is an $\alpha_0 < 2$ such that there is no walk of length k between any pair of points having the same color. For $k = 1$ this is just the properness of the coloring, for $k = 5$ it means that the coloring is wide. Thus by Lemma 21 we get the following result.

Theorem 24. *For $\alpha < 2$ large enough we have $\Psi(B(n,\alpha)) \leq \lfloor \frac{n+1}{2} \rfloor + 2$.*

For Schrijver graphs, the inequality $\Psi(SG(m,\ell)) \leq \lfloor \frac{\chi}{2} \rfloor + 2$ for large enough m, ℓ is given a direct, combinatorial proof in [19]. Using a result of Schultz [18], however, a different and conceptually simpler argument can be used based on the observations discussed above.

Sketch of Proof of Theorem 20. Schultz [18] proved that for every $\alpha < 2$ and fixed $t = m-2\ell+2$ there are large enough m and ℓ such that there exists a homomorphism from $SG(m,l)$ to $B(m-2\ell+1,\alpha)$. Since by Theorem 24 and Proposition 16 we have a homomorphism from $B(m-2\ell+1,\alpha)$ to $U(m-2l+3, \lfloor \frac{m-2l+2}{2} \rfloor + 2)$ if $\alpha < 2$ is large enough, it follows that there is a homomomorphism from $SG(m,\ell)$ to $U(m-2l+3, \lfloor \frac{m-2l+2}{2} \rfloor + 2)$, so $\Psi(SG(m,\ell)) \leq \lfloor \frac{t}{2} \rfloor + 2$.

The Zig-Zag Theorem (see below) implies that $\Psi(SG(m,l)) \geq \lceil \frac{t}{2} \rceil + 1$. This matches the upper bound for odd t and is one less for even t. To close this gap more involved topological tools are needed than those we use for the proof of the Zig-zag Theorem. These more involved tools are outside of our scope here, they may be found in the references of [21]. □

The remaining part of the section is devoted to a proof of the Zig-Zag Theorem and its consequences.

2.2.1 Box complexes and the Zig-Zag Theorem.

Definition 25. A Z_2-space is a pair (W,μ), where W is a topological space and μ is a continuous map from W to itself for which $\mu(\mu(x))$ is the identity map. (Such a μ is called an *involution.*)

A Z_2-map is a continuous map f between two Z_2-spaces (W,μ) and (Y,ν) that "respects" the involution, that is, for which $f(\mu(x)) = \nu(f(x))$ for every $x \in W$.

In all these lectures the Z_2-spaces are *free*, meaning that no point is mapped into itself by the defining involution. When we refer to the sphere S^d as a Z_2-space we always mean it being equipped with the involution mapping every $x \in S^d$ to its antipodal pair $-x$.

Definition 26 (Box complex $B_0(G)$)**.** Let G be a graph. $B_0(G)$ is the simplicial complex with vertices $V(G) \times \{0,1\}$ and faces $\{(S \times \{0\}) \cup (T \times \{1\}) \mid S,T \subseteq V(G), S \cap T = \emptyset, \forall u \in S, v \in T, uv \in E(G)\}$ (the faces are defined by the complete bipartite subgraphs of G with sides S and T).

For example, $B_0(K_t)$ is homotopy equivalent (meaning "essentially the same" in our context) to S^{t-1} (check examples for $t \in \{2,3\}$).

Definition 27 (Box simplex $B(G)$)**.** Let G be a graph. $B(G)$ is the simplicial complex with vertices $V(G) \times \{0,1\}$ and faces $\{(S \times \{0\}) \cup (T \times \{1\}) \mid S,T \subseteq V(G), S \cap T = \emptyset, \forall u \in S, v \in T, uv \in E(G),$ *and* $($*if* $S = \emptyset$ *then* $\cap_{v \in T} N(v) \neq \emptyset)$ *and vice versa*$\}$

The map exchanging the sides S and T makes both $B_0(G)$ and $B(G)$ a Z_2-complex (a simplicial complex equipped with a *simplicial involution*, i.e., one mapping simplices to simplices.)

Lemma 28. *If there is a homomorphism from G to H, then there is a Z_2-map from $B_0(G)$ to $B_0(H)$ and also from $B(G)$ to $B(H)$.*

Choosing $H = K_t$ we obtain that $\chi(G) \leq t$ implies the existence of a Z_2-map from $B_0(G)$ to S^{t-1} (and from $B(G)$ to S^{t-2}).

Definition 29. Let (W,μ) be a $\mathbb{Z}_2$-space.
$coindex(W,\mu) = \max\{d \mid \exists\ Z_2 - map\ : S^d \to_{\mathbb{Z}_2} (W,\mu)\}$.
$index(W,\mu) = \min\{d \mid \exists\ Z_2 - map\ : (W,\mu) \to_{\mathbb{Z}_2} S^d\}$.

A not yet stated version of the Borsuk-Ulam theorem is that if a Z_2-map from S^d to S^h exists, then $d \leq h$. This gives

Proposition 30. *$coindex(W, \mu) \leq index(W, \mu)$.*

Proposition 31. *$coindex(B_0(KG(m,l)) \geq m - 2l + 1$.*
$coindex(B(SG(m,l)) \geq m - 2l$.

Proof. The first statement can be proven from Greene's proof of the Lovász-Kneser Theorem.
The second statement can be proven from the Strong Gale Lemma. □

Given a topological space W, the *suspension* $susp(W)$ is obtained by taking the product $W \times [0,1]$ and identifying the points in $W \times \{0\}$ and also in $W \times \{1\}$. Geometrically it belongs to erecting a cone on "both sides" of w. As an example $susp(S^d) \cong S^{d+1}$. (The sign $\cong$ stands for homeomorphism, while we denote homotopy equivalence by $\simeq$.) If W is a Z_2-space with involution μ, then μ is extended appropriately to $\mu^* : (x,t) \mapsto (\mu(x), -t)$.

Proposition 32 (Csorba [3])**.** *For any graph G, $B_0(G) \simeq susp(B(G))$.*

Proposition 33. *If there is a map $A \to_{\mathbb{Z}_2} B$ then there is a map $susp(A) \to_{\mathbb{Z}_2} susp(B)$.*

Lemma 34. *$coindex(B_0(G)) \geq coindex(B(G)) + 1$.*
$index(B_0(G)) \leq index(B(G)) + 1$.

Proof. By definition, there is a Z_2-map from $S^{coindex(B(G))}$ to $B(G)$. Hence, there is a Z_2-map from $S^{coindex(B(G))+1}$ to $susp(B(G)) \simeq B_0(G)$ by Prop. 32. Thus $coindex(B_0(G)) \geq coindex(B(G)) + 1$. The other statement can be proved similarly. □

Putting all inequalities together, it follows:

Theorem 35. *For any graph G,*
$coindex(B(G)) + 2 \leq coindex(B_0(G)) + 1 \leq index(B_0(G)) + 1 \leq index(B(G)) + 2 \leq \chi(G)$.

Theorem 36 (Zig-Zag Theorem, cf. Ky Fan [10], Meunier [15], Simonyi and Tardos [19])**.** *Let G be a graph such that $coindex(B_0(G)) \geq t - 1$. For any proper coloring of G, there is a $K_{\lfloor t/2 \rfloor, \lceil t/2 \rceil}$ subgraph colored with t colors such that the colors appear alternating on the two sides of the bipartition.*

Corollary 37. *$coindex(B_0(G)) \geq t - 1 \Rightarrow \Psi(G) \geq \lceil t/2 \rceil + 1$.*

Proof. Consider a vertex on the $\lfloor t/2 \rfloor$ size side of our multicolored complete bipartite graph. It has $\lceil t/2 \rceil$ differently colored vertices among its neighbours. □

To prove the above theorem, we will use the second version of Ky Fan's theorem stated below.

Theorem 38 (Ky Fan [9])**.** *Let $A_1, \cdots, A_m$ be antipodal-free open sets covering S^d. Then, there exist $1 \leq k_1 < k_2 < \cdots < k_{d+2} \leq m$ such that there exists $x \in S^d$ with $x \in \bigcap_{1 \leq i \leq d+2} (-1)^{i+1} A_{k_i}$.*
If $A_1, \cdots, A_m$ are antipodal-free open sets such that $A_1, \cdots, A_m, -A_1, \cdots, -A_m$ together cover S^d, then there exist $1 \leq k_1 < k_2 < \cdots < k_{d+1} \leq m$ such that there exists $x \in S^d$ with $x \in \bigcap_{1 \leq i \leq d+1} (-1)^{i+1} A_{k_i}$.

Observe the difference between the two statements above: The price for the looser condition in the second statement is that we find only $d+1$ rather than d+2 special sets in the conclusion. It is worth noting the similarity of this difference to that between the proofs of Bárány and Greene: To get the same bound for the necessary number of colors Greene allows a less strict cover but goes one dimension higher.

Proof of Theorem 36. Let G be *topologically t-chromatic*, i.e., such that $coindex(B_0(G)) \geq t-1$. Consider a proper coloring of it with m colors and let $f : S^{t-1} \to_{\mathbb{Z}_2} B_0(G)$.

For any $i \leq m$, let A_i be the set of points $x \in S^{t-1}$ such that the minimum simplex containing $f(x)$ in the interior has a vertex colored i in $V(G) \times \{0\}$, while the whole simplex is not entirely in $V \times \{0\}$. If the minimum simplex containing $f(x)$ is a subset of $V(G) \times \{0\}$, then set $x \in A_{m+1}$. One can check that the conditions of the second version of Ky Fan's theorem are satisfied, so its conclusion holds. So by Ky Fan's Theorem (Theorem 38), there exists x for which $f(x)$ is in a simplex of $B_0(G)$ having all the colors i (odd) in the $V(G) \times \{0\}$ side and all colors i (even) in the $V(G) \times \{1\}$ side. □

The next theorem is also a consequence of the Borsuk-Ulam theorem. (See Bacon's paper [1] for many more equivalent forms of the Borsuk-Ulam theorem, where it also appears.)

Theorem 39. *If S^d is covered by open antipodal-free sets $A_1, \cdots, A_{d+2} \subseteq S^d$, then for any $j \in \{1, \cdots, d+1\}$, there exists $x \in S^d$ such that $x \in \bigcap_{1\leq i\leq j} A_i$ and $-x \in \bigcap_{j+1\leq i\leq d+2} A_i$.*

Using Theorem 39 and the techniques developed above the following can be proven. (Try it!)

Theorem 40 ([20])**.** *If $coindex(B(G)) + 2 \geq d + 2$ and $\chi(G) = d + 2$, then for any optimal proper coloring of G and for any bipartition (A, B) of the color set, there exists a multicolored $K_{|A|,|B|}$ such that the colors satisfy the partition.*

Exercise 1. Recall the definition of *Shift graphs H_m* from Remark 12: $V(H_m) = \{(i,j) \mid i,j \in [m]\ i < j\}$ and $E(H_m) = \{(i,j),(k,l) \mid j = k \text{ or } i = l\}$. Recall that H_m is isomorphic to the line graph of the transitive tournament on m vertices and have chromatic number $\lceil \log_2 m \rceil$. Prove $coindex(B_0(H_m)) + 1 < 4$. (This means that in case of H_m the left hand side is a very poor lower bound on the chromatic number.)

Proof. Color each node (i,j) with color j. All complete bipartite graphs have a monochromatic side. So the result follows from Theorem 36. □

Remark 41. We have seen that the lower bound on $\psi(SG(m,\ell))$ obtained from the Zig-Zag Theorem was not sharp for even-chromatic Schrijver graphs: a gap of 1 remained, that was closed by proving that if $coindex(B(G))+2 \geq t$ for some graph G, then it already implies the matching lower bound $\psi(G) \geq \lfloor t/2 \rfloor + 2$. The proof of this needed more involved topological tools (cf. [21] and the references therein). It turns out, however, that the bound obtained via the Zig-Zag Theorem is also sharp: It is possible that the graph does not satisfy $coindex(B(G)) + 2 \geq t$ but does satisfy $coindex(B_0(G)) + 1 \geq t$ (the assumption we had in the Zig-Zag Theorem, which is a weaker property of the graph, cf. the chain of inequalities in Theorem 35), and the local chromatic number $\psi(G)$ is equal to $\lceil t/2 \rceil + 1$ also for even t. Thus it matches the lower bound obtained from the Zig-Zag Theorem. An example for such a graph is $U(5,3)$ defined in Definition 15. This means that the local chromatic number is sensitive for the difference between the weaker graph property $coindex(B_0(G)) + 1 \geq t$ and the stronger $coindex(B(G)) + 2 \geq t$, while the two are not distinguished by the chromatic number.

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