Coloring Powers of Planar Graphs*

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Abstract

We give nontrivial bounds for the chromatic number of power graphs G^k of a planar graph G. This is done in terms of the inductiveness of the graph, which naturally relates to a greedy algorithm. We derive a sharp upper bound for the inductiveness of the square graph G^2, in terms of the maximum degree Δ of G, for large values of Δ. In general, we show the inductiveness and chromatic number of G^k to be $O(\Delta^{\lfloor k/2 \rfloor})$, which is tight. This leads to a 2-approximation for coloring squares of planar graphs, improving on the previous best factor of 9.

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1 Introduction

The k-th power G^k of a graph G is defined on the same set of vertices as G, and has an edge between any pair of vertices of distance at most k in G. The topic of this paper is the coloring of power graphs, or equivalently coloring the underlying graphs so that vertices of distance at most k receive different colors. We focus on the planar case, long the center of attention for graph coloring. We upper-bound the chromatic number by the *inductiveness* of the graph, $\mathrm{ind}(G)$, defined to be $\max_{H \subseteq G}\{\min_{v \in H}(d_H(v))\}$, where H runs through all induced subgraphs of G. Inductiveness leads to an ordering of the vertices, $\{v_1, \ldots, v_n\}$, such that the number of pre-neighbors of any v_i, $d^+(v_i) = |\{v_j \in N_G(v_i) : j > i\}|$, is at most $\mathrm{ind}(G)$.

The problem of coloring squares of graphs has been studied recently for its applications to frequency allocation. Transceivers in a radio network communicate using channels at given radio frequencies. Graph coloring formalizes this problem well when the constraint is that nearby pairs of transceivers cannot use the same channel due to interference. However, if two transceivers are using the same channel and both are adjacent to a third station, a clashing of signals is experienced at that third node. This can be avoided by additionally requiring all neighbors of a node to be assigned different colors, i.e. that vertices of distance at most two receive different colors. This is equivalent to coloring the square of the underlying network. Another application of this problem, from a completely different direction, is that of approximating certain Hessian matrices, see [9].

Observe that neighbors of a node in a graph form a clique in the square of the graph. Thus, the minimum number of colors needed to color any square graph is at least $\Delta + 1$, where $\Delta = \Delta(G)$ is the maximum degree of the original graph. As a result, the number of colors used by our algorithms on power graphs will necessarily be a function of Δ. We are particularly interested in the asymptotic behavior as Δ grows.

The first reference on coloring squares of planar graphs is by Wegner [16], who gave bounds on the clique number of such graphs. In particular, he gave an instance for which the clique number is at least $3\Delta/2 + 1$ (which is largest possible), and conjectured this to be an upper bound on the chromatic number, for Δ large. Some work has been done on the case $\Delta = 3$, as listed in [5, Problem 2.18].

McCormick [9] showed that the problem of coloring the power of a graph is NP-complete, for any fixed power, and a later proof was given by Lin and Skiena [8]. McCormick gave a greedy algorithm with a $O(\sqrt{n})$-approximation for squares of general graphs. Heggernes and Telle [4] showed that determining if the square of a cubic graph can be colored with 4 colors or less is NP-complete, while it is easily determined if 3 colors suffice.

Ramanathan and Lloyd [13, 12] showed the problem of coloring squares of planar graphs to be NP-complete. They also gave an algorithm with a performance ratio of 9, which was the best result previously known. More generally, they gave a $O(q)$-ratio for graphs of inductiveness q. Krumke, Marathe and Ravi [7] showed more precisely that the ratio is $2q - 1$. They also gave a polynomial algorithm for graphs of both bounded treewidth and bounded degree, and used that to give a 2-approximation for bounded-degree planar graphs. Sen and Huson [14] showed that coloring squares of unit-circle graphs is NP-complete, while a constant approximation algorithm was given in [15].

This paper attempts to further the knowledge on the colorability and inductiveness of powers of planar and general graphs. We first show that for large values of Δ, squares of planar graphs are $9\Delta/5+1$-inductive, implying a $9\Delta/5+2$-coloring. This is the tightest possible, since there are graphs attaining this inductiveness. We combine this with previous results for bounded-degree graphs to obtain a 2-approximation for coloring that holds for all values of Δ.

We next show that the power G^k of a planar graph G is $O(\Delta^{\lfloor k/2 \rfloor})$-inductive, for any $k \geq 1$. This gives an asymptotically tight algorithmic bound for the chromatic number of the power graph. This yields the first constant factor approximation for coloring cubes of planar graphs. However, the real strength of the current bounds are in giving absolute bounds on the number of colors used by the algorithm, as opposed to relative approximations, and thus implicitly bounding the number of colors used by an optimal solution.

Note the fine distinction between coloring the power graph G^k, and finding a *distance-k* coloring of G. The resulting coloring is naturally the same. However, in the latter case, the original graph is given. While it is easy to compute the power graph G^k from G, Motwani and Sudan [10] showed that it is NP-hard to compute the k-th root G of a graph G^k. All of the algorithm presented in this paper work without knowledge of the underlying root graph.

Zhou et al. [17] have in independent work given a polynomial algorithm for distance-d coloring partial-k trees, for any constants d and k. As indicated in Section 4, this immediately implies a 2-approximation for distance-d coloring planar graphs for any d, thus improving and generalizing our approximation results in that section. Their algorithm, however, appears to be only of a theoretical interest, with a large polynomial complexity. In comparison, the results given here apply to the most natural greedy algorithm.

The rest of the paper is organized as follows. We bound the inductiveness of squares of planar graphs in Section 2, and general powers of planar graphs in Section 3. We consider the implications of these bounds to approximate colorings of powers of planar graphs in Section 4.

NOTATION: The degree of a vertex v within a graph G is denoted by $d_G(v)$ or simply by $d(v)$ when there is no danger of ambiguity. The maximum degree of G is denoted by $\Delta = \Delta(G)$. For a vertex v denote by $d_k(v)$ the degree of v in G^k. Distance between two vertices u and v in a graph is the number of edges on the shortest path from u to v, and is denoted by $d_G(u, v)$. Let $G[W]$ denote the subgraph of G induced by vertex subset W.

2 Squares of Planar Graphs

We first take a look at the main technique we use to derive bounds on the inductiveness of a square graph (and more generally, power graphs). The argument that is used, e.g., to show that planar graphs are 5-inductive is the following. By Euler's theorem, any planar graph contains a vertex of degree at most 5. Place one such node first in the inductive ordering, and remove it from the graph. Now the remaining graph is planar, so inductively we obtain a 5-inductive ordering.

The bound of 5 on the minimum degree of a planar graph also implies that squares of planar graphs are of minimum degree at most 5Δ. That would seem to imply a 5Δ-ordering. However, when a vertex is deleted from the graph, its incident edges are deleted as well, so vertices originally distance two apart may become much further apart in the remaining graph. An example of this is shown in Figure 1. Namely, the problem is that an induced subgraph does not preserve the paths of length two between vertices within the subgraph. The upshot is that degrees in the remaining graph do not adequately characterize the degree in the remaining part of the square of the graph. Our solution is to replace the *deletion* of vertex by the *contraction* of an incident edge.

The *contraction* of an edge uv in graph G is the operation of collapsing the vertices u and v into a new vertex, giving the simple graph G/uv defined by $V(G/uv) = V(G) \setminus \{v\}$ and

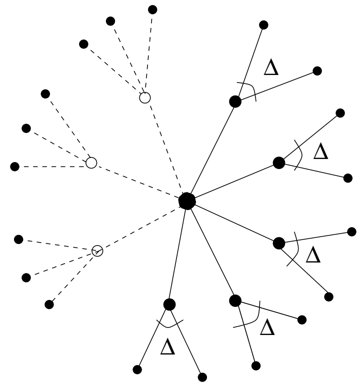

Figure 1: The center vertex has degree 5 but $5\Delta + 9$ distance-2 neighbors. Whitened vertices have been deleted from the graph.

$E(G) = \{ww' \in E(G) : w, w' \neq v\} \cup \{uw : vw \in E(G)\}$. Observe that if G is planar, then G/uv is also planar. This is a property of various classes of graphs that are *closed under minor operations*. By the classic theorem of Kuratowski, planar graphs are precisely those graphs for which repeated contractions do not yield supergraphs of K_5 or $K_{3,3}$. Minor-closedness holds for various other classes of graphs, e.g. partial-k trees, but not d-inductive graphs in general.

Since our bounds on the inductiveness are functions of Δ, it is imperative that the contraction operations do not increase the maximum degree. In summary, in order to show that a power graph G^k is q-inductive, where q is necessarily a function of the maximum degree Δ, we show the existence of a vertex $v \in V(G^k) = V(G)$ such that

- $d_k(v) \leq q$, and
- v has a neighbor u such that $d(u) + d(v) - 2 \leq \Delta$. (1)

If such an edge uv exists, then the contraction of uv in G yields yield a simple planar graph G/uv whose distance function is dominated by the one on G (i.e. distances in G/uv are at most those in G). Further, by the second condition, the maximum degree of G/uv stays at most Δ.

2.1 Example applications of the contraction technique

We first illustrate the technique on simpler examples.

Consider graphs that are 2-inductive and minor-closed, e.g. partial-2 trees or series-parallel graphs. We inductively choose a vertex of degree at most 2 in the graph and contract one of its incident edges. In this case, the degree of each of its remaining neighbors does not not increase. Thus, in each step, at most 2Δ vertices are within distance at most 2 of the selected vertex. That is, we satisfy (1) with $q = 2\Delta$, leading to a 2Δ-inductive ordering of the square graph.

Theorem 2.1 *Squares of partial-2 trees are 2Δ-inductive.*

Our second example yields a bound on the inductiveness of planar graphs of small degree that improves on the 9Δ-bound of [13] for 5-inductive graphs. A theorem of Kotzin [6] states that a maximal planar graph contains an edge uv such that $d(u) + d(v) \leq 13$. We first argue that this implies that any maximal planar graph G with $\Delta(G) \geq 11$ is $5\Delta + 6$-inductive. We find an edge as guaranteed by Kotzin's theorem, select the vertex of lower degree, contract the edge, and inductively apply the argument on the resulting maximal planar graph. The degree of the lower degree vertex u is at most 6, and that of v at most $13 - d(u)$ (including the edge uv), thus the number of distance-2 neighbors of u is at most $(d(u)-1)\Delta + (13 - d(u) - 1) \leq 5\Delta + 6$. The degree of the new contracted edge is at most $(d(u)-1)+(d(v)-1) \leq 11$, hence maximum degree does not increase. The contracted graph is also maximally planar, hence this yields a $5\Delta + 6$-inductive ordering of G^2.

For a non-maximal planar graph G, we first form an arbitrary maximal supergraph G', find an inductive ordering as above, and use that to color G^2. Consider a vertex u and let G'_v be the contracted subgraph when v was selected. u had at most 6 neighbors in G'_u (including v of degree at most $13 - d_{G'_v}(u)$). Each neighbor w was either a contracted node of degree at most 11, or a node that had not received any new neighbors. In the latter case, the degree of w in G is at most $\Delta(G)$; the other neighbors of w do not count as neighbors of u in G^2, unless it is through some other path. Hence, we have a $5\Delta + 6$-inductive ordering of the square of any planar graph with $\Delta \geq 11$. We can use that to improve the 9Δ inductiveness bound of [13] for every value of Δ. For smaller values of Δ, we know that any graph is trivially Δ^2-inductive, and the above also gives us an upper bound of 61. In particular, we have that the square of any planar graph is 8Δ-inductive. We summarize these arguments in the following theorem.

Theorem 2.2 *If G is a planar graph, then* $\text{ind}(G^2) \leq 8\Delta(G)$. *If $\Delta(G) \geq 11$, then* $\text{ind}(G^2) \leq 5\Delta(G) + 6$.

2.2 Sharp upper bound for large degree graphs

We now turn to the main result of this section, which is that when G is planar and Δ large enough, then G^2 is $\lfloor \frac{9\Delta(G)}{5} \rfloor + 1$-inductive. The following lemma is the key to this result.

Lemma 2.3 *Let G be a simple planar graph of maximum degree $\Delta \geq 26$. Then there exists a vertex $v \in V(G)$ satisfying one of the following conditions:*

1. *$d(v) \leq 25$ and at most one neighbor of v has degree ≥ 26.*
2. *$d_2(v) \leq \lfloor \frac{9}{5}\Delta \rfloor + 1$ and only two neighbors of v in G have degree ≥ 26.*

Proof. We assume that we have a fixed planar embedding of G, and hence G is a plane graph. Let $V_h = \{v \in V(G) : d(v) \geq 26\}$ and $V_l = V(G) \setminus V_h$. If there is a vertex in V_l with at most one neighbor in V_h, then we are done, so assume the contrary.

Call a cycle of four vertices in G *forbidden*, if exactly two opposite vertices of the cycle are in V_h and the enclosed region formed by the cycle in the plane properly contains at least one vertex in V_h. If G contains a forbidden 4-cycle then let G' be the subgraph of G induced by the region bounded by a minimal such 4-cycle. (Here, minimal means that no other 4-cycle is inside.) If G contains no such cycle then let G' be G.

Consider now the multigraph H with vertex set $V_h \cap V(G')$ and with colored edges defined as follows. For each edge uw in $E(G')$ with both $u, w \in V_h$ connect u and w with a red edge. For each vertex $v \in V_l$ adjacent to u and $w \in V_h$ in G' and to no other vertex in V_h, connect u and w in H with a green edge. Finally for $v \in V_l$ adjacent to $u_1, u_2, \ldots, u_k \in V_h$ in G' in a clockwise order for $k \geq 3$, connect u_1 to u_2, u_2 to u_3,…,u_{k-1} to u_k and u_k to u_1 with blue edges in H.

Since G is planar, we note that H is also a planar multigraph. Hence, we can assume we have a drawing of H in the plane such that

1. The vertices of H have the same configuration as they have in the plane graph G.

2. For every pair $\{u, w\}$ of vertices of H connected by green or blue edges, their order with respect to u and w is the same as the order of the corresponding vertices of V_l.

By our assumption there is no vertex in V_l with at most one neighbor in V_h in G and hence in G'. Therefore, the degree of a vertex in H is at least that in G'.

Using Euler's formula for planar graphs, it is easy to show that there are at least three vertices of $V(H) = V_h \cap V(G')$ with at most 5 neighbors in H, and hence there is such a vertex $v \in V(H) \subseteq V(G')$ that is not on the 4-cycle defining G' (if G' was so defined.)

Consider now a neighbor u of this $v \in V(H)$. Let m_{uv} be the multiplicity of the edge uv in H. By our definition of G' there are at most two blue edges connecting u and v since the third one would imply a forbidden 4-cycle within G'. Also, there is only one red edge connecting u and v. Hence, if $m_{uv} \geq 4$ there are at least $m_{uv} - 3 \geq 1$ green edges connecting u and v in H. We note that all the blue and green edges connecting u and v in H correspond to different vertices of V_l in G'.

Let c_{uv} be the number of common neighbors of u and v in G' (if u and v are connected in G', then both u and v are counted as well.) The combined closed neighborhood of u and v in G' has precisely $(d_{G'}(u) + 1) + (d_{G'}(v) + 1) - c_{uv}$ vertices. Since $m_{uv} \leq c_{uv}$ (in fact, $m_{uv} + 1 \leq c_{uv}$, if u and v are connected in G'), we have that this closed neighborhood of u and v in G' contains at most $(d_{G'}(u) + 1) + (d_{G'}(v) + 1) - m_{uv}$ vertices.

Letting w run through all the neighbors of v in H, we note that $\sum_w m_{vw} = d_H(v) \geq d_{G'}(v)$. Since v has at most 5 neighbors in G', there must be a neighbor u of v such that $m_{uv} \geq \lceil d_{G'}(v)/5 \rceil$ and hence the combined neighborhood of u and v is at most

$$\begin{aligned} & d_{G'}(v) + d_{G'}(u) + 2 - \left\lceil \frac{d_{G'}(v)}{5} \right\rceil \\ = & \left\lfloor \frac{4d_{G'}(v)}{5} \right\rfloor + d_{G'}(u) + 2 \\ \leq & \left\lfloor \frac{9\Delta(G')}{5} \right\rfloor + 2 \\ \leq & \left\lfloor \frac{9\Delta(G)}{5} \right\rfloor + 2. \end{aligned}$$

Since $v \in V(H) \subseteq V_h$ is in the interior of the 4-cycle defining G', we have $d_{G'}(v) = d_G(v) \geq 26$ and hence $m_{uv} \geq \lceil d_{G'}(v)/5 \rceil \geq 6$. Hence, u and v are connected by at least 5 non-red edges. Choose 5 consecutive non-red edges between u and v, and let $z_1, z_2, \ldots, z_5$ be the neighbors of u and v in G', in a clockwise order, corresponding to these chosen non-red edges. The edges corresponding to z_2, z_3 and z_4 are green, since otherwise we would have a forbidden 4-cycle within G'.

As reference, we show in Fig. 2 the common neighborhood in G of two vertices u and v, along with the the corresponding multigraph. Vertices in V_h are in black, blue vertices are grey, and green vertices are white. Here c_{uv} is 6, including one red edges, two blue, and three green.

Now, if z_i, $i \in \{2, 3, 4\}$, is adjacent to a vertex in V_l that does not represent a green nor blue edge between u and v, then by our assumption that every vertex in V_l has at least two neighbors in V_h in the graph G', one of these neighbors in V_h must be distinct from u and v and therefore contained in the region formed by the 4-cycle (u, z_{i-1}, v, z_{i+1}). Again this would imply a forbidden 4-cycle and contradict our definition of G'.

Therefore, the only vertices of V_l that z_i can possibly be adjacent to in G' are z_{i-1} and z_{i+1}. In particular, the neighbors of z_3 in G' are among $\{u, v, z_2, z_4\}$, and the neighbors of z_2 and z_4

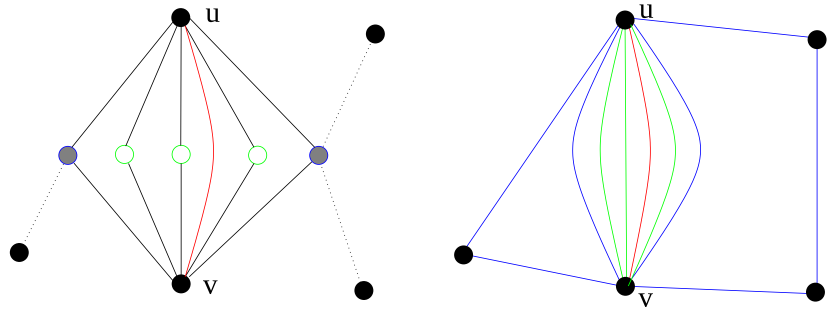

Figure 2: Example of a common neighborhood and the corresponding multigraph

are among $\{u, v, z_1, z_3\}$ and $\{u, v, z_3, z_5\}$ respectively. In any case, the combined neighborhood of z_2 and z_4 is contained in the closed combined neighborhood of u and v. Hence the number of vertices of distance at most 2 from z_3 are at most $\lfloor \frac{9\Delta(G)}{5} \rfloor + 2$ (including z_3 itself). □

Theorem 2.4 *If G is a planar graph with maximum degree $\Delta \geq 749$, then G^2 is $\lfloor \frac{9}{5}\Delta \rfloor + 1$-inductive.*

Proof. Assume that $\Delta \geq 25 + 25 - 2$ and that we have a vertex v of G which satisfies the first condition of Lemma 2.3. If v has a neighbor u of degree 25 or less, then $d_2(v) \leq 600 + \Delta$, and moreover $d(v) + d(u) - 2 \leq \Delta$. If v has no neighbor of degree 25 or less, then it has only one neighbor u. In this case $d_2(v) \leq \Delta$ and $d(v) + d(u) - 2 \leq \Delta$.

In the proof of Lemma 2.3 we assumed that there is no vertex in V_l with at most one neighbor of V_h. In that case there is a vertex of G, called z_3 in the last paragraph of the proof, with $d_2(z_3) \leq \lfloor \frac{9}{5}\Delta \rfloor + 1$. Also, z_3 has at most two neighbors z_2 and z_4 of V_l. If z_3 has no neighbors of V_l (that is, is connected to neither z_2 nor z_4), then since the only neighbors of z_3 in V_h are u and v, we have $d(z_3) + d(v) - 2 = d(z_3) + d(u) - 2 \leq \Delta$. If z_3 has a neighbor z_1 or z_2 of V_l, say z_1, then, $d(z_3) + d(z_1) - 2 \leq \Delta$.

In any case, we see that we can always find a vertex w of G with $d_2(w) \leq \max\{600 + \Delta, \lfloor \frac{9}{5}\Delta \rfloor + 1\}$, and such that w has a neighbor w' with $d(w) + d(w') - 2 \leq \Delta$. □

It turns out that $\frac{9}{5}\Delta + 1$ is a *sharp* upper bound.

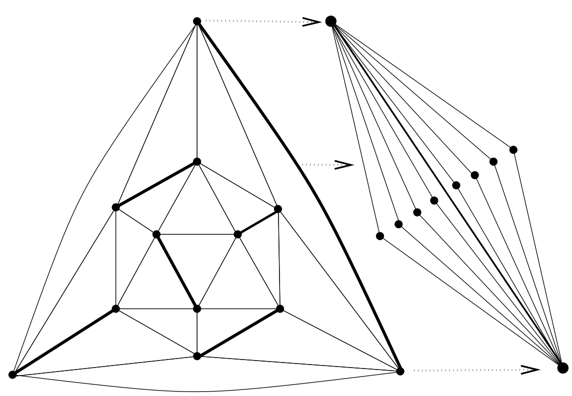

Figure 3: Icosahedron graph, and split edges

Observation 2.5 *For any Δ, there exists a planar graph G of maximum degree Δ such that G^2 is of minimum degree $\lfloor \frac{9}{5}\Delta \rfloor + 1$.*

Proof. Let $k = \lfloor \Delta/5 \rfloor$. Start with a 5-regular planar graph that contains a perfect matching. An example is the graph corresponding to the regular icosahedron in Fig. 3, with the edges of the perfect matching shown in bold. Parallel to each edge of the perfect matching, we add $k-1$ paths of length two. Each edge not in the perfect matching is replaced by k parallel paths of length two. This completes the construction of a simple planar graph G.

Observe that vertices in G are either of degree 2 or degree $5k$. The two neighbors of a degree-2 node share either k or $k+1$ neighbors (including themselves). Thus, the distance-2 degree of a degree-2 node is at least $2(5k+1)-(k+1) = 9k+1$. On the other hand, high-degree nodes have $5k-1$ low-degree neighbors, which have together 5 additional neighbors, and one high-degree neighbor, which itself has $5k$ neighbors. This counts k nodes twice, and one node three times, but still gives a distance-2 degree of $9k+3$. $\square$

3 General Powers of Planar Graphs

In this section we prove the following theorem.

Theorem 3.1 *Let G be a planar graph with maximum degree Δ. For any $k \geq 1$, G^k is $O(\Delta^{\lfloor k/2 \rfloor})$-colorable. Also, there is a family of graphs that attains this bound. This bound is also asymptotically tight for the clique number, inductiveness, arboricity, and minimum degree of G^k.*

Let us first give a construction that matches the bound of the theorem. Given $k, \Delta \geq 1$, consider the tree T of height $\lfloor k/2 \rfloor$ where internal vertices have degree Δ. The number of vertices in T is

$$D_{\Delta,k} = 1 + \Delta + \Delta(\Delta-1) + \Delta(\Delta-1)^2 + \cdots + \Delta(\Delta-1)^{\lfloor k/2 \rfloor - 1} = \frac{\Delta(\Delta-1)^{\lfloor k/2 \rfloor} - 2}{\Delta - 2}.$$

Observe that T^k is a complete graph, thus thus $\chi(T^k) = D_{\Delta,k}$.

We now turn to proving the upper bound of the theorem. The rest of this section is divided up into several subsections, each of which deals with necessary tools to complete the proof of our main Theorem 3.1 above. First let us set forth some useful terminology.

Notation

A *k-path* is a path of length exactly k. A *$(k, \leq)$-path* is a path of length k or less. If u and v are two given vertices then an *$(k; u, v)$-path* is a path between u and v of length exactly k, and finally a *$(k, \leq; u, v)$-path* is a path between u and v of length k or less. A vertex w is called a *$(k, \leq; u, v)$-link* if w is on every $(k, \leq; u, v)$-path. $N(v)$ will denote the set of the neighbors of v in G, and $N[v]$ the closed neighborhood of v, that is $N(v) \cup \{v\}$.

Definition 3.2 *For a simple planar graph G, an integer $k \geq 1$ and a subset $U \subseteq V(G)$, denote by $\mathcal{P}_k(G; U)$ the set of all W with $U \subseteq W \subseteq V(G)$ and such that any two vertices in U connected by a $(k, \leq)$-path in G, are also connected by a $(k, \leq)$-path in $G[W]$.*

We will derive the following bound on the size of each minimal element of $\mathcal{P}_k(G; U)$, that is linear in $|U|$, for any fixed k.

Theorem 3.3 *There exists an integer sequence $(d_k)_{k \geq 1}$ with $d_k \leq 10^{k-1}$, such that for every connected simple planar graph G, every integer $k \geq 1$ and every $U \subseteq V(G)$, each minimal element of $\mathcal{P}_k(G; U)$ has at most $d_k|U|$ vertices.*

Let us get a better grasp of this by examining the first two cases $k = 1, 2$. Clearly U itself is the only minimal element in $\mathcal{P}_1(G; U)$, for any U and G, thus $d_1 = 1$.

For the case $k = 2$, let W be a minimal element in $\mathcal{P}_2(G; U)$, for a given U. We form a graph G' on vertex set U as follows. For each $w \in W \setminus U$, select a pair u_1, u_2 in U for which w is a 2-link, and add an edge $u_1 u_2$ to G'. Note that G' is a simple graph, since each w represents the only path in $G[W]$ between the endpoints of the corresponding edge in G', and it is planar, since it is an edge contraction of a subgraph of $G[W]$ (where all vertices in $W \setminus U$ are of degree 2). By Euler's formula, $|E(G')| \leq 3|U| - 6$. Since each edge of G' corresponds to a distinct vertex of $W \setminus U$, we have that $|W| \leq 4|U| - 6$. Thus, $d_2 \leq 4$.

Before proving the general case of Theorem 3.3, let us continue and derive our conclusions.

Arboricity

For a graph G, define its *arboricity* as $\mathrm{arb}(G) = \max_{H \subseteq G} \left\lceil \frac{|E(H)|}{|V(H)|-1} \right\rceil$. By the Nash-Williams theorem [11] there are $\mathrm{arb}(G)$ edge-disjoint subforests of G that cover all the edges of G.

Arboricity is closely related to inductiveness.

Lemma 3.4 *For any graph G, we have* $\mathrm{arb}(G) \leq \mathrm{ind}(G) < 2\,\mathrm{arb}(G)$.

Proof. Assume first $\mathrm{ind}(G) = q$. We will show that $E(G)$ can be partitioned into q forests. Given a linear arrangement of the vertices, such that the pre-order is at most q, we arbitrarily color the q edges from a vertex v_i to later vertices with q colors. In this way, each color class is acyclic – since two edges of the same color cannot have the same first-labeled endpoint – and thus a forest. Therefore $\mathrm{arb}(G) \leq q$, proving the first inequality.

For the other inequality, let $\mathrm{ind}(G) = q$. Let H be a subgraph of G such that $\min_v(d_H(v)) = q$. Since $2|E(H)| = \sum_v d_H(v) \geq q|V(H)|$, we have $\mathrm{arb}(G) > |E(H)|/|V(H)| \geq q/2$, which completes our lemma. □

From Lemma 3.4 we have in particular from [13] that $\mathrm{arb}(G^2) \leq 9\Delta$.

Consider now the power graph G^k of G. For a vertex set $U \subseteq V(G)$, let $E^k(U)$ be the edgeset of the subgraph of G^k induced by U. Then, the arboricity of G^k is

$$\mathrm{arb}(G^k) = \max_{U \subseteq V(G)} \left\lceil \frac{|E^k(U)|}{|U| - 1} \right\rceil.$$

Note that every edge in $E^k(U)$ is represented by at least one $(k, \leq)$-path between vertices of U. Let $W_U \in \mathcal{P}_k(G; U)$ be a minimal element. By Theorem 3.3, $|W_U| \leq 10^{k-1}|U|$ and we have that $|E^k(U)|$ is less than the number of $(k, \leq)$-paths in $G[W_U]$. We note that all $(k, \leq)$-paths in $G[W_U]$ connecting two vertices of U, except the $(2, \leq)$-paths, are represented by an edge uv in $G[W_U]^{k-2}$ together with edges e and e' in G, with one endpoint u and v respectively. Hence,

$$|E^k(U)| \leq |E^2(U)| + \sum_{uv \in G[W_U]^{k-2}} d(u)d(v). \tag{2}$$

Degree products over edges

The following lemma will be used in our inductive argument.

Lemma 3.5 *If G is a simple graph of maximum degree Δ and F is a forest with $V(F) \subseteq V(G)$, then*

$$\sum_{uv \in E(F)} d_G(u)d_G(v) \leq 2\Delta|E(F)|.$$

Proof. For any graph H, with $V(H) \subseteq V(G)$ let

$$S(H) = \sum_{uv \in E(H)} d_G(u) d_G(v).$$

For each tree T of F, direct its edge away from an arbitrarily chosen root r. Thus, T becomes a directed tree T^d in which every vertex but the root has indegree one. For each arc $\vec{uv}$ in T^d bound the summand $d_G(u)d_G(v)$ from above by $\Delta d_G(v)$. Then,

$$S(T) \leq \sum_{\vec{uv} \in E(T^d)} \Delta d_G(v) = \Delta \left(\sum_{v \in V(T) \setminus \{r\}} d_G(v) \right) \leq \Delta \left(\sum_{v \in V(T)} d_G(v) \right).$$

As F is a disjoint union of trees T_i, we have that

$$S(F) = \sum_{i=1}^{k} S(T_i) \leq 2\Delta |E(F)|.$$

□

Arboricity of power graphs.

We now want to show inductively that there is a sequence $(\alpha_k)_{k=1}^{\infty}$ such that for every planar G with maximum degree Δ we have

$$\text{arb}(G^k) \leq \alpha_k \Delta^{\lfloor k/2 \rfloor}. \tag{3}$$

We know at this point that $\alpha_1 = 3$ and $\alpha_2 = 9$ satisfy (3). We proceed by induction and consider general $k \geq 3$. By (2) we get

$$|E^k(U)| \leq 9\Delta(|U| - 1) + \sum_{uv \in G[W_U]^{k-2}} d(u)d(v),$$

where now W_U is our minimal element of $\mathcal{P}_k(G; U)$. By the induction hypothesis we have that $\text{arb}(G[W_U]^{k-2}) \leq \alpha_{k-2} \Delta^{\lfloor \frac{k-2}{2} \rfloor} = a_{k-2}$ and hence by the Nash-Williams theorem [11] there are a_{k-2} edge-disjoint forests $F_1, F_2, \ldots, F_{a_{k-2}}$ covering all the edges of $G[W_U]^{k-2}$. By Lemma 3.5, and Theorem 3.3,

$$\begin{aligned}
\sum_{uv \in G[W_U]^{k-2}} d(u)d(v) &= \sum_{i=1}^{a_{k-2}} \sum_{uv \in E(F_i)} d(u)d(v) \\
&\leq \sum_{i=1}^{a_{k-2}} 2\Delta |E(F_i)| \\
&\leq a_{k-2}(2\Delta(|W_U| - 1)) \\
&< 2a_{k-2}\Delta |W_U| \\
&\leq 2 \cdot 10^{k-1} \alpha_{k-2} \Delta^{\lfloor k/2 \rfloor} |U|.
\end{aligned}$$

Since $k \geq 3$ and $\alpha_k \geq 1$, we can assume $|U| \geq 3$. Thus,

$$\begin{aligned}
|E^k(U)| &\leq 9\Delta(|U| - 1) + 2 \cdot 10^{k-1} \alpha_{k-2} \Delta^{\lfloor k/2 \rfloor} |U| \\
&\leq 4 \cdot 10^{k-1} \alpha_{k-2} \Delta^{\lfloor k/2 \rfloor} (|U| - 1).
\end{aligned}$$

Thus, $\text{arb}(G^k) \leq \alpha_k \Delta^{\lfloor k/2 \rfloor}$, where $\alpha_1 = 3, \alpha_2 = 9$ and $\alpha_k = 4 \cdot 10^{k-1} \alpha_{k-2}$. By an easy induction, we obtain the following lemma.

Lemma 3.6 *If G is a planar graph with a maximum degree Δ, and $k \geq 1$ is an integer, then we have* $\operatorname{arb}(G^k) \leq \alpha_k \Delta^{\lfloor k/2 \rfloor}$, *where* $\alpha_k = 2^k 10^{k^2/4}$. □

Letting α_k be as in the previous lemma, we get by Lemma 3.4 the following corollary.

Corollary 3.7 *For a simple planar graph G and an integer $k \geq 1$, we have that G^k is $2\alpha_k \Delta^{\lfloor k/2 \rfloor}$-inductive.* □

Proof of Theorem 3.3

We have already proved the theorem in the case where $k \in \{1, 2\}$. When considering the general case of $(k, \leq)$-paths, we proceed by induction on k and assume $k \geq 3$. Let $U \subseteq V(G)$ be given. Let $W \in \mathcal{P}_k(G; U)$ be a minimal element. Note that every vertex $w \in W \setminus U$ is *nonremovable*, in that there is a pair of vertices $\{u_{w1}, u_{w2}\}$ in U such that w is a $(k, \leq; u_{w1}, u_{w2})$-link in $G[W]$.

Let $U' \subseteq W \setminus U$ be the set of vertices of W that are connected to some vertex in U by an edge. We want to show that there is a constant c such that $|U'| \leq c|U|$. We can partition U' as $U' = U'_1 \cup U'_2 \cup U'_3$, where

$$\begin{aligned} U'_1 &= \{v \in U' : |N(v) \cap U| = 1\}, \\ U'_2 &= \{v \in U' : |N(v) \cap U| = 2\}, \\ U'_3 &= \{v \in U' : |N(v) \cap U| \geq 3\}. \end{aligned}$$

When estimating the sizes of U'_1, U'_2 and U'_3, the easiest case to deal with is U'_3. By the following Lemma 3.8 we have that $|U'_3| \leq 2|U| - 4$.

Lemma 3.8 *For a simple planar graph with vertex set $U \cup V$ such that every vertex in V is connected to at least three vertices of U, we have that $|V| \leq 2|U| - 4$.*

Proof. The bipartite subgraph on (U, V, E) has at most $2(|U|+|V|)-4$ edges by Euler's formula, but at least $3|V|$ edges by the degree bound on V. □

Lemma 3.9 $|U'_2| \leq 9|U|$.

Proof. The idea here is to consider the vertices of U'_2 together with their adjacent vertices of U. Every 2-path between vertices of U via vertices of U'_2 can be replaced by a single edge. In that way we get a planar multigraph with vertex set U, in which every edge corresponds to a vertex of U'_2. In this multigraph, many vertices of U could be isolated.

If we now consider one edge (of possible many) between given two vertices of U, then that edge corresponds uniquely to a vertex w of U'_2. Since w is nonremovable, it is crucial either for connecting the two neighbors of U with a 2-path, or for connecting either of its two neighbors of U to a third vertex of U. That third vertex must then lie in the face which has w on its boundary. This allows us to estimate the number of vertices of U'_2 in comparison with $|U|$.

Let us do this in a more precise manner. First note that the number of pairs of vertices of U that are connected by a 2-path via a vertex from U'_2 is at most $3|U| - 6$ by the planarity of $G[W]$ and Euler's formula. Consider now a fixed pair u and u' of U, connected by such a 2-path. By the planarity of $G[W]$, we can label all the vertices of U'_2 connecting u and u' as $v_1, v_2, \ldots, v_k$ such that this listing is clockwise with respect to u, and $v_2, \ldots, v_{k-1}$ are all inside the 4-cycle (u, v_1, u', v_k). Consider the graph $G(u, u')$ consisting of the vertices $u, u', v_1, v_2, \ldots, v_k$ along with all the edges connecting each v_i to both u and u'. The union of all the subgraphs $G(u, u')$, where $u, u' \in U$, together with the rest of the vertices of U, will form a simple subgraph G_2 of $G[W]$ with vertex set $U \cup U'_2$, in which each vertex of U'_2 has degree 2. Let $U = U_c \cup U_i$, where U_c is the set of nonisolated vertices of U in G_2, and U_i is the set of the isolated ones.

Every 2-path between vertices of U_c via a vertex of U_2' can be replaced with an edge, giving a planar multigraph G_2' in which each edge corresponds to a vertex in U_2'. By Euler's formula for simple planar graphs, we get the following claim.

Claim 3.10 *Consider a connected planar multigraph on n vertices and e edges. For a plane embedding of it, we call a face a* 2-face *if it is bounded by 2 edges and 2 vertices. The number of edges bounding two such 2-faces, is at least $e - 6n + 12$.* □

Let U_2'' be the set of vertices of U_2' bounding two 4-faces of G_2, each of which is bounded by 2 vertices of U and 2 vertices of U_2'. By Claim 3.10 we have

$$|U_2''| \geq |U_2'| - 6|U_c| + 12. \tag{4}$$

Consider $v \in U_2''$, and let $u, u' \in U$ be its neighbors. Call the two 4-faces that v bounds, f_{v1} and f_{v2}. Since at most one vertex of U_2' is crucial for connecting u to u', we may assume that v is not so. Since, however, v is nonremovable in $G[W]$, there is a vertex $u'' \in U \setminus \{u, u'\}$ such that v is either a $(k, \leq; u, u'')$-link or a $(k, \leq; u', u'')$-link in $G[W]$. By the planarity of $G[W]$, this u'' must be contained either within f_{v1} or f_{v2}, since otherwise v is removable. This holds for every v in U_2''.

To summarize, we see that at most $3|U_c| - 6$ vertices of U_2'' are actual links between their neighbors. Also, each v that is not a link between its neighbors, has a neighboring 4-face, which includes an isolated vertex of U_i. Therefore

$$|U_2''| - (3|U_c| - 6) \leq 2|U_i|. \tag{5}$$

By (4) and (5) we get

$$|U_2'| \leq 2|U_i| + 9|U_c| - 18 < 9|U|.$$

□

We now derive the the final step towards completion of the proof of Theorem 3.3. Assume we have successfully found $d_1, d_2, \ldots, d_{k-1}$ as in Theorem 3.3, we now prove the following lemma.

Lemma 3.11 *For a minimal element W of $\mathcal{P}_k(G; U)$, $|W| \leq 84 d_{k-2}|U|$.*

Proof. Let $U_1 \subseteq U$ be the set of vertices that have neighbors in U_1'. We now have the following partition

$$U_1' = \bigcup_{u \in U_1} N_{U_1'}(u)$$

where $N_{U_1'}(u) = \{v \in U_1' : uv \in E(G[W])\}$. Consider the planar graph $C[W]$ we get from $G[W]$ by contracting $N_{U_1'}[u]$ to a single vertex u^*, for each $u \in U_1$. Let $U^* = \{u^* : u \in U_1\} \cup (U \setminus U_1)$. If we let $U'' = W \setminus (U \cup U')$, then clearly W is a disjoint union of U, U_1', U_2', U_3' and U''. In view of this, $C[W]$ will become a graph whose vertices are a disjoint union of U^*, U_2', U_3' and U''. For convenience define a map $c : W \to U^* \cup U_2' \cup U_3' \cup U''$ by

$$c(w) = \begin{cases} u^* & \text{if } w \in N_{U_1'}[u], \text{ for } u \in U_1, \\ w & \text{otherwise.} \end{cases}$$

Note that every $(k, \leq)$-path between a pair of vertices of U in $G[W]$ gives a $(k-2, \leq)$-path between a pair of vertices of $U^* \cup U_2' \cup U_3'$.

Let us now show that every vertex of U'' is nonremovable in $C[W]$ when considering $(k-2, \leq)$-paths between pairs of vertices of $U^* \cup U_2' \cup U_3'$. Let $u'' \in U''$. Since u'' is nonremovable in $G[W]$

there is a pair u, u' of vertices of U such that u'' is a $(k, \leq; u, u')$-link. Pick a fixed $(k, \leq; u, u')$-path γ and let v and v' be the endpoints of $\gamma \setminus \{u, u'\}$. Now u'' is a $(k-2, \leq; v, v')$-link in $G[W]$, since otherwise u'' would not be a $(k, \leq; u, u')$-link in $G[W]$. That u'' is a $(k-2, \leq; c(v), c(v'))$-link in $C[W]$ can be seen as follows. If u'' is not such a link, then there is a $(k-2, \leq; c(v), c(v'))$-path γ' not including the vertex u'' in $C[W]$. It then gives a $(k, \leq; u, u')$-path γ'' in $G[W]$ not including u'', which is a contradiction.

By induction hypothesis on k, we now have that the number of vertices of $C[W]$ are bounded, that is $|U^* \cup U'_2 \cup U'_3 \cup U''| \leq d_{k-2}|U^* \cup U'_2 \cup U'_3|$. By previous arguments and the fact that $|U^*| = |U|$, we have

$$|U^* \cup U'_2 \cup U'_3 \cup U''| \leq d_{k-2}(|U| + 9|U| + 2|U|) = 12d_{k-2}|U|.$$

The only thing left to conclude our inductive argument is to show that $|W \setminus (U \cup U'_2 \cup U'_3 \cup U'')| = |U'_1| \leq c|U|$ for some constant c.

Let $u \in U$ be fixed. For each neighbor v of u in $G[W]$, let $p_u(v) \in U$ be a vertex such that v is a $(k, \leq; u, p_u(v))$-link. We assume further that for a fixed u and distinct v, all the $p_u(v)$ are distinct.

Claim 3.12 *With the notation from above, for each neighbor v of u in $G[W]$, let γ_v be a $(k, \leq; u, p_u(v))$-path. Except for the vertex u, all these paths are vertex-disjoint.*

Proof. Assume γ_{v_1} and γ_{v_2} have a common vertex x other than u. Hence, for $i = 1, 2$, $\gamma_{v_i} = \gamma_{ix}\beta_i$, where γ_{ix} is the path from u to x along γ_{v_i} and β_i the path from x to $p_u(v_i)$ along γ_{v_i}. If now $l(\gamma_{1x}) \leq l(\gamma_{2x})$, then $\gamma' = \gamma_{1x}\beta_2$ is a $(k, \leq; u, p_u(v_2))$-path not including the vertex v_2 (since if v_2 lies on γ_{1x} then v_2 is not a link). This contradicts the definition of $p_u(v_2)$. □

By Claim 3.12, all the $(k, \leq; u, p_u(v))$-paths from u to each of the $p_u(v)$, are vertex disjoint. Therefore the number of vertices of $N_{U'_1}(u)$ is less than the number of edges going out of u^* in the contracted graph $C[W]$. Since each edge in $C[W]$ connects to at most two vertices of U^*, we have that $|U'_1| \leq 2|E(C[W])|$. Since $|E(C[W])| \leq 3|V(C[W])| - 6$ and $V(C[W]) = U^* \cup U'_2 \cup U'_3 \cup U''$, we have

$$\begin{aligned} |W| &= |U^* \cup U'_2 \cup U'_3 \cup U''| + |U'_1| \\ &\leq 12d_{k-2}|U| + 2|E(C[W])| \\ &\leq 12d_{k-2}|U| + 6|U^* \cup U'_2 \cup U'_3 \cup U''| \\ &\leq 84d_{k-2}|U|. \end{aligned}$$

We see from the above display that $d_k = 84d_{k-2}$ is sufficient in the case for general k provided that d_{k-2} is known. Therefore the sequence $(d_k)_{k=1}^{\infty}$ defined inductively by $d_1 = 1$, $d_2 = 4$ and $d_k = 84d_{k-2}$ will give us the desired constants. A straightforward induction implies that $d_k \leq 10^{k-1}$, and hence we have Theorem 3.3. □

4 Approximation Algorithms

We can improve the best approximation factor known for coloring squares of planar graphs. Recall that since neighbors in G must be colored differently in G^2, $\chi(G^2) \geq \Delta + 1$. Thus, for $\Delta \geq 749$, Corollary 3.7 yields a 1.8-approximation. Hence, we obtain an *asymptotic* ratio of 1.8.

For constant values of Δ, we can use a result of Krumke, Marathe and Ravi [7]. They stated a 3-approximation, but actually a 2-approximation easily follows from their approach which is based on an often-used decomposition due to Baker [2]. The complexity of their approach is equivalent to the complexity of coloring a partial $O(\Delta)$-tree. Combined, we obtain a 2-approximation for any value of Δ.

Theorem 4.1 *The problem of coloring squares of planar graph has a 2-approximation.*

Theorem 3.1 also immediately gives a $O(1)$-approximation to coloring cubes of planar graphs. However, better factors are possible.

Zhou et al. [17] independently gave a polynomial algorithm for distance-d coloring partial k-trees, for any constant d and k. The complexity of their algorithm is $O(n(\alpha+1)^{2^{2(k+1)(d+2)+1}} + n^3)$, where $\alpha = O(\min(\Delta^{d/2}, n))$ is the number of colors needed. When combined with the decomposition of Baker, this result yields a 2-approximation for coloring G^d, for any constant d.

Baker's result states that the vertex set V of a planar graph can be partitioned into layers $V_1, V_2, \ldots$, such that all edges are between adjacent layers or within the same layer, i.e. if $u \in V_i$ and $uv \in E$, then $v \in V_{i-1} \cup V_i \cup V_{i+1}$. Now, let $V' = \cup_{i \bmod 2d < d} V_i$, $V'' = V - V'$, and G', G'' be the subgraphs induced by V' and V''. Observe that both G' and G'' consist of a collection of disjoint subgraphs U_i, corresponding to $V_{di} \cup V_{di+1} \cup \cdots \cup V_{d(i+1)-1}$. Further, notice that the subgraphs induced by the U_i will also be disjoint in G'^d and G''^d, since distance between any pair of nodes in different subgraphs U_i is at least $d+1$. Thus, G'^d can be computed by considering each U_i separately. Now, G^d restricted to U_i is a subgraph of the graph H_i^d, where $H_i = G[\cup_{j=di-(d-1)}^{d(i+1)-(d-2)} U_i]$. H_i is a $3d-2$-outerplanar graph, which means that it is a partial $9d-8$-tree by a result of Bodlaender [3]. Hence, we can compute the optimal coloring of each H_i in time $O(n^{2^{(9d+7)(d+2)+1}+1})$. Thus, we can solve G'^2 and G''^2 exactly, and in total, using at most twice the optimal number of colors.

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