Problem session SGT 2015

June 21, 2015

Homology with integer coefficients

Communicated by F. Lazarus.

Let C_0 be the set of all linear combination of vertices, i.e. $C_0 = \{\sum_{v \in V(G)} \alpha_v v\}$, and let C_1 be the set of all linear combinations of edges $C_1 = \{\sum_{e \in E(G)} \alpha_e e\}$. (Every edge has a default orientation.)

Let f be the function $C_1 \to C_0$, $vw \mapsto w - v$. $H_1(G) = Ker(f)$ is the *cycle space* of G.

A basis of the cycle space may be easily found using a spanning tree and taking all cycles C_e in $T \cup \{e\}$ for all $e \in E(G) \setminus E(T)$. If we add weights, then there is a matroid structure and a minimum-weight basis can be found using the greedy algorithm.

But what happens when we only consider linear combinations of C_1 with integer coefficients ? There are graphs, like the generalized Petersen graph $P_{11,4}$, such that no minimum-weight basis has integer coefficient.

List-colouring peculiar graphs

Communicated by L. Pastor.

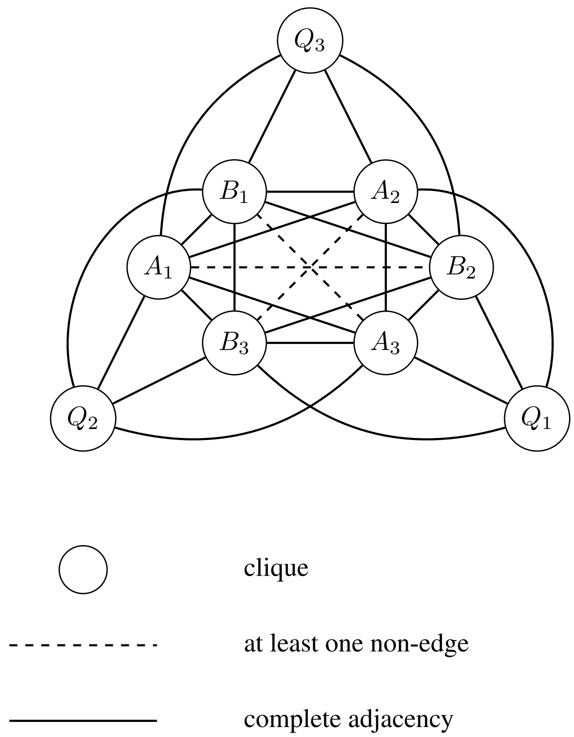
$\chi(G)$ is the chromatic number of G, $\mathrm{ch}(G)$ is the choosability of G.

The following conjecture generalizes the celebrated List Colouring Conjecture.

Conjecture 1 (Gravier and Maffray)**.** For any claw-free graph G, $\chi(G) = \mathrm{ch}(G)$.

We are interested in claw-free perfect graphs. Those graphs can be decomposed by clique-cutsets into *elementary* or *peculiar* graphs.

Peculiar graphs are those that can be partitioned as in the picture below.



Problem 2. Show that if G is a peculiar perfect graph, then $\mathrm{ch}(G) = \omega(G)$.

Orienting a graph so that all labeling have a monochromatic directed cycle

Communicated by A. Harutyunyan.

Conjecture 3 (Neumann-Lara, 1985)**.** There exists an integer-valued function f such that if $\chi(G) \geq f(k)$, then there exists an orientation D of G such that any k-labelling of the vertices of D will create a monochromatic directed cycle.

Even the existence of $f(2)$ is unknown. Another question is whether $f(k) = O(k \log k)$?

This is motivated by the *dichromatic number*. The *dichromatic number* of D, denoted $\vec{\chi}(D)$ is the minimum k such that there is a partition of $V(D)$ into k sets $V_1, \dots, V_k$ such that $D[V_i]$ is acyclic.

Conjecture 4 (Erdős, Neumann-Lara)**.** $\vec{\chi}(D) \leq O\left(\frac{\Delta(D)}{\log \Delta(D)}\right)$

Conjecture 5 (Aharoni, Berger, ??)**.** For all digraph D, there is an acylic induced subdigraph of order at least $\Omega\left(\frac{\Delta(D)}{\log \Delta(D)}\right)$.

Helly graphs

Communicated by J. Chalopin.

Let $\mathcal{F}$ be a family of sets. It has the *Helly Property* if for any subfamily $\mathcal{F}' \subseteq \mathcal{F}$ of pairwise intersecting sets (i .e. $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{F}'$), then $\bigcap_{A \in \mathcal{F}'} A \neq \emptyset$.

A graph G is *Helly* if the family of all balls (of any radius) have the Helly property.

A graph is 1-*Helly*, if the family of balls of radius 1 has the Helly . A graph G is *clique-Helly* if the family of all maximal cliques have the Helly property.

1-Helly graphs are clique-Helly but the opposite does not always hold.

Theorem 6. *The following statements are equivalent.*

- *G is a Helly graph;*
- *G is 1-Helly and its clique complex is simply connected;*
- *G is clique-Helly and G is dismantable.*

A graph G is *dismantable* if there is an ordering $v_1, \ldots, v_n$ of $V(G)$ such that for every $i > 1$, there exists $j < i$ such that $N_{G_i}[v_i] \subseteq N_G[v_j]$, with $G_i = G[V_1, \ldots, v_n]$.

Problem 7. Is it true that Helly graphs are precisely clique-Helly graphs with a simply connected clique complex ?

Orientations of triangulations on surfaces

Communicated by B. Lévêque.

Theorem 8 (Barát and Thomassen, 2006)**.** *If G is the triangulation of any surface, then there is an orientation of G such that $d^+(v) = 0 \mod 3$ for every vertex v.*

This corresponds to a decomposition of the edge set into claws ($K_{1,3}$).

Theorem 9 (Albar, Goncalves, Knauer)**.** *If G is the triangulation of any surface, then there is an orientation of G such that $d^+(v) = 0 \mod 3$ and $d^+(v) \neq 0$ for every vertex v.*

Problem 10. If G is the triangulation of any surface, does there exists an orientation such that across any close curve on the surface, all arcs are not going in the same direction ?

True when the genus is 1.

Cooperative colouring

Communicated by R. Aharoni.

Theorem 11. *If C_1, C_2, C_3 are cycles on the same set V, then one can choose independent sets A_i of C_i $i = 1, 2, 3$ such that $A_1 \cup A_2 \cup A_3 = V$.*

It is easy to fond an example showing that two cycles are not enough (even if the cycles are even).

The proof of this theorem is topological.

Problem 12. Give a combinatorial proof of this theorem.

Directed local chromatic number

Communicated by G. Simonyi.

Let D be a digraph. $\vec{\psi}(D) = \min_{c \text{ proper colouring}} \max_{v \in V(D)} |c(N^+(v))| + 1$.

Problem 13. How large can be the difference between $\psi(G)$ and $\max\{\vec{\psi}(D) \mid D \text{ orientation of } G\}$?

There is a construction where the difference is at least 1. Can it be at least 2 ?

Colouring k-regular graphs whose neighbourhood are perfect matchings

Communicated by A. Munaro.

Let G be a k-regular graph such that each edge is in exactly one triangle. (equivalently G is k-regular and $G[N(v)]$ is a perfect matching.

Problem 14. If $k = 6$, is it true that such a graph is 3-colourable ? For any larger k ?

The answer is 'No'. Actually for any k, there is a regular graph such that each edge is in exactly one triangle and whose chromatic number is at least k.

For convenience, let us say that a graph is *nice* if every graph is in exactly one triangle. Recall that a graph is *eulerian* if every vertex has even degree.

Lemma 15. *If there is an eulerian graph with maximum degree* Δ*, then there is a nice* Δ*-regular graph.*

Proof. Let G be an eulerian graph with maximum degree Δ, and let S be the set of vertices of degree $\delta(G)$. Let H be the graph obtained from Δ copies $G_1, \dots, G_\Delta$ of G as follows. For every vertex $x \in S$, let x_i be the vertex corresponding to x in G_i. Add a new vertex z_x connected to $x_1, \dots, x_\Delta$ and the edges $x_1x_2, \dots, x_{\Delta-1}x_\Delta$. One easily checks that H is nice, has maximum degree Δ and minimum degree $\delta(G) + 2$.

Repeating the process $(\Delta - \delta(G))/2$ times we obtain a nice Δ-regular graph. □

Now by a well-known result of Erdős, there exists a $k$colourable graph F with chromatic number k and girth 4. Let G be the graph obtained form F by ading for every edge $e = uv$ a new vertex x_e connected to u and v. Then F is nice, eulerian and has maximum degree $2\Delta(F)$. Now by Lemma **??**, there a $2\Delta(F)$-regular nice graph with chromatic number k.

Now the question is the following:

Problem 16. What is the maximum chromatic number of an r-regular nice graphs ?

Partition of the infinite path into distance-independent sets

Communicated by N. Gastineau.

A set S of vertices in a graph G is *s-independent* if any two vertices of S are at distance at least s in G.

Problem 17 (Goddard and Xu, 2012)**.** For which lists $(s_1, \ldots, s_\ell)$, can you partition the infinite path P_∞ into k sets $X_1, \ldots, X_\ell$ such that each X_i is s_i-independent.

One can easily see that it is impossible for $(1, 3, 7)$.

Number of induced cycles in a graph with no $2K_3$-subdivision

Communicated by J.-F. Raymond.

Problem 18. Let G be a graph that contains no induced subdivision of $2K_3$. Is is true that G contains a polynomial number of induced cycles ?

Colour-change-distance between antipodal vertices in 2-edge-coloured hypercube

Communicated by D. Soltész.

Q_n s the n-dimensional hypercube. Colour its edges in red or blue

The distance between two vertices u and v is the minimum number of colour changes in a (u, v)-path.

Problem 19. Is it always true that there exists two antipodal vertices at distance at most 1 ?

It is known that if there is no alternating C_4, then there are two antipodal vertices at distance 0.