Theoretical Computer Science 243 (2000) 289–305

Theoretical Computer Science

www.elsevier.com/locate/tcs

Power consumption in packet radio networks ☆

Lefteris M. Kirousis [a], Evangelos Kranakis [b,*,1], Danny Krizanc [b,1], Andrzej Pelc [c,1]

[a] *University of Patras, Department of Computer Engineering and Informatics, Rio 26500, Patras, Greece*

[b] *Carleton University, School of Computer Science, Ottawa, Ont., K1S 5B6, Canada*

[c] *Département d'Informatique, Université du Québec à Hull, Hull, Qué., Canada J8X 3X7*

Received April 1997; revised July 1998

Communicated by D. Peleg

Abstract

In this paper we study the problem of assigning transmission ranges to the nodes of a multi-hop packet radio network so as to minimize the total power consumed under the constraint that adequate power is provided to the nodes to ensure that the network is strongly connected (i.e., each node can communicate along some path in the network to every other node). Such assignment of transmission ranges is called complete. We also consider the problem of achieving strongly connected bounded diameter networks.

For the case of $n+1$ colinear points at unit distance apart (the unit chain) we give a tight asymptotic bound for the minimum cost of a range assignment of diameter h when h is a fixed constant and when $h \geqslant (1+\varepsilon)\log n$, for some constant $\varepsilon > 0$. When the distances between the colinear points are arbitrary, we give an $O(n^4)$ time dynamic programming algorithm for finding a minimum cost complete range assignment.

For points in three dimensions we show that the problem of deciding whether a complete range assignment of a given cost exists, is NP-hard. For the same problem we give an $O(n^2)$ time approximation algorithm which provides a complete range assignment with cost within a factor of two of the minimum. The complexity of this problem in two dimensions remains open, while the approximation algorithm works in this case as well.

Keywords: Multi-hop packet radio networks; Transmission range assignments; Power consumption

☆ An extended abstract of this paper has appeared in the proceedings of the 14th Annual Symposium on Theoretical Aspects of Computer Science (STACS 97), R. Reischuk, M. Morvan (Eds.), pp. 363–374, vol. 1200, Springer, Lecture Notes in Computer Science, 1997.

* Corresponding author.

E-mail address: kranakis@scs.carleton.ca (E. Kranakis).

[1] Research supported in part by NSERC (Natural Sciences and Engineering Research Council of Canada) grant.

PII: S0304-3975(98)00223-0

1. Introduction

A packet radio network is a network where the nodes consist of radio transmitter/receiver pairs distributed over a region. Communication takes place by a node broadcasting a signal over a fixed range (the size of which is proportional to the power expended by the node's transmitter). Any receiver within the range of the transmitter can receive the signal assuming no other nodes are transmitting signals that reach the receiver simultaneously. For a message to be sent to a node outside of the range of the message originator, multiple "hops" may be required, whereby intermediate nodes pass on (re-broadcast) the message until the ultimate destination node is reached.

Such networks have applications in many situations, over many different scales, where traditional networks are too expensive or even impossible to build. Some examples include: (1) setting up a LAN in a historic building where adding wiring would destroy or obscure valuable features of the building; (2) battlefield or disaster situations where temporary WANs are required but the infrastructure for a traditional network does not exist; (3) networks which include nodes in outer space (e.g., satellites, space stations, the moon).

A key issue in setting up and running such a network is the amount of power required by each of the nodes for its transmission. It is well-established [11] that the power of the signal received at a node is inversely proportional to the distance the receiver is from the transmitter, raised to an exponent known as the distance–power gradient, i.e.,

$$P_{\mathrm{r}} = \frac{P_{\mathrm{o}}}{d^{\alpha}},$$

where P_{r} is the power of the received signal, P_{o} is the power of the transmitted signal, d is the distance between the receiver and the transmitter, and α is the distance–power gradient. In an ideal situation $\alpha = 2$. However, due to various environmental factors such as building materials, street layouts, terrain characteristics, etc., the measured value of α may vary from less than two to more than six. (Here we will assume $\alpha = 2$ though all of our results are easily adjusted for any constant $\alpha \geqslant 1$.) This distance–power relationship implies there is a tradeoff between the power used by the nodes of the network (i.e., the size of the node ranges) and the diameter of the network (i.e., the number of hops in a path between communicating pairs of nodes if such a path exists).

In this paper we study the problem of assigning transmission ranges to the nodes of a multi-hop packet radio network so as to minimize the total power consumed under the constraint that adequate power is provided to the nodes to ensure that the network is strongly connected (i.e., each node can communicate along some path in the network to every other node). We also consider the problem of achieving strongly connected bounded diameter networks.

1.1. Terminology and problem statement

Let $V = \{x_1, \ldots, x_n\}$ be a set of n points in a Euclidean space. For two points $x_i, x_j \in V$, let $d(x_i, x_j)$ denote their Euclidean distance. We also refer to the points of V as *vertices*.

A *broadcasting range assignment* (or *range assignment*, for short) on the vertices in V is a function from V into the set of nonnegative real numbers.

If R is a range assignment on V, the *cost* of R is defined to be the sum $\sum_i (R(x_i))^2$. (Note that the exponent 2 was chosen for convenience and our results are easily adjusted for other choices of the distance–power gradient.)

The *communication graph* associated with a range assignment R (denoted by G_R) is a directed graph with V as its set of vertices and a directed edge from x_i to x_j iff $R(x_i) \geqslant d(x_i, x_j)$. In other words, a directed edge (x_i, x_j) indicates that x_j is within the range of x_i. A range assignment R is called *complete* iff G_R is strongly connected. A complete range assignment R has *diameter h* iff G_R has diameter h.

The problems we consider in this work, which is an extended version of [10], are those of finding a minimum cost complete range assignment and a minimum cost range assignment with a given diameter, for a given set of points.

1.2. Our results

The results described in this paper deal with range assignment problems in one- and three-dimensional Euclidean space.

For the case of $n+1$ colinear points at unit distance apart (the unit chain) we give a tight asymptotic bound for the minimum cost of a range assignment of diameter h when h is a fixed constant and when $h \geqslant (1+\varepsilon)\log n$, where $\varepsilon > 0$ is a constant (Section 2.1). When the distances between the colinear points are arbitrary, we give an $O(n^4)$ time dynamic programming algorithm for finding a minimum cost complete range assignment (Section 2.2).

For points in three dimensions we show that the problem of deciding whether a complete range assignment of a given cost exists, is NP-hard (Section 3.1). For the same problem we give an $O(n^2)$ time approximation algorithm which provides a complete range assignment with cost within a factor of two of the minimum (Section 3.2). The complexity of this problem in two dimensions remains open, while the approximation algorithm works in this case as well.

1.3. Related results

Studies of multi-hop packet radio networks have mainly concentrated on the problem of scheduling communication so as to avoid simultaneous broadcast to the same receiver which results in a scrambled signal. A number of authors have shown that the problem of minimizing the number of rounds required to realize communication between an arbitrary set of neighboring nodes is NP-hard and/or have provided heuristics for it [2, 3, 6, 12, 13]. Sen and Huson [15] point out these previous authors assumed that the underlying graphs were arbitrary (and therefore the NP-hardness easily follows from known graph coloring problems). They show that the problem remains NP-hard when restricted to the domain of possible packet radio graphs and they give an $O(n \log n)$ time algorithm for the case of vertices located on a line. Asymptotically efficient algorithms for the problem of scheduling a broadcast from a single source to all other nodes of a packet radio network are discussed in [1, 4, 5, 7, 14]. Takagi and Kleinrock [16] consider the problem of assigning transmission radii so as to maximize the expected one-hop progress of a packet assuming randomly distributed packet radio terminals are broadcasting packets with fixed probability of transmission. A survey of packet radio network technology appears in [9, 11] contains useful background information on wireless networks in general.

2. Range assignments in one dimension

In this section we study range assignments when the points are arranged on a line.

2.1. The unit chain

Consider a set $N = \{0, 1, \dots, n\}$ of $n+1$ colinear points at unit distances. Let $Cost_h(n)$ be the minimum cost of a range assignment for N, of diameter at most h, for a positive integer h. We establish the exact order of magnitude of $Cost_h(n)$ for any fixed integer h and construct a range assignment of diameter h corresponding to this cost.

Lemma 2.1. *There exists a range assignment for N of diameter at most h, with cost $C_h(n) \in O(n^{E(h)})$, where $E(h) = (2^{h+1} - 1)(2^h - 1)$, for any fixed positive integer h.*

Proof. Induction on h. For $h = 1$ assign to every point its distance from the farthest end of the segment N. This is a range assignment of diameter 1, with cost $O(1^2 + \cdots + n^2) = O(n^3)$.

Suppose that the lemma is true for h. Choose $k-1$ equidistant points in the segment N, for $k = n^{1-2/E(h)}$ (for simplicity assume that k divides n: it is easy to modify the argument in the general case). To each of these points assign the range equal to its distance from the farthest end of N. For each (closed) segment I between these consecutive points choose a range assignment of diameter h with cost $O(|I|^{E(h)})$. Since there are k segments of length n/k, the total cost is at most

$$\begin{aligned} kn^2 + kC_h\left(\frac{n}{k}\right) &\in O\left(k\left(n^2 + \left(\frac{n}{k}\right)^{E(h)}\right)\right) \\ &= O\left(kn^2 + k\frac{n^{E(h)}}{n^{E(h)-2}}\right) \\ &= O(kn^2) \end{aligned}$$

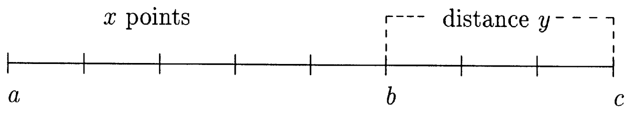

Fig. 1. Chain.

$$= \mathrm{O}(n^{3-2/E(h)})$$
$$= \mathrm{O}(n^{E(h+1)}). \qquad \square$$

Let $a < b$ be integers. Consider a set M of x colinear integer points in the segment (a,b). Call them *senders*. Let c be an integer point $b + y$, for $y \geqslant 1$ (cf. Fig. 1). Let $C(h,x,y)$ be the minimum cost of a range assignment for (a,b), for which c can be reached from any sender in at most h hops.

Lemma 2.2. *$C(h,x,y) \in \Omega(xy^{e(h)})$, where $e(h) = (2^h/2^h - 1)$, for any fixed positive integer h.*

Proof. Induction on h. For $h = 1$, the point c must be in the range of every sender, hence $C(h,x,y) \geqslant xy^2$.

Suppose that the lemma is true for h. Let $C(h,x,y) \geqslant c_h x y^{e(h)}$, where c_h depends only on h. Consider a range assignment r for (a,b), for which c can be reached from any sender in at most $h+1$ hops. Assume that there are l integers in the segment (a,b) such that c is in the range of each of them. Call these integers *transmitters*. The part of the cost of r charged for the transmitters is at least ly^2. Let $t_1 < \cdots < t_l$ be consecutive transmitters and let $t_{l+1} = b$. Let x_i be the number of senders strictly between t_i and t_{i+1}. Every sender in the segment (t_i, t_{i+1}) must reach one of its ends in at most h hops. For at least $x_i/2$ senders this end is common, without loss of generality t_{i+1}. Hence, at least $x_i/8$ senders must reach the point t_{i+1} which is at distance at least $y = x_i/4$ from each of them, in at most h hops. The part of the cost of r charged for those senders is at least

$$\sum_{i \leqslant l} C\left(h, \frac{x_i}{8}, \frac{x_i}{4}\right) \geqslant \sum_{i \leqslant l} c_h \frac{x_i}{8} \left(\frac{x_i}{4}\right)^{e(h)} = \sum_{i \leqslant l} d_h x_i^{e(h)+1},$$

where $d_h = c_h / 2^{2e(h)+3}$.

Since $e(h) + 1 > 1$ and $\sum_{i \leqslant l} x_i \geqslant x - l$, the value of $\sum_{i \leqslant l} d_h x_i^{e(h)+1}$ is the smallest when all x_i are equal to $(x-l)/(l)$. Hence, we get

$$C(h+1,x,y) \geqslant ly^2 + d_h l \left(\frac{x-l}{l}\right)^{e(h)+1}.$$

If $l \geqslant x/2$ then

$$C(h+1,x,y) \geqslant ly^2 \geqslant \tfrac{1}{2} x y^{e(h)},$$

which proves the inductive conclusion. Otherwise,

$$ly^2 + d_h l\left(\frac{x-l}{l}\right)^{e(h)+1} \geqslant ly^2 + d_h\frac{x}{2}\left(\frac{x}{2l}\right)^{e(h)}.$$

The latter is smallest when the summands are equal. This implies

$$ly^2 = d_h\frac{x^{e(h)+1}}{2^{e(h)+1}l^{e(h)}},$$
$$l = \frac{d_h^{1/(e(h)+1)}x}{2y^{2/(e(h)+1)}}.$$

Hence,

$$C(h+1,x,y)\geqslant 2ly^2 = d_h^{1/(e(h)+1)}xy^{2-2/(e(h)+1)} = c_{h+1}xy^{e(h+1)},$$

where $c_{h+1} = d_h^{1/(e(h)+1)}$. This proves the lemma by induction. □

Theorem 2.1. *$Cost_h(n) \in \Theta(n^{E(h)})$, where $E(h) = (2^{h+1} - 1/2^h - 1)$, for any fixed positive integer h.*

Proof. By Lemma 2.1. $Cost_h(n) \in O(n^{E(h)})$. By Lemma 2.2,

$$Cost_h(n)\geqslant C\left(h,\frac{n}{2},\frac{n}{2}\right) \in \Omega(n^{e(h)+1}) = \Omega(n^{E(h)}). \qquad \square$$

The previous theorem deals with constant diameter range assignments. In the sequel we consider the case when the size of the diameter is $\Omega(\log n)$. First we will need to prove two lemmas.

Lemma 2.3. *For any diameter h, $Cost_h(n)\geqslant n^2/h$.*

Proof. Consider a range assignment of diameter k on the chain with vertices $0,1,2,\ldots,n$. By assumption, it should be possible to reach vertex n from vertex 1 in k hops, where $k\leqslant h$. Let the sizes of the corresponding hops be $x_1,x_2,\ldots,x_k$. By definition we have that

$$Cost_h(n)\geqslant x_1^2 + x_2^2 + \cdots + x_k^2.$$

By Hölder's inequality and since $x_1+x_2+\cdots+x_k = n$ the right-hand side of the above inequality must exceed n^2/k. This completes the proof of the lemma. □

The following lemma that will be used in the sequel concerns minimum cost range assignments of at least logarithmic diameter.

Lemma 2.4. *$Cost_h(n) \in O(n^2)$, where $h\geqslant\lceil\log n\rceil$.*

Proof. We construct a "tree-layout" (cf. Fig. 2) range assignment by induction on n. Assume we have constructed the range assignment for the subchains $0..\lfloor n/2\rfloor$ and

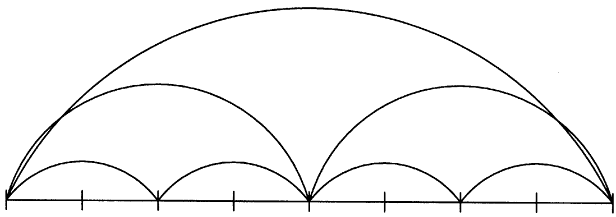

Fig. 2. Tree-layout on a chain.

$\lfloor n/2 \rfloor..n$. We extend the range assignment by adding a station at vertex $\lfloor n/2 \rfloor$ with range $\lceil n/2 \rceil$. Regarding the cost we observe that

$$Cost_{\lceil \log n \rceil}(n) \leqslant \sum_{i=1}^{\lceil \log n \rceil} 2^{i-1} \left(\frac{n}{2^i} + 1\right)^2 \in \mathrm{O}(n^2).$$

This completes the proof of Lemma 2.4. □

Theorem 2.2. *If* $h \geqslant (1+\varepsilon)\log n$, *for some constant* $\varepsilon > 0$, *then* $Cost_h(n) \in \Theta(n^2/h)$,

Proof. The lower bound of $\Omega(n^2/h)$ is immediate from Lemma 2.3. To prove the upper bound we need to construct the corresponding range assignments for the chain. Let $x = h - \log n$. Construct a range assignment as follows. Divide the whole chain into x subchains by placing stations in locations $1 + jn/x$, where $j \leqslant x$ each with range n/x. In each subchain use the range assignment of Lemma 2.4. The resulting diameter is

$$x + \log(n/x) = h - \log n + \log n - \log x \leqslant h$$

and, by Lemma 2.4, the total cost is

$$Cost_h(n) \leqslant x\left(\frac{n}{x}\right)^2 + x\,Cost_{\log(n/x)}(n/x) \leqslant 2 \cdot \frac{n^2}{x}.$$

If we define $\delta = (1+\varepsilon)/\varepsilon$ then

$$h \geqslant (1+\varepsilon)\log n = \frac{\delta}{\delta - 1}\log n.$$

Multiplying out by $\delta - 1$ and simplifying we obtain that

$$x = h - \log n \geqslant \frac{1}{\delta} h.$$

This shows that

$$Cost_h(n) \in \mathrm{O}(n^2/h)$$

and completes the proof of the theorem. □

2.2. Arbitrary point-arrangements on a line

Theorem 2.3. *If the vertices in V lie on a line, then there is a* $O(n^4)$ *time algorithm that finds a minimum cost complete range assignment.*

We will construct a minimum cost complete range assignment recursively. Suppose, without loss of generality, that the vertices $x_1,\dots,x_n$ lie on the line from left to right in the order indicated by their subscripts. A natural first attempt towards a recursive definition of a minimum cost range assignment would be to assume that at a stage $k = 1,\dots,n$ we know a minimum cost assignment R_k for $x_1,\dots,x_k$ and try to extend R_k to include x_{k+1}. Unfortunately, this does not work because the range that will be assigned to x_{k+1} may render some of the ranges of R_k unnecessarily large. A second approach would be to assume that at stage k, we know for any given vertex $x_l, l \geqslant k$ an assignment which is minimum cost among those that establish communication between any pair in $x_1,\dots,x_k$ and, additionally, have the property that x_l is within the reach of at least one vertex from $x_1,\dots x_k$. Then, in order to establish communication between any pair in $x_1,\dots,x_{k+1}$, it would be sufficient to assign to x_{k+1} a range equal to $d(x_{k+1},x_k)$. However, this also fails because for the recursive construction to be correct, it is necessary to examine the case that x_l is within the reach of x_{k+1}. The range of x_{k+1} that would guarantee this may, again, render some of the ranges of the $x_1,\dots,x_k$ unnecessarily large. Fortunately, and despite the sometimes vicious circle of induction strengthening, an even stronger recursive assumption carries through: we assume that for any $l \geqslant k$ and any $i \leqslant k$, we have an assignment which is minimum cost among those such that (i) in the communication graph, there is a path between any pair from $x_1,\dots,x_k$, (ii) x_l is within the reach of a vertex in $x_1,\dots,x_k$, and (iii) in the communication graph, any backwards edge from x_k up to x_i is free of cost. (We note that these edges enable connectivity without adding to the cost.) Below we formalize and then prove the correctness of this approach.

We start with some definitions:

Let (V,E) be a directed (not necessarily strongly connected) graph with vertices V. Let x be an additional vertex which may or may not belong to V and which we call the *receiver* vertex. A range assignment R is called *total* for $((V,E);x)$ if (i) the graph on V obtained by adding to the set E the set of edges $\{(x_i,x_j) : R(x_i) \geqslant d(x_i,x_j)\}$ is strongly connected, and (ii) there is a vertex $x_i \in V$ such that $R(x_i) \geqslant d(x_i,x)$. The cost of such an assignment is $\sum_i (R(x_i))^2$, as usual. An *optimal* assignment with respect to $((V,E);x)$ is an assignment of minimum cost which is total for $((V,E);x)$. Intuitively, such an assignment has zero cost for the edges in E and establishes communication paths between any pair of vertices in V and also between a vertex in V and the receiver vertex x, in this direction only. We define $R \in Feas((V,E);x)$ iff R is total for $((V,E);x)$ and $R \in Opt(V,E);x)$ iff R is optimal with respect to $((V,E);x)$.

If the points $x_1,\dots,x_n$ lie on a line from left to right in this order, and if x_i,x_j are any two of them with $j \geqslant i$ then $E_{i,j}$ is defined to be the set of edges $\{(x_s,x_r) : i \leqslant r < s \leqslant j\}$.

Intuitively, $E_{i,j}$ is the set of edges from *right to left* which have their endpoints among $x_i, x_{i+1}, \dots, x_j$.

We now prove the following two technical lemmas:

Lemma 2.5. *Fix a k such that $1 \leqslant k \leqslant n$. Let j, m be such that $j \leqslant k+1 \leqslant m$ and let R be an assignment on $x_1, \dots, x_{k+1}$. Finally, let $r = R(x_{k+1})$ and let R_k be the restriction of R on the set $\{x_1, \dots, x_k\}$. Assuming that not both $r = 0$ and $j = k+1$ hold, then*

- *If $r < d(x_{k+1}, x_m)$ and $r < d(x_{k+1}, x_j)$, then*

$$R \in Feas((\{x_1, \dots, x_{k+1}\}, E_{j,k+1}); x_m) \text{ iff } R_k \in Feas((\{x_1, \dots, x_k\}, E_{j,k}); x_m).$$

- *If $r \geqslant d(x_{k+1}, x_m)$ and $r < d(x_{k+1}, x_j)$, then*

$$R \in Feas((\{x_1, \dots, x_{k+1}\}, E_{j,k+1}); x_m) \text{ iff } R_k \in Feas((\{x_1, \dots, x_k\}, E_{j,k}); x_{k+1}).$$

- *If $r < d(x_{k+1}, x_m)$ and $r \geqslant d(x_{k+1}, x_j)$, then if i is the least positive integer such that $r \geqslant d(x_{k+1}, x_i)$, we have that*

$$R \in Feas((\{x_1, \dots, x_{k+1}\}, E_{j,k+1}); x_m) \text{ iff } R_k \in Feas((\{x_1, \dots, x_k\}, E_{i,k}); x_m).$$

- *If $r \geqslant d(x_{k+1}, x_m)$ and $r \geqslant d(x_{k+1}, x_j)$, then if i is the least positive integer such that $r \geqslant d(x_{k+1}, x_i)$, we have that*

$$R \in Feas((\{x_1, \dots, x_{k+1}\}, E_{j,k+1}); x_m) \text{ iff } R_k \in Feas((\{x_1, \dots, x_k\}, E_{i,k}); x_{k+1}).$$

Proof. We will only prove the third case and for this case we will only prove one direction. All other cases are analogous and even easier. So assume that

$$R \in Feas((\{x_1, \dots, x_{k+1}\}, E_{j,k+1}); x_m).$$

We will prove that

$$R_k \in Feas((\{x_1, \dots, x_k\}, E_{i,k}); x_m),$$

where i is defined as in the statement of the lemma. Let E_R be the set of edges induced on $\{x_1, \dots, x_{k+1}\}$ by R, i.e., $(x, y) \in E_R$ iff $R(x) \geqslant d(x, y)$. Define similarly E_{R_k}. We have to prove that (i) the set of edges $E_{R_k} \cup E_{i,k}$ defines a strongly connected graph on $\{x_1, \dots, x_k\}$ and (ii) there is a vertex in $\{x_1, \dots, x_k\}$ whose range is at least equal to its distance from x_m. The latter conclusion is obvious because $r < d(x_{k+1}, x_m)$ and $R \in Feas((\{x_1, \dots, x_{k+1}\}, E_{j,k+1}); x_m)$ (in this case the only possibility to reach out to the receiver vertex x_m is from a vertex in $\{x_1, \dots, x_k\}$.) To prove that conclusion (i) also holds, consider a pair x and y of vertices in $\{x_1, \dots, x_k\}$. Since $R \in Feas((\{x_1, \dots, x_{k+1}\}, E_{j,k+1}); x_m)$, there is a path p from x to y which uses edges from $E_R \cup E_{j,k+1}$. Consider an occurrence of x_{k+1} in this path, say in the consecutive edges (w, x_{k+1}) and (x_{k+1}, z). Then (w, x_{k+1}) must belong to E_R (the edges in $E_{j,k+1}$ are directed left). Therefore, (w, x_k) belongs to E_{R_k}. On the other hand, since $r \geqslant d(x_{k+1}, x_j)$, we conclude that independently of whether (x_{k+1}, z) is in E_R or in $E_{j,k+1}$, $r \geqslant d(x_{k+1}, z)$.

Therefore, by the definition of i, z is to the right of x_i, and hence $(x_k, z) \in E_{i,k}$. This proves that p can be modified to contain only edges in $E_{R_k} \cup E_{i,k}$. But then this shows conclusion (i) and also concludes the proof of the lemma. □

One more piece of notation: If $y \in V$ and r is a positive real, then $Opt((V,E);x;(y,r))$ is the class of assignments that have the least cost among the assignments R that (i) belong to $Feas((V,E);x)$ and (ii) $R(y) = r$.

The next lemma is an immediate corollary of the previous one.

Lemma 2.6. *Using the notation of the previous lemma, we have that if* $r = 0$ *and* $j = k+1$ *then*

$$Opt(\{x_1,\ldots,x_{k+1}\},E_{j,k+1});x_m;(x_{k+1},r)) = \emptyset,$$

otherwise we have:

- *If* $r < d(x_{k+1},x_m)$ *and* $r < d(x_{k+1},x_j)$, *then*

$$R \in Opt(\{x_1,\ldots,x_{k+1}\},E_{j,k+1});x_m;(x_{k+1},r))$$

$$\textit{iff } R_k \in Opt(\{x_1,\ldots,x_k\},E_{j,k});x_m).$$

- *If* $r \geqslant d(x_{k+1},x_m)$ *and* $r < d(x_{k+1},x_j)$, *then*

$$R \in Opt(\{x_1,\ldots,x_{k+1}\},E_{j,k+1});x_m;(x_{k+1},r))$$
$$\textit{iff } R_k \in Opt(\{x_1,\ldots,x_k\},E_{j,k});x_{k+1}).$$

- *If* $r < d(x_{k+1},x_m)$ *and* $r \geqslant d(x_{k+1},x_j)$, *then if* i *is the least positive integer such that* $r \geqslant d(x_{k+1},x_i)$, *we have that*

$$R \in Opt(\{x_1,\ldots,x_{k+1}\},E_{j,k+1});x_m;(x_{k+1},r))$$
$$\textit{iff } R_k \in Opt(\{x_1,\ldots,x_k\},E_{i,k});x_m).$$

- *If* $r \geqslant d(x_{k+1},x_m)$ *and* $r \geqslant d(x_{k+1},x_j)$, *then if* i *is the least positive integer such that* $r \geqslant d(x_{k+1},x_i)$, *we have that*

$$R \in Opt(\{x_1,\ldots,x_{k+1}\},E_{j,k+1});x_m;(x_{k+1},r))$$
$$\textit{iff } R_k \in Opt(\{x_1,\ldots,x_k\},E_{i,k});x_{k+1}).$$

We are now in a position to give our recursive construction:

Proof of Theorem 2.3. Assume that at stage k we know a range assignment in

$$Opt(\{x_1,\ldots,x_k\},E_{i,k});x_l)$$

for any i,l such that $i \leqslant k \leqslant l$. Under this assumption, for any j,m such that $j \leqslant k+1 \leqslant m$, we will recursively construct a range assignment

$$R \in Opt(\{x_1,\ldots,x_{k+1}\},E_{j,k+1});x_m).$$

Of course, this will prove the theorem, since an optimal assignment for V is one in

$$Opt(\{x_1,\ldots,x_n\},E_{n,n});x_n).$$

To construct the required R, we examine all possible values of $R(x_{k+1})$. There are $k+2$ of them as it only makes sense to have a range that extends from x_{k+1} to either one of the $x_1,\ldots,x_k$ or to x_m or one which is zero (this second case occurs only when $x_k = x_{k+1}$). For each such possible value r of $R(x_{k+1})$, find an assignment (if any) in $\mathrm{Opt}((\{x_1,\ldots,x_k\},E_{j,k+1});x_m;(x_{k+1},r))$ by making use of the recursion stack and Lemma 2.6. The one with least cost among all of them is obviously the required R. Notice that this algorithm takes time $O(n^4)$. □

3. Range assignments in three dimensions

3.1. NP-hardness

This section is devoted to the proof that the problem of finding a minimum cost complete range assignment for a given set of points in the three-dimensional Euclidean space is NP-hard. We formulate the corresponding decision problem as follows.

Problem **RANGE**
Instance: A set of points A in the three-dimensional space, a positive number q.
Question: Does there exist a complete range assignment for A of total cost at most q?

We will show a reduction from the vertex cover problem for connected planar cubic graphs, known to be NP-hard (cf. [8]). The decision version of it is formulated as follows.

Problem **COVER**
Instance: An undirected connected planar cubic graph G, a positive integer k.
Question: Does G have a vertex cover of size at most k, i.e., is there a set of vertices S of size at most k, such that each edge of G has at least one endpoint in S ?

We first need some auxiliary notions, facts and constructions. A *subdivision* of a graph G is a graph H resulting from G by adding new vertices of order 2 on edges of G (every edge is replaced by a chain and distinct edges are replaced by vertex disjoint chains). The new vertices of order 2 are called *subdivision vertices*. An *even subdivision* is a subdivision in which an even number of vertices are added on every edge.

Lemma 3.1. *Let G be a graph with edges $e_1,\ldots,e_k$. Let H be a subdivision of G such that $2x_i$ new vertices are added on edge e_i, for all $i \leqslant k$. Let $x = \sum_{i=1}^{k} x_i$. Then G has a vertex cover of size $\leqslant r$ iff H has a vertex cover of size $\leqslant r + x$.*

Proof. Suppose that G has a vertex cover C of size $\leqslant r$. Construct the following set D of vertices of H. Take all vertices from C and on every edge e_i of G take x_i

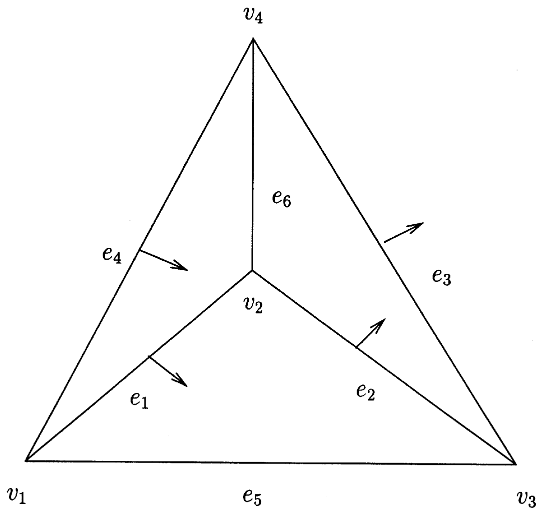

Fig. 3. Sector assignment.

subdivision vertices starting from the end belonging to C and skipping every other vertex. The resulting set D is a vertex cover of H of size $\leqslant r+x$.

Consider a vertex cover D of H of size $\leqslant s$ and consider any edge e_i of G. If an end of e_i is in D then at least x_i subdivision vertices of e_i must be in D. If no end of e_i is in D then at least x_i+1 subdivision vertices of e_i must be in D. In both cases remove x_i subdivision vertices of e_i from D and in the latter case replace the remaining subdivision vertex by one of the ends of e_i. The resulting set C of vertices is a vertex cover of G of size $\leqslant s-x$. □

Until the end of this section, G denotes a connected planar cubic graph. For any such G consider a fixed planar representation P. Edges incident to any vertex yield a division of the plane into 3 *sectors*. Take any vertex v_1 of G and any edge e_1 incident to it and assign to e_1 one of the two neighboring sectors. Consider the other end v_2 of e_1 and the other edge e_2 neighboring the chosen sector. Assign to e_2 the other neighboring sector and go to the other end v_3 of e_2. Proceed in this way, at each vertex trying to assign a free sector to the new edge neighboring the previously assigned sector. This can be done iff the graph G is bipartite. If G is not bipartite, call edges for which a conflict occurred – *special*. In Fig. 3 we show a non-bipartite graph and a sector assignment depicted by arrows, with the special edges e_5 and e_6. For any planar representation of G this *sector assignment* can be constructed in polynomial time with at most one special edge incident to each vertex.

Consider a rectangular grid on the plane with grid points having integer coordinates. Call unit length grid edges – *grid lines*. Consider a planar representation of a graph G such that vertices are mapped to grid points and edges are mapped to polygonal lines composed of grid lines. Call this a *Valiant representation* of G. It follows from

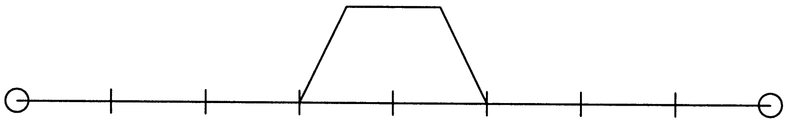

Fig. 4. Augmented chain in a normal picture.

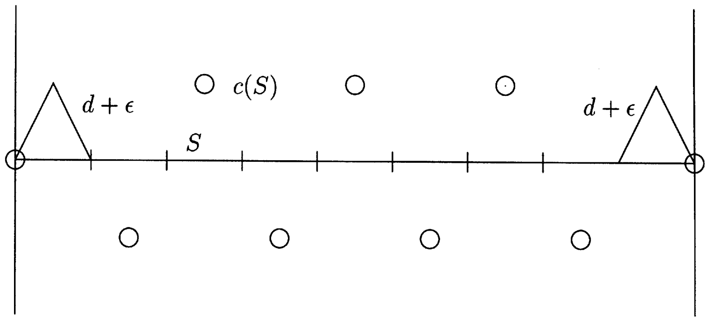

Fig. 5. Companion points in the ε-envelope. The figure depicts the sector assigned to e and the chain corresponding to e.

[17] that a Valiant representation of G always exists and can be constructed in polynomial time. (We note that in a Valiant representation edges correspond to distinct non-overlapping polygonal lines.)

Take a Valiant representation R of G. Subdivide every grid line in R into 8 equal *segments* of length $d=\frac{1}{8}$. In the resulting figure every edge e_i of G is mapped to a rectangular polygonal line c_i consisting of an even number of segments. Take 2 middle segments of each chain c_i and replace them by a chain of 3 segments as shown in Fig. 4.

Call the resulting figure a *normal picture of* G. Every edge of G is mapped to a polygonal line consisting of an *odd* number of segments, thus a normal picture of G yields a graph $P(G)$ which is an even subdivision of G.

Let P be a normal picture of G in the plane $\mathscr{P}$. Let X be the set of vertices of $P(G)$ and ε a positive number less than d. We define the *ε-envelope* of G as the set $E=X\cup Y$, where Y is a set of points in the three-dimensional space constructed as follows.

First, construct a sector assignment corresponding to P (P is homeomorphic to a planar representation of G). Next, for any segment S of P construct a *companion point* $c(S)$. This point is at distance $d+\varepsilon$ from both ends of S. If S is a segment corresponding to a special edge in the sector assignment, $c(S)$ is in the plane perpendicular to $\mathscr{P}$, containing S. Otherwise, $c(S)$ is in one of the two sectors neighboring S, determined as follows. If S is one of the end segments of its chain then $c(S)$ is in the sector assigned to the corresponding edge. For other segments sectors alternate (cf. Fig. 5). Recall that each chain has an odd number of segments. The set Y consists of companion points for all segments of P.

It follows from the construction that the distance between any two points in the ε-envelope of G is at least d. Moreover, every point $c(S) \in Y$ is at distance $d+\varepsilon$ from the ends s_1, s_2 of the segment S. Consider any point t in E, different from $c(S)$, s_1 and s_2. Considering all possible cases it is easy to see that the distance between $c(S)$ and t is smallest when $t = c(S')$, segments S and S' are adjacent and perpendicular, and one of the points $c(S)$ or $c(S')$ is in the plane perpendicular to $\mathscr{P}$ (situated there because the corresponding edge was special). This smallest distance is then at least αd, where

$$\alpha = \sqrt{2 - \frac{\sqrt{3}}{2}} \approx 1.06.$$

Theorem 3.1. *The problem* **RANGE** *is* NP-*hard.*

Proof. We show a polynomial reduction from **COVER**. Let G be a connected planar cubic graph and k a positive integer. Let N be the set of vertices of G. Let d and α be as before. Let P be a normal picture of G in the plane $\mathscr{P}$ and $H = P(G)$. Let X be the set of vertices of $P(G)$, $|X| = m$, $X_1 = X \setminus N$, $|X_1| = 2x$. Let ε be a positive number satisfying the following two conditions:

$$\varepsilon < (\alpha - 1)d, \tag{1}$$

$$\varepsilon < \frac{(\alpha^2 - 1)d^2}{(m+1)(2d+1).} \tag{2}$$

Inequality (1) implies $\varepsilon < 1$. Hence inequality (2) implies

$$m(2d+\varepsilon)\varepsilon + 2d\varepsilon + \varepsilon^2 < (\alpha^2 - 1)d^2$$

and consequently for all y,

$$md^2 + m(2d\varepsilon + \varepsilon^2) + y(d+\varepsilon)^2 < md^2 + (y-1)(d+\varepsilon)^2 + (\alpha d)^2. \tag{3}$$

By Lemma 3.1 G has a vertex cover of size at most k iff H has a vertex cover of size at most $z = k + x$. Construct an ε-envelope $E = X \cup Y$ of G. The following claim concludes the proof of the theorem. $\square$

Claim. *The graph* H *has a vertex cover of size at most* z *iff the envelope* E *has a complete range assignment of cost at most*

$$q_z = md^2 + z(2d\varepsilon + \varepsilon^2) + y(d+\varepsilon)^2.$$

Proof. Suppose that H has a vertex cover C of size at most z. Assign range $d + \varepsilon$ to all points in $C \cup Y$ and range d to all points in $X \setminus C$. Every point of X can be reached from any other point of X via a path in H. Any end of a segment S is in the range of $c(S)$ and $c(S)$ is in the range of the end of S belonging to the cover C. Hence this is a complete range assignment for E. Its cost is clearly at most q_z.

Conversely, suppose we are given a complete range assignment for E, of cost at most q_z. Every point in X must have range at least d because it is at distance at least d from any other point in E. Inequality (1) implies $d + \varepsilon < \alpha d$. Hence every point in Y is at distance at least $d + \varepsilon$ from any other point in E. Consequently, the range of every point in Y must be at least $d + \varepsilon$.

Suppose that some point $c(S)$ in Y is in the range of a point u which is not an end of S. Then the range of u is at least αd and hence the cost of the range assignment is at least

$$md^2 + (y-1)(d+\varepsilon)^2 + (\alpha d)^2.$$

(Here we put $y = |Y|$.) In view of inequality (3) this is larger than q_m. Since m is an upper bound on the size of any vertex cover of H, this yields a contradiction.

Hence every point $c(S)$ in Y is in the range of at least one of the ends of segment S. It follows that for every segment S of P one of its ends must have range at least $d + \varepsilon$. Let $C \subset X$ be the set of vertices of P that have range at least $d + \varepsilon$. Hence C is a vertex cover of H. The cost of the range assignment is at least

$$|C|(d+\varepsilon)^2 + |X \setminus C|d^2 + y(d+\varepsilon)^2 = md^2 + |C|(2d\varepsilon + \varepsilon^2) + y(d+\varepsilon)^2 = q_{|C|}.$$

It follows that $q_{|C|} \leqslant q_z$ and consequently $|C| \leqslant z$, i.e., H has a vertex cover of size at most z. This concludes the proof of the claim and of the theorem. □

3.2. An approximation algorithm

As was noted above, the minimum cost complete range assignment problem for a set of points in three dimensions is NP-hard. In this section we describe an $O(n^2)$ time approximation algorithm for this problem with a ratio bound of 2, i.e., the algorithm finds a solution within a factor of 2 of the optimal.

Given a set $V = \{x_1, \dots, x_n\}$ of points in 3-space the algorithm proceeds as follows:

1. Construct an undirected weighted complete graph $G(V)$ with vertices V and where the weight of the edge between x_i and x_j is $d(x_i, x_j)^2$ for all i and j.
2. Find a minimum weight spanning tree T of $G(V)$.
3. For $i = 1, \dots, n$ assign the range of x_i to be the maximum of $d(x_i, x_j)$ over j such that $\{x_i, x_j\}$ is an edge in T.

Clearly, the algorithm runs in $O(n^2)$ time and the resulting range assignment is complete (since at the very least it contains all of the edges of the spanning tree in both directions). Further we can establish:

Theorem 3.2. *Let $OPT(V)$ be the minimum cost of a complete range assignment for V and let $APP(V)$ be the cost of the complete range assignment for V found by the above algorithm. Then $APP(V) < 2 \cdot OPT(V)$.*

Proof. Let $MST(V)$ be the cost of the minimum weight spanning tree T of $G(V)$. The theorem follows immediately from the following claims:

Claim 1. $OPT(V) > MST(V)$.

Proof. From any optimal assignment we can construct a spanning tree of $G(V)$ by choosing any vertex, constructing a shortest path destination tree with the chosen vertex as the destination (i.e., a tree rooted at the chosen vertex, with all edges directed towards the root representing a minimum hop path from each vertex to the root) and changing the directed edges of the destination tree to the corresponding undirected edges in $G(V)$. Since each of the $n-1$ vertices other than the root vertex must have a range assigned which establishes the edges of the destination tree, $OPT(V)$ is greater than the weight of the resulting spanning tree which in turn is greater than or equal to $MST(V)$. □

Claim 2. $APP(V) < 2 \cdot MST(V)$.

Proof.

$$APP(V) = \sum_{i=1}^{n} \max_{\{j \mid \{x_i, x_j\} \in T\}} d(x_i, x_j)^2 < \sum_{i=1}^{n} \sum_{\{j \mid \{x_i, x_j\} \in T\}} d(x_i, x_j)^2 = 2 \cdot MST(V). \quad \square$$

4. Conclusion and open problems

In one dimension, we gave asymptotically tight bounds on the minimum cost of a range assignment with diameter h on equidistant points when h is constant and when $h \geqslant (1+\varepsilon)\log n$, for some constant $\varepsilon > 0$. When h is between these ranges the precise bound is unknown. For the case of arbitrarily distributed points on a line we gave an $O(n^4)$ algorithm for finding the minimum cost complete range assignment. We believe our techniques may be extendable to find the minimum cost assignment of a given diameter.

In three dimensions we showed the problem of finding the minimum cost complete range assignment is NP-hard and gave an approximation algorithm optimal to within a factor of two. We conjecture the problem remains NP-hard in two dimensions. Note that the approximation algorithm works in two dimensions as well.

Acknowledgements

Many thanks to the anonymous referee for pointing to an error and for many helpful suggestions that improved the presentation of the paper.

References

[1] N. Alon, A. Bar-Noy, N. Linial, D. Peleg, A lower bound for radio broadcast, J. Comput. System Sci. 43 (1991) 290–298.

[2] N. Alon, A. Bar-Noy, N. Linial, D. Peleg, Single round simulation on radio networks, J. Algorithms 14 (1992) 188–210.
[3] E. Arikan, Some complexity results about packet radio networks, IEEE Trans. Inform. Theory IT-30 (1984) 681–685.
[4] R. Bar-Yehuda, O. Goldreich, A. Itai, On the time-complexity of broadcast in multi-hop radio networks: an exponential gap between determinism and randomization, J. Comput. System Sci. 45 (1992) 104–126.
[5] I. Chlamtac, S. Kutten, On broadcasting in radio networks, IEEE Trans. Commun. COM-33 (1985) 1240–1246.
[6] A. Ephremides, T. Truong, Scheduling broadcasts in multihop radio networks, IEEE Trans. Commun. 30 (1990) 456–461.
[7] I. Gaber, Y. Mansour, Broadcast in radio networks, Proc. 6th Ann. ACM-SIAM Symp. on Discrete Algorithms, 1995, pp. 577–585.
[8] M.R. Garey, D.S. Johnson, Computers and Intractability – A Guide to the Theory of NP-Completeness, Freeman, New York, 1979.
[9] R. Kahn, S. Gronemeyer, J. Burchfiel, R. Kunzelman, Advances in packet-radio technology, Proc. of the IEEE 66 (1978) 1468–1496.
[10] L. Kirousis, E. Kranakis, D. Krizanc, A. Pelc, Power consumption in packet radio networks, Proc. 14th Symp. on Theoretical Aspects of Computer Science (STACS'97), Lecture Notes in Computer Science, Vol. 1200, Springer, Berlin, pp. 363–374.
[11] K. Pahlavan, A. Levesque, Wireless Information Networks, Wiley, New York, 1995.
[12] S. Ramanathan, E. Lloyd, Scheduling broadcasts in multi-hop radio networks, IEEE/ACM Trans. Networking 1 (1993) 166–172.
[13] R. Ramaswami, K. Parhi, Distributed scheduling of broadcasts in a radio network, Proc. 8th Annual Joint Conf. of IEEE Computer and Communication Societies INFOCOM, 1989, pp. 497–504.
[14] K. Ravishankar, S. Singh, Broadcasting on [0,L], Discrete Appl. Math. 53 (1994) 299–320.
[15] A. Sen, M. Huson, A New Model for Scheduling Packet Radio Networks, Proc. 15th Annual Joint Conf. of the IEEE Computer and Communication Societies INFOCOM, 1996, pp. 1116–1124.
[16] H. Takagi, L. Kleinrock, Optimal transmission ranges for randomly distributed packet radio terminals, IEEE Trans. Commun. COM-32 (1984) 246–257.
[17] L. Valiant, Universality considerations in VLSI circuits, IEEE Trans. on Comput. C-30 (1981), 135–140.