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# Graph Imperfection

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We are interested in colouring a graph $G=(V, E)$ together with an integral weight or demand vector $\mathbf{x}=(x_v: v \in V)$ in such a way that $x_v$ colours are assigned to each node $v$, adjacent nodes are coloured with disjoint sets of colours, and we use as few colours as possible. Such problems arise in the design of cellular communication systems, when radio channels must be assigned to transmitters to satisfy demand and avoid interference.

We are particularly interested in the ratio of chromatic number to clique number when some weights are large. We introduce a relevant new graph invariant, the "imperfection ratio" imp$(G)$ of a graph $G$, present alternative equivalent descriptions, and show some basic properties. For example, imp$(G)=1$ if and only if $G$ is perfect, imp$(G)=$ imp$(\bar{G})$ where $\bar{G}$ denotes the complement of $G$, and imp$(G)= g/(g-1)$ for any line graph $G$ where $g$ is the minimum length of an odd hole (assuming there is an odd hole). © 2001 Academic Press

*Key Words:* weighted colouring; imperfection ratio; perfect graphs; stable set polytope; radio channel assignment.

## 1. INTRODUCTION

We are interested in colouring weighted graphs, that is, in assigning colours to the nodes of a graph $G=(V, E)$ together with an integral weight or demand vector $\mathbf{x}=(x_v: v \in V)$ in such a way that $x_v$ colours are assigned

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to each node $v$, adjacent nodes are coloured with disjoint sets of colours, and we aim to minimise the total number of colours used. There is a natural graph $G_{\mathbf{x}}$ associated with a pair $(G, \mathbf{x})$ as above, obtained by replacing each node $v$ by a clique of size $x_v$. Colourings of the pair $(G, \mathbf{x})$ correspond to usual proper node colourings of the graph $G_{\mathbf{x}}$, where adjacent nodes must receive distinct colours.

Weighted colouring problems arise in the design of cellular radio communication systems such as mobile telephone networks, where sets of radio channels must be assigned to transmitters [13, 19, 24] and one wants to use the smallest possible part of the spectrum. In one of the basic models one must assign $x_v$ channels to transmitter $v$ in order to satisfy the estimated local demand in the cell served by $v$, and two transmitters must not be assigned the same channel if this would result in excessive interference. We may thus construct a weighted "interference" graph $G$, with nodes the transmitters, where nodes $u$ and $v$ are adjacent when the corresponding transmitters must be assigned disjoint sets of channels, and the weight $x_v$ at node $v$ equals the demand at the corresponding transmitter. The problem of finding an assignment of frequencies to the transmitters which minimises the number of channels used then translates to finding a colouring of the pair $G_{\mathbf{x}}$ using as few colours as possible. The recent dramatic growth in demand for radio spectrum has made such problems increasingly important.

The clique number $\omega(G_{\mathbf{x}})$ is a lower bound on the chromatic number $\chi(G_{\mathbf{x}})$, as is well known. For problems arising in channel assignment, typically $\omega(G_{\mathbf{x}})$ can be found or approximated quickly, even though this is not true for general graphs [31]; and typically some demands are large, see for example [7]. Here, we want to compare the chromatic number $\chi(G_{\mathbf{x}})$ and the clique number $\omega(G_{\mathbf{x}})$ of a graph $G$ with weight vector $\mathbf{x}$ when the maximum weight $x_{\max} = \max\{x_v : v \in V\}$ is large. We let

$$r_k(G) = \max\left\{\frac{\chi(G_{\mathbf{x}})}{\omega(G_{\mathbf{x}})} : \mathbf{x} \in \mathbb{N}^V \text{ with } x_{\max} = k\right\}.$$

Of course $r_k(G) \geqslant 1$. Also observe that $\chi(G_{\mathbf{x}}) \leqslant \chi(G)\, x_{\max} \leqslant \chi(G)\, \omega(G_{\mathbf{x}})$, and therefore $r_k(G) \leqslant \chi(G)$.

We are interested in the values $r_k$ for large $k$ and not in the maximum value over all $k$. It turns out that $r_k(G)$ always tends to a limit as $k \to \infty$. This limit is the quantity which we next introduce and which is the focus of this paper. Note that we are considering the ratio of chromatic number to clique number and not the difference between these quantities. Thus our development does not follow along lines like integer rounding—see, for example, Section 22.10 of [30] and the references there.

The *imperfection ratio* imp($G$) for a graph $G$ is defined as

$$\mathrm{imp}(G)=\sup\left\{\frac{\chi_f(G_{\mathbf{x}})}{\omega(G_{\mathbf{x}})}:\mathbf{0}\neq\mathbf{x}\in\mathbb{N}^V\right\}. \tag{1}$$

Here $\chi_f(G)$ denotes the *fractional chromatic number* of $G$, that is, the value of the following linear program with a variable $y_S$ for each stable (or independent) set $S$ of $G$: min $\sum_S y_S$ subject to $\sum_{S\ni v} y_S\geqslant 1$ for each node $v\in V$, and $y_S\geqslant 0$ for each stable set $S$ of $G$. Since $\chi_f(G)\geqslant\omega(G)$, we have $\mathrm{imp}(G)\geqslant 1$. It turns out that the "supremum" in the definition (1) may be replaced by "maximum"—see Theorem 2.1 below.

The plan of the paper is as follows. In Section 2 we show that $r_k(G)\to\mathrm{imp}(G)$ as $k\to\infty$, as well as introduce equivalent polyhedral definitions of imp($G$). In certain models for channel assignment, only a subset of the channels may be available at each transmitter. We obtain a 'list colouring' problem (see, for example, [5, 15, 36]), and we are led to consider a list colouring variant $r_k^l(G)$ of $r_k(G)$. In Theorem 2.2 below we show that there is no need to introduce a new quantity, the 'list imperfection ratio,' as there is a limiting result like that mentioned above for $r_k(G)$ with the same limit imp($G$); that is $r_k^l(G)\to\mathrm{imp}(G)$ as $k\to\infty$.

In Section 3 we determine the imperfection ratio for graphs in certain classes, including line graphs, triangle-free graphs, and minimal imperfect graphs. We also derive various results concerning the imperfection ratio. For example we see that $\mathrm{imp}(G)=1$ if and only if $G$ is perfect, and $\mathrm{imp}(G)=\mathrm{imp}(\bar{G})$ where $\bar{G}$ denotes the complement of $G$. The former property gave imp($G$) its name, and the latter is clearly desirable for any proposed measure of how "imperfect" a graph is.

In Section 4 we exhibit examples such that certain coordinates have to be large for any vector $\mathbf{x}$ attaining the maximum in (1), that is, for any nonnegative integer-valued vector $\mathbf{x}$ with $\mathrm{imp}(G)=\chi_f(G_{\mathbf{x}})/\omega(G_{\mathbf{x}})$.

Further results on the imperfection ratio appear in [9]. For example, we investigate properties of the imperfection ratio under some graph operations including the lexicographic product of two graphs, explore random and extremal behaviour, and see that it is NP-hard to determine the imperfection ratio.

## 2. EQUIVALENT DESCRIPTIONS

Before we can present the polyhedral descriptions of the imperfection ratio we need some further notation and definitions, following [11]. The *stable set polytope* $STAB(G)\subseteq[0,1]^V$ is the convex hull of the incidence vectors of the stable sets in $G$. The *fractional stable set polytope*

$QSTAB(G) \subseteq [0, 1]^V$ is the set of non-negative real vectors $\mathbf{x} = (x_v : v \in V)$ such that

$$\sum_{v \in K} x_v \leqslant 1 \qquad \text{for every clique } K \text{ in } G.$$

(This polytope is also called the "clique-constrained stable set polytope" or the "fractional node-packing polytope.") Observe that $STAB(G) \subseteq QSTAB(G)$, since each stable set and each clique meet in at most one node. The two polyhedra are equal if and only if the graph is perfect [3] (or see, for example, [11, 28]). For a polytope $P$ we denote by $tP$ the scaled set $\{t\mathbf{x} : \mathbf{x} \in P\}$.

For a non-zero rational vector $\mathbf{x}$ indexed by the nodes of $G = (V, E)$, define $\omega(G, \mathbf{x})$ to be the maximum value of $\sum_{v \in K} x_v$ over all the cliques $K$ of $G$. If $\mathbf{x}$ is integral then this equals $\omega(G_{\mathbf{x}})$. We next define $\chi_f(G, \mathbf{x})$ in such a way that if $\mathbf{x}$ is integral then $\chi_f(G_{\mathbf{x}}) = \chi_f(G, \mathbf{x})$. Introduce a variable $y_S$ for each stable set $S$ of $G$. Then $\chi_f(G, \mathbf{x})$ equals the value of the linear program: $\min \sum_S y_S$ subject to $\sum_{S \ni v} y_S \geqslant x_v$ for each node $v$ of $G$, and $y_S \geqslant 0$ for every stable set $S$ of $G$.

We may now state the main theorem of this section, which provides alternative equivalent definitions of the imperfection ratio, as well as give the limiting result mentioned in the Introduction. After we prove this theorem we introduce a list colouring variant of the limiting result.

THEOREM 2.1. *For any graph $G$,*

$$\operatorname{imp}(G) = \max\{\chi_f(G, \mathbf{x}) : \mathbf{x} \in QSTAB(G)\} \tag{2}$$

$$= \min\{t : QSTAB(G) \subseteq t\, STAB(G)\} \tag{3}$$

$$= \max\{\mathbf{x} \cdot \mathbf{y} : \mathbf{x} \in QSTAB(G), \mathbf{y} \in QSTAB(\bar{G})\} \tag{4}$$

$$= \lim_{k \to \infty} r_k(G). \tag{5}$$

*In* (2) *above we can restrict* $\mathbf{x}$ *to being a vertex of* $QSTAB(G)$. *In addition, there exists an integral weight vector* $\mathbf{y}$ *such that for any positive integer multiple* $\mathbf{x}$ *of* $\mathbf{y}$,

$$\operatorname{imp}(G) = \frac{\chi_f(G_{\mathbf{x}})}{\omega(G_{\mathbf{x}})} = \frac{\chi(G_{\mathbf{x}})}{\omega(G_{\mathbf{x}})}. \tag{6}$$

*Proof.* The first three equations follow from simple scaling arguments. Observe that $\omega(G, \mathbf{x})$ and $\chi_f(G, \mathbf{x})$ both scale, in the sense that $\omega(G, k\mathbf{x}) = k\omega(G, \mathbf{x})$ and $\chi_f(G, k\mathbf{x}) = k\chi_f(G, \mathbf{x})$. Observe also that

$$\omega(G, \mathbf{x}) \leqslant t \qquad \text{if and only if} \quad \mathbf{x} \in t\, QSTAB(G),$$

and

$$\chi_f(G, \mathbf{x}) \leqslant t \qquad \text{if and only if} \quad \mathbf{x} \in t\, STAB(G). \tag{7}$$

Consider any graph $G$ and $t > 0$. By the definition of $\text{imp}(G)$ we have

$$\text{imp}(G) \leqslant t \Leftrightarrow \chi_f(G, \mathbf{x})/\omega(G, \mathbf{x}) \leqslant t \qquad \text{for all integral} \quad \mathbf{x} \geqslant \mathbf{0}.$$

By the scaling result noted above we may replace "integral" above by "rational," and by continuity we may drop the "rational." Thus by (7)

$$\begin{aligned} \text{imp}(G) \leqslant t \Leftrightarrow \chi_f(G, \mathbf{x}) \leqslant t \qquad & \text{for all} \quad \mathbf{x} \in QSTAB(G) \\ \Leftrightarrow \mathbf{x} \in t\, STAB(G) \qquad & \text{for all} \quad \mathbf{x} \in QSTAB(G) \\ \Leftrightarrow QSTAB(G) \subseteq t\, STAB(G). & \end{aligned}$$

Further note that in the second to last line we could restrict the points $\mathbf{x}$ to be in the finite set of vertices of $QSTAB(G)$. These results establish (2) and (3) in the theorem and the later comment about (2). The next equation (4) now follows, since by linear programming duality

$$\chi_f(G, \mathbf{x}) = \max\{\mathbf{x} \cdot \mathbf{y} \colon \mathbf{y} \in QSTAB(\bar{G})\},$$

where $\bar{G}$ denotes the complement of $G$.

To prove (6), let $\tilde{\mathbf{x}}$ be a vertex of $QSTAB(G)$ such that $\text{imp}(G) = \chi_f(G, \tilde{\mathbf{x}})$. Since the LP defining $\chi_f(G, \tilde{\mathbf{x}})$ has a rational optimal solution, there is an integral vector $\mathbf{y}$ which is a positive multiple of $\tilde{\mathbf{x}}$ and satisfies $\chi_f(G_\mathbf{y}) = \chi(G_\mathbf{y})$. Then $\mathbf{y}$ satisfies (6).

Finally we prove (5). Let

$$r'_k(G) = \max\left\{\frac{\chi_f(G, \mathbf{x})}{\omega(G, \mathbf{x})} : \mathbf{x} \in \mathbb{N}^V \text{ with } x_{\max} = k\right\}.$$

Then $r'_k(G) \leqslant r_k(G)$ for all positive integers $k$. Let $\tilde{\mathbf{x}}$ be a fixed integral weight vector such that $\chi_f(G, \tilde{\mathbf{x}})/\omega(G, \tilde{\mathbf{x}}) = \text{imp}(G)$. Let $\tilde{k} = \tilde{x}_{\max}$. Now consider any integer $k \geqslant \tilde{k}$. Let $\mathbf{y} = \lfloor (k/\tilde{k})\, \tilde{\mathbf{x}} \rfloor$. (Here of course we mean that each coordinate is rounded down.) Then $y_{\max} = k$, and so $r'_k(G) \geqslant \chi_f(G, \mathbf{y})/\omega(G, \mathbf{y})$. But $\omega(G, \mathbf{y}) \leqslant (k/\tilde{k})\, \omega(G, \tilde{\mathbf{x}})$. Also $(k - \tilde{k})\, \tilde{\mathbf{x}} \leqslant \tilde{k}\, \mathbf{y}$, and so $(k - \tilde{k})\chi_f(G, \tilde{\mathbf{x}}) \leqslant \tilde{k}\chi_f(G, \mathbf{y})$. Hence

$$r'_k(G) \geqslant \frac{\chi_f(G, \mathbf{y})}{\omega(G, \mathbf{y})} \geqslant \frac{k - \tilde{k}}{\tilde{k}}\, \frac{\tilde{k}}{k}\, \frac{\chi_f(G, \tilde{\mathbf{x}})}{\omega(G, \tilde{\mathbf{x}})} = \left(1 - \frac{\tilde{k}}{k}\right) \text{imp}(G).$$

Thus for any integer $k \geqslant \tilde{k}$ we have $r'_k(G) > (1 - \tilde{k}/k)\, \text{imp}(G)$.

Next note that $\chi(G, \mathbf{x}) < \chi_f(G, \mathbf{x}) + |V(G)|$ for any weight vector $\mathbf{x}$. For, consider a basic optimal solution $y^*_S$ for the LP defining $\chi_f(G, \mathbf{x})$. At most $|V(G)|$ coordinates $y^*_S$ are non-zero, and so

$$\chi(G, \mathbf{x}) \leqslant \sum_S \lceil y^*_S \rceil < \sum_S y^*_S + |V(G)| = \chi_f(G, \mathbf{x}) + |V(G)|.$$

Hence if $x_{\max} = k$ then

$$\frac{\chi(G, \mathbf{x})}{\omega(G, \mathbf{x})} < \frac{\chi_f(G, \mathbf{x}) + |V(G)|}{\omega(G, \mathbf{x})} \leqslant \operatorname{imp}(G) + \frac{|V(G)|}{k},$$

and it follows that $r_k(G) < \operatorname{imp}(G) + |V(G)|/k$. We have now shown that for any integer $k \geqslant \tilde{k}$

$$(1 - \tilde{k}/k) \operatorname{imp}(G) < r'_k(G) \leqslant r_k(G) < \operatorname{imp}(G) + |V(G)|/k,$$

and (5) follows. ∎

Let us return to the colouring problem mentioned earlier, when not all colours are available at each node. We introduce a weighted version of list colouring, a concept which was introduced in [5, 36]; see also [15]. As usual we have a graph $G = (V, E)$ together with an integral weight vector $\mathbf{x}$. Suppose that we have a list $L_v$ of available colours for each node $v$. We say that the pair $(G, \mathbf{x})$ is $(L_v: v \in V)$-*choosable* if for each $v \in V$ there is a subset $X_v$ of $L_v$ of size $x_v$ such that $X_u$ and $X_v$ are disjoint whenever the nodes $u$ and $v$ are adjacent. The *list chromatic number* $\chi^l(G, \mathbf{x})$ of the pair $(G, \mathbf{x})$ is the least $t$ such that the pair $(G, \mathbf{x})$ is $(L_v: v \in V)$-choosable whenever each set $L_v$ has size at least $t$. Thus certainly $\chi(G, \mathbf{x}) \leqslant \chi^l(G, \mathbf{x})$.

Corresponding to the earlier definition of $r_k(G)$, let

$$r^l_k(G) = \max\{\chi^l(G, \mathbf{x})/\omega(G, \mathbf{x}) \colon \mathbf{x} \in \mathbb{N}^V \text{ with } x_{\max} = k\}.$$

The following result shows that we do not need to introduce a new quantity, the 'list imperfection ratio.'

THEOREM 2.2. *For any graph $G$, $r^l_k(G) \to \operatorname{imp}(G)$ as $k \to \infty$.*

*Proof.* Fix a graph $G$. Clearly $r_k(G) \leqslant r^l_k(G)$ for each positive integer $k$. Thus by (5), it suffices to show that for any $\varepsilon > 0$,

$$r^l_k(G) < (1 + \varepsilon) \operatorname{imp}(G) \tag{8}$$

for all $k$ sufficiently large.

It is known [1] that the list fractional chromatic number equals $\chi_f(G)$, in other words that $\chi^l(G, s\mathbf{1})/s \to \chi_f(G)$ as $s \to \infty$. Now let $0 < \varepsilon \leqslant 1$. Then for each integral weight vector $\mathbf{x}$, there exists a constant $t(G, \mathbf{x}, \varepsilon)$ such that for each $t \geqslant t(G, \mathbf{x}, \varepsilon)$ we have

$$\chi^l(G_{\mathbf{x}}, t\mathbf{1}) \leqslant (1+\varepsilon)\, t\chi_f(G_{\mathbf{x}}) = (1+\varepsilon)\, \chi_f(G_{t\mathbf{x}}).$$

Call the integral weight vectors $\mathbf{x}$ with $x_{\max} \leqslant 1/\varepsilon$ the *small* weight vectors. Let $t_\varepsilon = t_\varepsilon(G)$ be the maximum of the values $t(G, \mathbf{x}, \varepsilon)$ over all the small weight vectors $\mathbf{x}$. Now consider any integral weight vector $\mathbf{x}$ with $x_{\max} \geqslant t_\varepsilon/\varepsilon$. Let $t = \lceil \varepsilon x_{\max} \rceil$, and note that $t \geqslant t_\varepsilon$ and $\lceil \mathbf{x}/t \rceil$ is a small weight vector. Hence

$$\begin{aligned}\chi^l(G, \mathbf{x}) &\leqslant \chi^l(G_{\lceil \mathbf{x}/t \rceil}, t\mathbf{1})\\ &\leqslant (1+\varepsilon)\, \chi_f(G_{t\lceil \mathbf{x}/t \rceil})\\ &\leqslant (1+\varepsilon)(\chi_f(G_{\mathbf{x}}) + t\chi_f(G)).\end{aligned}$$

But

$$t/\omega(G_{\mathbf{x}}) \leqslant t/x_{\max} < \varepsilon + 1/x_{\max} \leqslant 2\varepsilon.$$

Therefore

$$\begin{aligned}\frac{\chi^l(G, \mathbf{x})}{\omega(G_{\mathbf{x}})} &\leqslant (1+\varepsilon)\left(\frac{\chi_f(G_{\mathbf{x}})}{\omega(G_{\mathbf{x}})} + 2\varepsilon\chi_f(G)\right)\\ &\leqslant (1+\varepsilon)\,\mathrm{imp}(G) + 4\varepsilon\chi_f(G).\end{aligned}$$

Hence $r_k^l$ is at most this bound for all $k \geqslant t_\varepsilon/\varepsilon$. By replacing $\varepsilon$ here by $\varepsilon/(5\chi_f(G))$ we obtain (8), which completes the proof. ∎

## 3. BOUNDS AND EXAMPLES

Let us look back to Theorem 2.1 for a moment. The following proposition is an immediate consequence of the definition (1) of the imperfection ratio, of (3), (4) in Theorem 2.1, and of the fact mentioned earlier that $QSTAB(G) = STAB(G)$ if and only if $G$ is perfect. Recall that $\alpha(G)$ denotes the maximum size of a stable set in $G$, and that $\chi_f(G) \geqslant |V(G)|/\alpha(G)$ for every graph $G$.

PROPOSITION 3.1. *For any graph G*

(i) $\mathrm{imp}(H) \leqslant \mathrm{imp}(G)$ *for any induced subgraph H of G*;

(ii) $\mathrm{imp}(G) \geqslant \chi_f(G)/\omega(G) \geqslant |V(G)|/(\alpha(G)\,\omega(G))$;

(iii) $\text{imp}(G) = \text{imp}(\bar{G})$ *where* $\bar{G}$ *denotes the complement of* $G$;

(iv) $\text{imp}(G) = 1$ *if and only if* $G$ *is perfect.*

By (iv) of Proposition 3.1 and (4) in Theorem 2.1, we note that a graph is perfect if and only if $\mathbf{x} \cdot \mathbf{y} \leqslant 1$ for each $\mathbf{x} \in QSTAB(G)$ and $\mathbf{y} \in QSTAB(\bar{G})$. This characterisation of perfect graphs may remind the reader of Lehman's length–width inequality [20] if we are willing to reverse all inequalities except non-negativity.

Another way to prove (iv) in Proposition 3.1 is to consider minimal imperfect graphs. For if $G$ is perfect then $\chi(G_{\mathbf{x}})/\omega(G_{\mathbf{x}}) = 1$ for any non-zero integral weight vector $\mathbf{x}$ of $G$ [21], and thus $\text{imp}(G) = 1$. If $G$ is not perfect, then it has an induced subgraph $H$ which is a minimal imperfect graph and so satisfies $|V(H)| = \alpha(H)\,\omega(H) + 1$ ([22], or see [11]). Therefore by Proposition 3.1

$$\text{imp}(G) \geqslant \text{imp}(H) \geqslant \frac{|V(H)|}{\alpha(H)\,\omega(H)} = \frac{|V(H)|}{|V(H)| - 1} > 1. \tag{9}$$

This leads to the following proposition.

Proposition 3.2. *For any minimal imperfect graph* $H$ *of order* $n$, $\text{imp}(H) = n/(n-1)$.

*Proof.* We have already seen (9) that $\text{imp}(H) \geqslant n/(n-1)$. For the opposite inequality consider an integral weight vector $\mathbf{x}$ of $H$ with $\text{imp}(H) = \chi_f(H, \mathbf{x})/\omega(H, \mathbf{x})$. The removal of a node $v$ of $H$ yields a perfect graph $H \backslash v$. By [21] one can cover a perfect graph $G$ with integral weight vector $\mathbf{y}$ with $\omega(G, \mathbf{y})$ stable sets. Hence each node $u$ of $H \backslash v$ can be covered $x_u$ times with at most $\omega(H, \mathbf{x})$ stable sets. Putting together these coverings, one for each node $v$ of $H$, we find that each node $v$ of $H$ can be covered $(n-1)\,x_v$ times using at most $n\omega(H, \mathbf{x})$ stable sets. Hence $\chi_f(H, \mathbf{x}) \leqslant n\omega(H, \mathbf{x})/(n-1)$ and the result follows. ∎

We used in this proof a method which always yields an upper bound on the imperfection ratio: if we can cover every node of a graph $G$ $b$ times by $a$ induced perfect subgraphs, then $\text{imp}(G) \leqslant a/b$. As we have seen one can cover every node of a minimal imperfect graph $n-1$ times with $n$ perfect graphs.

With this perfect graph covering method we can also bound the imperfection ration of unit disk graphs. A *unit disk graph* can be embedded in the plane such that two nodes are adjacent if and only if the Euclidean distance between them is at most 1, that is when the closed unit diameter disks around them intersect—see [4]. Such graphs are of particular interest for modelling radio channel assignment problems, because one obtains a unit disk graph if one assumes that the system consists of omni-directional antennas with equal power. A set of nodes of this representation which each lie in a stripe of width $\sqrt{3}/2$ forms a perfect graph, indeed a co-comparability graph [10]. This leads to the following proposition.

PROPOSITION 3.3. *If G is a unit disk graph, then*

$$\mathrm{imp}(G) \leqslant 1 + 2/\sqrt{3} \approx 2.155.$$

*Proof.* Let $G=(V, E)$ be a unit disk graph embedded in the plane such that two nodes are adjacent when the Euclidean distance between them is at most 1. Let $s=\sqrt{3}/2$, and let $w=1+s$. For a fixed number $r \in \mathbb{R}$, let $V_r$ consists of all the nodes of $G$ of which the $x$-coordinate can be written in the form $r+tw+x$ where $t \in \mathbb{Z}$ and $x \in [0, s)$. Then the graph induced by $V_r$ is perfect for any $r \in \mathbb{R}$ as we saw above. If we pick $r$ uniformly at random from $[0, w)$, then the probability that a node $v$ is covered by $V_r$ equals

$$p = \mathrm{Prob}(v \text{ covered by } V_r) = \frac{s}{w} = \frac{\sqrt{3}}{2+\sqrt{3}}.$$

If we independently choose $t$ such perfect graphs, then we expect that a node $v$ is covered $tp$ times. Let $0<p'<p$. Then for sufficiently large $t$, we have (using well-known results for sums of independent Bernoulli random variables, e.g., Chernoff bounds) that

$$\mathrm{Prob}(v \text{ is covered less than } tp' \text{ times}) \leqslant \frac{1}{2\,|V|}.$$

Therefore the probability that all the nodes are covered at least $tp'$ times is greater than 1/2. Hence there must exist a family of $t$ perfect graphs covering every node $tp'$ times, which shows that $\mathrm{imp}(G) \leqslant 1/p'$. ∎

There are unit disk graphs $G$ with $\mathrm{imp}(G)$ arbitrarily close to 3/2. For example, for a positive integer $w$, consider the cycle power $G=C_{3w-1}^{w-1}$ consisting of $3w-1$ nodes on a circle where each node is adjacent to the $w-1$ nearest nodes in each direction of the circle. This is a unit disk graph [23]. Also $\omega(C_{3w-1}^{w-1})=w$, and all maximal independent sets have cardinality 2. Therefore by (ii) of Proposition 3.1, $\mathrm{imp}(C_{3w-1}^{w-1}) \geqslant (3w-1)/(2w)$. Perhaps the bound in Proposition 3.3 can be improved to 3/2?

For a subclass of unit disk graphs, namely the class of induced subgraphs of the triangular lattice, the bound we have just derived can be improved. These graphs are of importance for channel assignment, since a pattern of omni-directional transmitters in two dimensions laid out like nodes of a triangular lattice in the plane gives good coverage in the sense of providing universal coverage with minimum overlap between service areas of adjacent transmitters. It is known ([25], see also [17, 27]) that

$$\chi(G_{\mathbf{x}}) \leqslant \frac{4\omega(G_{\mathbf{x}})+1}{3},$$

and therefore $\mathrm{imp}(G) \leqslant 4/3$ for any induced subgraph $G$ of the triangular lattice. It has been conjectured that this bound can be improved to 9/8 [25]. That bound would be best possible, since the cycle $C_9$ on nine nodes is an induced subgraph of the triangular lattice. Havet [14] has shown that if an induced subgraph $G$ of the triangular lattice is triangular-free then $\mathrm{imp}(G) \leqslant 7/6$.

Another upper bound on the imperfection ratio for any graph $G$ is $\mathrm{imp}(G) \leqslant \chi(G)$. This follows from the fact that $r_k(G) \leqslant \chi(G)$ which was noted in the Introduction and the limiting result (5) of the previous section. This bound can be improved as in the following proposition.

PROPOSITION 3.4. *For any graph G with at least one edge, and any positive integer k, we have* $r_k \leqslant \chi(G)/2$ *and* $\mathrm{imp}(G) \leqslant \chi_f(G)/2$.

Since $\mathrm{imp}(G) \geqslant \chi_f(G)/\omega(G) = \chi_f(G)/2$ for any triangle-free graph, we obtain

PROPOSITION 3.5. *If G is a triangle-free graph, then* $\mathrm{imp}(G) = \chi_f(G)/2$.

*Proof of Proposition* 3.4. For a weighted graph $G$ with weight vector $\mathbf{x}$ define $\omega_2(G, \mathbf{x})$ to be the maximum of $\max\{x_u + x_v\colon u, v \text{ adjacent in } G\}$ and $\max\{x_v\colon v \in V\}$. Thus $\omega_2(G, \mathbf{x}) \leqslant \omega(G, \mathbf{x})$.

Let $\mathbf{x}$ be an integral weight vector with $\mathrm{imp}(G) = \chi_f(G, \mathbf{x})/\omega(G, \mathbf{x})$ and $\omega_2(G, \mathbf{x})$ even. By removing fractions $y_S$ of stable sets $S$ of $G$ such that $\sum_{S \ni u} y_S = 2$ for each isolated node $u$, $\sum_{S \ni v} y_S = 1$ for all other nodes $v$, and $\sum_S y_S = \chi_f(G)$, we construct a weight vector $\mathbf{x}'$ with $\omega_2(G, \mathbf{x}') = \omega_2(G, \mathbf{x}) - 2$. Repeating this process yields

$$\chi_f(G, \mathbf{x}) \leqslant \frac{\omega_2(G, \mathbf{x})}{2} \chi_f(G).$$

Therefore

$$\mathrm{imp}(G) = \frac{\chi_f(G, \mathbf{x})}{\omega(G, \mathbf{x})} \leqslant \frac{\omega_2(G, \mathbf{x})}{2} \frac{\chi_f(G)}{\omega(G, \mathbf{x})} \leqslant \frac{\chi_f(G)}{2}.$$

The result on $r_k(G)$ can be proved in a similar fashion by removing $\chi(G)$ stable sets covering every isolated node twice and every other node once. ∎

A well-known construction to form triangle-free graphs with high chromatic number is due to Mycielski [26]. Starting with $G_1$ as a single edge we may iteratively form a sequence of triangle-free graphs $G_1$, $G_2$, $G_3$, ... where $G_i$ has $3 \cdot 2^{i-1} - 1$ nodes, $\chi(G_i) = i + 1$, and $\chi_f(G_i) \sim \sqrt{2i}$. The last result is from [18]. The graph $G_2$ is $C_5$ and $G_3$ is the Grötsch graph shown in Fig. 3. We obtain $\text{imp}(G_i) = \chi_f(G_i)/2 \sim \sqrt{i/2} \sim \sqrt{\log |V(G_i)|}$. The graphs $G_i$ are of interest because they have high chromatic number despite being triangle free, but in a sense they are not very imperfect, as the imperfection ratio of a random graph $G_{n,1/2}$ is close to $n/(4 \log^2 n)$ with high probability; see [9].

Next we use Proposition 3.5 to consider planar triangle-free graphs.

Proposition 3.6. *For any triangle-free planar graph G,*

$$\text{imp}(G) \leqslant \frac{3}{2},$$

*and there exists a sequence* $G_1$, $G_2$, ... *of triangle-free planar graphs with* $\text{imp}(G_k) \to 3/2$ *as* $k \to \infty$.

*Proof.* By Grötzsch's theorem [12] (see also [32]), $\chi(G) \leqslant 3$ for any triangle-free planar graph $G$. Thus by Proposition 3.4,

$$\text{imp}(G) \leqslant \frac{\chi_f(G)}{2} \leqslant \frac{\chi(G)}{2} \leqslant \frac{3}{2}.$$

There is a family of planar graphs $G_k$ with girth 5 such that $G_k$ has $3k+2$ nodes and $\alpha(G_k) = k+1$ ([6]; see also [35, p. 283, exercise 7.3.7]). Let $G_1$ be the 5-cycle with nodes $a$, $x_0$, $x_1$, $y_1$, $z_1$ in order. For $k > 1$, $G_k$ is obtained from $G_{k-1}$ by adding three nodes $x_k$, $y_k$, $z_k$ and the five edges $\{x_{k-1}, x_k\}$, $\{x_k, y_k\}$, $\{y_k, z_k\}$, $\{z_k, y_{k-1}\}$, and $\{z_k, x_{k-2}\}$. The graph $G_3$ is shown in Fig. 1. We have $\chi_f(G_k) \geqslant (3k+2)/(k+1)$ and so $\chi_f(G_k) \to 3$ as $k \to \infty$, and the result follows from Proposition 3.5. ▌

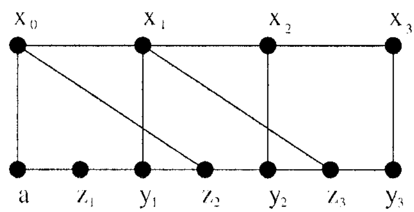


**FIG 1.** The graph $G_3$ on 11 nodes with $\alpha(G_3) = 4$.

A related result concerns planar graphs with no short odd cycles. It was shown in [16] that for any $\varepsilon > 0$, there is a $g$ such that any planar graph $G$ with no odd cycle of length less than $g$ satisfies $\chi_f(G) < 2 + \epsilon$ and so $\mathrm{imp}(G) < 1 + \epsilon$.

Observe that by Proposition 3.4 and the four color theorem, we have $\mathrm{imp}(G) \leqslant 2$ for any planar graph $G$. It is not hard to improve a little on this (though perhaps the truth in 3/2).

PROPOSITION 3.7. *If $G$ is planar, then*

$$\mathrm{imp}(G) \leqslant \frac{11}{6}.$$

To prove the above result we may consider colouring $G_{\mathbf{x}}$ by repeatedly using 4-colourings to reduce the weight vector $\mathbf{x}$ while there remain three pairwise adjacent nodes with positive weights, and then using 3-colourings similarly; see [8].

We commented above that perhaps $\mathrm{imp}(G) \leqslant 3/2$ for any planar graph $G$. Indeed perhaps there is always a 3-colouring of $G$ (not necessarily proper) such that each odd hole has a node of each colour. This latter result would imply the former by the covering result given after Proposition 3.2 and the fact that any planar graph with no odd holes is perfect [34].

We have seen that for a minimal imperfect graph $G$ of order $n$, $\mathrm{imp}(G) = n/(n-1)$. A *hole* in a graph $G$ is an induced cycle of length at least 4. An odd hole is a minimal imperfect graph, so if a graph $G$ has an odd hole of length $g$, then $\mathrm{imp}(G) \geqslant g/(g-1)$. For some graph classes this inequality is in fact an equality. The next proposition states that this holds for any *line graph*. The line graph $L(H)$ of a graph $H$ can be obtained by introducing a node for each edge of $H$ and connecting two nodes by an edge if the corresponding two edges of $H$ are incident.

PROPOSITION 3.8. *Let $G$ be a line-graph. If $G$ has no odd holes then $G$ is perfect and so* $\mathrm{imp}(G) = 1$. *If $G$ has odd holes, and the minimum length of an odd hole is $g$, then* $\mathrm{imp}(G) = g/(g-1)$.

*Proof.* The first part is immediate from known results. If $G$ has no odd holes then $G$ is perfect [33], and therefore $\mathrm{imp}(G) = 1$ by Proposition 3.1.

For the second part we need some notation. For each multigraph $H$ and odd integer $k$ with $3 \leqslant k \leqslant |V(H)|$, let $\Lambda_k(H)$ be the maximum value of $2\,|E(A)|/(k-1)$ over all subsets $A$ of $V(H)$ with $|A| = k$. Here $E(A)$ denotes the set of edges with both ends in $A$. Also let $\Lambda(H)$ be the maximum value of $\Lambda_k(H)$ over the above $k$. It follows from Edmonds' characterisation of the matching polytope, see for example [29], that the fractional edge chromatic number $\chi'_f$ satisfies

$$\chi'_f(H) = \max\{\Delta(H), \Lambda(H)\}, \tag{10}$$

where $\Delta(H)$ is the maximum degree of a node in $H$.

Suppose now that $G = L(H)$ has odd holes. Then $\text{imp}(G) \geqslant g/(g-1)$ as we have seen. Hence it suffices to show that $\chi_f(G, \mathbf{x}) \leqslant (g/(g-1))\, \omega(G, \mathbf{x})$ for each integral weight vector $\mathbf{x}$. But since $H$ can be a multigraph we may assume without loss of generality each $x_v = 1$. So it suffices to show that $\chi_f(G) \leqslant (g/(g-1))\, \omega(G)$. But $\chi_f(G) = \chi'_f(H)$ and $\omega(G) \geqslant \Delta(H)$, and hence in order to complete the proof it suffices by (10) to show that

$$\Lambda(H) \leqslant \frac{g}{g-1}\, \omega(G). \tag{11}$$

First consider an odd integer $k$ with $3 \leqslant k < g$. Let $A$ be a set of $k$ nodes in $H$, and let $H_A$ denote the corresponding induced subgraph of $H$ with this set of nodes. Then $H_A$ has no odd cycles of length greater than 3: thus the line graph $L(H_A)$ has no odd holes and so is perfect by the result of [33] used above. Hence

$$2\,|E(A)|/(k-1) \leqslant \chi(L(H_A)) = \omega(L(H_A)) \leqslant \omega(G).$$

It follows that $\Lambda_k(H) \leqslant \omega(G)$.

Finally let $k$ be an odd integer with $k \geqslant g$. For any set $A$ of $k$ nodes in $H$, $2\,|E(A)| \leqslant k\Delta(H)$, and so

$$\frac{2\,|E(A)|}{k-1} \leqslant \frac{k}{k-1}\,\Delta(H) \leqslant \frac{k}{k-1}\,\omega(G).$$

Thus

$$\Lambda_k(H) \leqslant \frac{k}{k-1}\,\omega(G) \leqslant \frac{g}{g-1}\,\omega(G).$$

Hence (11) holds, as required. ∎

Since we can find for any line graph the length of a shortest odd hole in polynomial time, see for example [8], we can determine the imperfection ratio for line graphs in polynomial time.

Let us consider another class of graphs $G = (V, E)$ for which the imperfection ratio is given as for line-graphs. The *circuit-constrained polytope*

$CSTAB(G)$ is the set of all non-negative vectors $(x_v: v \in V)$ such that $x_u + x_v \leqslant 1$ for all adjacent nodes $u$, $v \in V$, and

$$\sum_{v \in C} x_v \leqslant \frac{|C|-1}{2} \qquad \text{for each odd hole } C.$$

The graph $G$ is called *h-perfect* [11] if $STAB(G) = QSTAB(G) \cap CSTAB(G)$; that is, if $STAB(G)$ is described by the non-negativity constraints, the clique constraints and the odd hole constraints—see [11, 15] for examples of such graphs and for further discussion. Let us simply note here that any series-parallel graph $G$ satisfies $STAB(G) = CSTAB(G)$ (that is, is '$t$-perfect') and hence is $h$-perfect [2].

PROPOSITION 3.9. *Let the graph G be h-perfect. If G has no odd holes then G is perfect, so* $\text{imp}(G) = 1$. *If G has an odd hole, and the shortest length of an odd hole is g, then* $\text{imp}(G) = g/(g-1)$.

*Proof.* If $G$ has no odd holes, then $STAB(G) = QSTAB(G)$: thus $G$ is perfect, and $\text{imp}(G) = 1$. Suppose now that $G$ has an odd hole and let $g$ be the length of a shortest odd hole. Then $\text{imp}(G) \geqslant g/(g-1)$ as before. Let $\mathbf{x} \in QSTAB(G)$, and let $\mathbf{y} = ((g-1)/g)\,\mathbf{x}$. To show that $\text{imp}(G) \leqslant g/(g-1)$, we shall show that $\mathbf{y} \in CSTAB(G)$ since then $\mathbf{y} \in STAB(G)$. Let $C$ be an odd hole: we must show that $\sum_{v \in C} y_v \leqslant (|C|-1)/2$. But, since $|C| \geqslant g$,

$$\sum_{v \in C} y_v = \frac{g-1}{g} \sum_{v \in C} x_v \leqslant \frac{g-1}{g} \frac{|C|}{2} \leqslant \frac{|C|-1}{|C|} \frac{|C|}{2} = \frac{|C|-1}{2},$$

as required. ∎

To close this section let us note a result related to the last one, which brings odd antiholes into the picture. An *antihole* in a graph $G$ is a hole in the complement $\bar{G}$ of $G$. Corresponding to $CSTAB(G)$, let us define $\bar{C}STAB(G)$ to be the set of all non-negative vectors $(x_v: v \in V)$ such that

$$\sum_{v \in A} x_v \leqslant 2 \qquad \text{for each odd antihole } A.$$

PROPOSITION 3.10. *Suppose that the graph G satisfies*

$$STAB(G) = QSTAB(G) \cap CSTAB(G) \cap \bar{C}STAB(G).$$

(*In particular this holds if G is h-perfect.*) *If G has no odd holes or antiholes then G is perfect, so* $\text{imp}(G) = 1$. *If G has an odd hole or antihole, and g is the least number of nodes in an odd hole or antihole, then* $\text{imp}(G) = g/(g-1)$.

*Proof.* We need to add just one step to the proof of Proposition 3.9. As there, let $\mathbf{x} \in QSTAB(G)$ and let $\mathbf{y} = ((g-1)/g)\,\mathbf{x}$. The earlier proof showed that $\mathbf{y} \in CSTAB(G)$. We must show that $\mathbf{y} \in \bar{C}STAB(G)$.

Let $A$ be an odd antihole: we must show that $\sum_{v \in A} y_v \leqslant 2$. But, since $|A| \geqslant g$ and each node of $A$ can be covered $(|A|-1)/2$ times by $|A|$ cliques,

$$\sum_{v \in A} y_v = \frac{g-1}{g} \sum_{v \in A} x_v \leqslant \frac{g-1}{g} \frac{2\,|A|}{|A|-1} \leqslant 2,$$

as required. ▮

## 4. LARGE INTEGER WEIGHTS ARE NEEDED TO ACHIEVE imp($G$)

In Section 3 we saw various examples where the imperfection ratio equals the *binary imperfection ratio* $\text{imp}_b(G)$, which we define to be the maximum of $\chi_f(H)/\omega(H)$ over all induced subgraphs $H$ of $G$, or equivalently

$$\text{imp}_b(G) = \max \left\{ \frac{\chi_f(G, \mathbf{x})}{\omega(G, \mathbf{x})} : \mathbf{x} \in \{0, 1\}^V \right\}.$$

In this section we investigate the binary imperfection ratio, and the integer weight vectors $\mathbf{x}$ which *achieve the imperfection ratio*, that is, are such that $\text{imp}(G) = \chi_f(G, \mathbf{x})/\omega(G, \mathbf{x})$. It turns out that in general $\text{imp}(G) \neq \text{imp}_b(G)$. In fact the integer weights $x_v$ required to achieve $\text{imp}(G)$ can grow exponentially with the order $n$ of a graph, though they can always be bounded by $n^{n/2}$. At the end of this section we exhibit a graph $G$ with $\text{imp}_b(G) = \text{imp}(G)$ but $\text{imp}_b(\bar{G}) \neq \text{imp}(\bar{G})$: it follows that the binary imperfection ratio has the unpleasant feature that we can have $\text{imp}_b(G) \neq \text{imp}_b(\bar{G})$.

Let $G$ and $H$ be two graphs, and let $v$ be a node of $G$. Let $G[v \leftarrow H]$ denote the graph where $v$ is replaced by $H$ and every node of $H$ is adjacent to the neighbours of $v$ (and to no other nodes of $G$). Let $G^0 = C_5$ and let $G^1 = C_5[v \leftarrow G^0]$, where $v$ is a node of $C_5$. The graph $G^1$ is pictured in Fig. 2. The next proposition concerns a sequence of graphs starting with these two. It shows in particular that $\text{imp}(G^1) = 21/16$. However, it is straightforward to check that $\text{imp}_b(G^1) = 5/4$. Thus we have

$$\text{imp}(G^1) = 21/16 > 5/4 = \text{imp}_b(G^1).$$

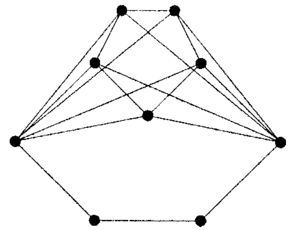

**FIG 2.** The graph $G^1 = C_5[v \leftarrow C_5]$.

We call a vector $\mathbf{x} \in QSTAB(G)$ *extremal* for $G$ if $\chi_f(G, \mathbf{x}) = \mathrm{imp}(G)$. We know from Theorem 2.1 that every graph $G$ has at least one extremal vector.

PROPOSITION 4.1. *Let $G^i$ be the graph on $5+4i$ nodes which is defined by setting $G^0 = C_5$ and for $i=0, 1, \ldots$ setting $G^{i+1} = C_5[v \leftarrow G^i]$, where $v$ is a node of $C_5$. Then for each $i = 0, 1, \ldots$*

(i) $\mathrm{imp}(G^i) = \frac{4}{3} - \frac{1}{12}(\frac{1}{4})^i$;

(ii) *there is a unique extremal vector $\mathbf{x}^i$ for $G^i$; and*

(iii) *integer weights up to $2^i$ are needed to achieve* $\mathrm{imp}(G^i)$*—indeed $2^{i+1}\mathbf{x}^i$ takes precisely the values* $1, 2, 4, \ldots, 2^i$, *and a non-zero integral vector $\mathbf{x}$ achieves* $\mathrm{imp}(G^i)$ *if and only if it is an integral multiple of this vector.*

Before we prove this result we establish a useful technical lemma (see also [3]), and we then investigate the extremal vectors $\mathbf{x}$ for the 5-cycle $C_5$.

LEMMA 4.1. *Let $G$ and $H$ be two graphs, let $v$ be a node of $G$, and let $\mathbf{x}$ be a weight vector for the graph $G[v \leftarrow H]$. Let $\mathbf{x}_H$ denote the restriction of $\mathbf{x}$ to the nodes of $H$. Then $\chi_f(G[v \leftarrow H], \mathbf{x}) = \chi_f(G, \hat{\mathbf{x}})$ where $\hat{\mathbf{x}}$ is the weight vector for $G$ with $\hat{x}_u = x_u$ for each $u \in V(G)\setminus\{v\}$ and $\hat{x}_v = \chi_f(H, \mathbf{x}_H)$.*

*Proof.* Denote the set of all stable sets of a graph $G$ by $\mathscr{S}_G$. For a graph $G$ with weight vector $\mathbf{x}$, we call a non-negative vector $\mathbf{y} = (y_S\colon S \in \mathscr{S}_G)$ a *fractional stable set cover* if $\sum_{S \ni v} y_S \geqslant x_v$ for all $v \in V(G)$. The *cost* of $\mathbf{y}$ is $\sum_{S \in \mathscr{S}_G} y_S$. A fractional stable set cover is called *optimal* if its cost is minimal, that is if the cost equals $\chi_f(G, \mathbf{x})$.

Let $\mathbf{y}$ be an optimal fractional stable set cover of $(H, \mathbf{x})$, and let $\mathbf{z}$ be an optimal fractional stable set cover of $(G, \hat{\mathbf{x}})$. Define the vector $\mathbf{z}'$ indexed by the stable sets $S$ of $G[v \leftarrow H]$ as

$$z'_S = z_S, \qquad \text{if } S \cap V(H) = \emptyset, \text{ and otherwise}$$

$$z'_S = \frac{z_{\bar{S}}\, y_{S^*}}{\chi_f(H, \mathbf{x})}, \qquad \text{where } \bar{S} = \{v\} \cup (S \setminus V(H)) \text{ and } S^* = S \cap V(H).$$

It is routine to check that $\mathbf{z}'$ is a fractional stable set cover of $(G[v \leftarrow H], \mathbf{x})$ with cost $\chi_f(G, \hat{\mathbf{x}})$. Therefore $\chi_f(G[v \leftarrow H], \mathbf{x}) \leqslant \chi_f(G, \hat{\mathbf{x}})$.

To show that $\chi_f(G[v \leftarrow H], \mathbf{x}) \geqslant \chi_f(G, \hat{\mathbf{x}})$ we consider an optimal fractional stable set cover $\mathbf{z}'$ of $(G[v \leftarrow H], \mathbf{x})$. Define the vector $\mathbf{z}$ indexed by the stable sets $S$ of $G$ as

$$z_S = z'_S, \qquad \text{if } v \notin S, \text{ and otherwise}$$

$$z_S = \sum z'_{\bar{S}} \qquad \text{where the sum is over all stable sets } \bar{S} \text{ of } G[v \leftarrow H] \text{ with } \bar{S} \cap V(H) \neq \emptyset,\ S = \{v\} \cup (\bar{S} \setminus V(H)).$$

As before, it is routine to check that $\mathbf{z}$ is a fractional stable set cover of $(G, \hat{\mathbf{x}})$ with cost $\chi_f(G[v \leftarrow H], \mathbf{x})$. Therefore $\chi_f(G[v \leftarrow H], \mathbf{x}) \geqslant \chi_f(G, \hat{\mathbf{x}})$. ∎

LEMMA 4.2. *Consider the 5-cycle $C_5$ on the node set $\{v_0, v_1, v_2, v_3, v_4\}$ with edge set $\{\{v_0,v_1\}, \{v_1,v_2\}, \{v_2,v_3\}, \{v_3,v_4\}, \{v_4,v_0\}\}$. For each $r \geqslant 0$, let $f(r) = \max\{\chi_f(C_5, (rx_0, x_1, x_2, x_3, x_4))\colon \mathbf{x} \in QSTAB(C_5)\}$. Then for $1 \leqslant r < 4/3$, $f(r) = 1 + r/4$, and the unique point of $QSTAB(C_5)$ which achieves this value is $\mathbf{z} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.*

*Proof.* For $\mathbf{x} \in QSTAB(C_5)$ and $r \geqslant 0$, let $r(\mathbf{x})$ denote the vector $(rx_0, x_1, x_2, x_3, x_4)$. Thus $f(r) = \max\{\chi_f(C_5, r(\mathbf{x}))\colon \mathbf{x} \in QSTAB(C_5)\}$. First consider the vertex $\mathbf{z} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ of $QSTAB(C_5)$. Let $0 \leqslant r \leqslant 2$. By assigning weight $r/4$ to the stable sets $\{v_0, v_2\}$, $\{v_0, v_3\}$, and $\{v_1, v_4\}$, and weight $1/2 - r/4$ to the stable sets $\{v_1, v_3\}$ and $\{v_2, v_4\}$, we see that $\chi_f(C_5, r(\mathbf{z})) \leqslant 1 + r/4$. Since $\sum_{i=0}^{4} r(z)_i = (4+r)/2$ and the maximum stable set size is 2, $\chi_f(C_5, r(\mathbf{z})) \geqslant 1 + r/4$. Therefore, $f(r) \geqslant \chi_f(C_5, r(\mathbf{z})) = 1 + r/4$ for all $0 \leqslant r \leqslant 2$.

Now let $1 \leqslant r < 4/3$. Consider any vertex $\mathbf{x}$ of $QSTAB(C_5)$ other than $\mathbf{z}$. Then some coordinate $x_i = 0$. (We are writing $x_i$ rather than $x_{v_i}$.) Let $\mathbf{y}$ denote the weight vector for the path $P_4$ on the four nodes other than $v_i$, where $y_j = r(\mathbf{x}_j)$ for each $j \neq i$. Then

$$\chi_f(C_5, r(\mathbf{x})) = \chi_f(P_4, \mathbf{y}) = \omega(P_4, \mathbf{y}) \leqslant r.$$

But $r < 1 + r/4$ since $r < 4/3$, and therefore $\chi_f(G, r(\mathbf{x})) < \chi_f(G, r(\mathbf{z}))$. We have now shown that $\chi_f(G, r(\mathbf{x})) < \chi_f(G, r(\mathbf{z}))$ for each vertex $\mathbf{x}$ of $QSTAB(C_5)$ other than $\mathbf{z}$, and it follows easily that this holds also for each point $\mathbf{x}$ of $QSTAB(C_5)$ other than $\mathbf{z}$. ∎

*Proof of Proposition* 4.1. The case $r = 1$ of Lemma 4.2 implies that $\mathrm{imp}(G^0) = \mathrm{imp}(C_5) = 5/4$ and that $\mathbf{x}^0 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is the unique extremal vector for $G^0$. Now let $i \geqslant 0$, and assume as our induction hypothesis that there is a unique extremal vector $\mathbf{x}^i$ for $G^i$ and that $\mathrm{imp}(G^i) < 4/3$.

Consider any weight vector $\mathbf{x}$ for $G^{i+1}$. Recall that $G^{i+1}$ is obtained by replacing a node of $C_5$, say $v_0$, by the graph $G^i$. Let $\mathbf{x}'$ denote the restriction of $\mathbf{x}$ to the nodes of $G^i$. Let $r = \chi_f(G^i, \mathbf{x}')/\omega(G^i, \mathbf{x}')$, so that $r \leqslant \text{imp}(G^i) < 4/3$. It follows from Lemma 4.1 that $\chi_f(G^{i+1}, \mathbf{x}) = \chi_f(C_5, \hat{\mathbf{x}})$ where $\hat{x}_0 = \chi_f(G^i, \mathbf{x}')$ and $\hat{x}_j = x_j$ for $j = 1, 2, 3, 4$. Now $(1/r)\,\hat{x}_0 = \omega(G^i, \mathbf{x}')$ and so $((1/r)\,\hat{x}_0, \hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4) \in QSTAB(C_5)$. Hence by Lemma 4.2,

$$\chi_f(G^{i+1}, \mathbf{x}) \leqslant 1 + r/4 \leqslant 1 + \text{imp}(G^i)/4.$$

Further

$$\chi_f(G^{i+1}, \mathbf{x}) = 1 + \text{imp}(G^i)/4$$

if and only if $r = \text{imp}(G^i)$ (that is, $\mathbf{x}'$ achieves the imperfection ratio for $G^i$), and $\omega(G^i, \mathbf{x}') = 1/2$ and $\hat{x}_j = 1/2$ for $j = 1, ..., 4$, that is, if and only if $\mathbf{x}' = (1/2)\,\mathbf{x}^i$ and $x_j = 1/2$ for $j = 1, ..., 4$, using the induction hypothesis. Thus $\text{imp}(G^{i+1}) = 1 + \text{imp}(G^i)/4$, and there is exactly one extremal vector for $G^{i+1}$, related to $\mathbf{x}^i$ as described. Note that $\text{imp}(G^{i+1}) < 4/3$ since $\text{imp}(G^i) < 4/3$. This proves the first two parts of the proposition.

The discussion above implies that for $i = 1, 2, ...$ the unique extremal vector $\mathbf{x}^i$ for $G^i$ has the following form: four nodes have weight $2^{-1}$ (the nodes in $V(G^i) \backslash V(G^{i-1})$, that is the remaining four nodes of the $C_5$ of $C_5[v \leftarrow G^{i-1}]$), four nodes have weight $2^{-2}$ (the nodes in $V(G^{i-1}) \backslash V(G^{i-2})$), ..., four nodes have weight $2^{-i}$ (the nodes in $V(G^1) \backslash V(G^0)$), and five nodes have weight $2^{-i-1}$ (the nodes originating from $G^0$). The final part of the proposition now follows. ∎

Proposition 4.2. *Let $G$ be a graph on $n$ nodes. Then there exists an integral weight vector* $\mathbf{x}$ *with* $\text{imp}(G) = \chi_f(G, \mathbf{x})/\omega(G, \mathbf{x})$ *and each coordinate at most* $2^{-n}(n+1)^{(n+1)/2} \leqslant n^{n/2}$.

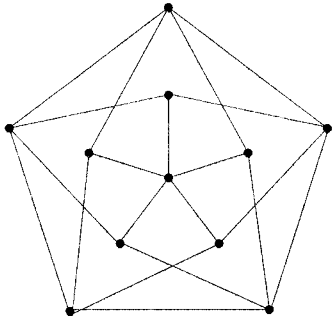

**FIG 3.** Grötsch graph.

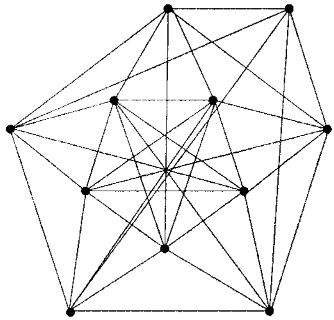

**FIG 4.** Complement of Grötsch graph.

*Proof.* By Theorem 2.1(2) there exists a vertex $\mathbf{y}$ of $QSTAB(G)$ with $\mathrm{imp}(G) = \chi_f(G, \mathbf{y})$. Any such vertex $\mathbf{y}$ is the unique solution of $A\mathbf{z} = \mathbf{b}$ for some $n \times n$ matrix $A$ with 0, 1 entries, and some 0, 1 vector $\mathbf{b}$. Therefore $\mathbf{y}$ has entries of the form $a_i/\det(A)$ for integers $0 \leqslant a_i \leqslant \det(A)$, $i = 1, ..., n$. But since $A$ is a 0, 1-matrix, $\det(A) \leqslant 2^{-n}(n+1)^{(n+1)/2}$ [1]. Setting $\mathbf{x} = \det(A)\,\mathbf{y}$ yields the first inequality. Finally, we have $2^{-n}(n+1)^{(n+1)/2} \leqslant 2^{-n}(n+1)^{1/2}\,(1+1/n)^{n/2}\,n^{n/2} \leqslant 2^{-n}(e(n+1))^{1/2}\,n^{n/2} \leqslant n^{n/2}$ for $n \geqslant 2$. ▮

Finally, consider the Grötsch (or Mycielski) graph $G$ shown in Fig. 3 and its complement $\bar{G}$ shown in Fig. 4. The Grötsch graph is triangle-free, and hence by Proposition 3.5, $\mathrm{imp}(G) = \mathrm{imp_b}(G) = \chi_f(G)/2 = 29/20$. One can calculate that $\mathrm{imp_b}(\bar{G}) = 4/3$ and thus the following proposition holds.

PROPOSITION 4.3. *There exists graphs $G$ with $\mathrm{imp_b}(G) = \mathrm{imp}(G)$ but $\mathrm{imp_b}(\bar{G}) \neq \mathrm{imp}(\bar{G})$.*

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