Chapter 5

Connectivity

5.1 Introduction

The (strong) connectivity corresponds to the fact that a (directed) (u,v)-path exists for any pair of vertices u and v. However, imagine that the graphs models a network, for example the vertices correspond to computers and edges to links between them. An important issue is that the connectivity remains even if some computers and links fail. This will be measured by the notion of k-connectivity.

Let G be a graph. Let W be an set of edges (resp. of vertices). If $G \setminus W$ (resp. $G - W$) is not connected, then W *separates* G, and W is called an *edge-separator* (resp. *vertex-separator*) or simply *separator* of G.

For any $k \geq 1$, G is *k-connected* if it has order at least $k+1$ and no set of $k-1$ vertices is a separator. In particular, the complete graph K_{k+1} is the only k-connected graph with $k+1$ vertices. The *connectivity* of G, denoted by $\kappa(G)$, is the maximum integer k such that G is k-connected. Similarly, a graph is *k-edge connected* if it has at least two vertices and no set of $k-1$ edges is a separator. The *edge-connectivity* of G, denoted by $\kappa'(G)$, is the maximum integer k such that G is k-edge-connected.

For any vertex x, if S is a vertex-separator of $G-x$ then $S \cup \{x\}$ is a vertex-separator of G, hence

$$\kappa(G) \leq \kappa(G-x)+1.$$

However, the connecivity of $G-x$ may not be upper bounded by a function of $\kappa(G)$; see Exercise 5.3 (ii).

Regarding egde-connectivity, things are a bit easier. Indeed, for any edge $e \in E(G)$, F is an edge-separator of $G \setminus e$ if and only if $F \cup \{e\}$ is an edge-separator of G. Hence

$$\kappa'(G) - 1 \leq \kappa'(G \setminus e) \leq \kappa'(G).$$

By definition, being 1-connected, 1-edge-connected or connected is equivalent. For larger value of k, k-connectivity implies k-edge-connectivity.

Proposition 5.1. *Let G be a graph with at least two vertices.*

$$\kappa(G) \leq \kappa'(G) \leq \delta(G).$$

Proof. Removing all edges incident to a vertex makes the graph disconnected. Hence, $\kappa'(G) \leq \delta(G)$.

Let us assume that $\kappa'(G) = k$. Let $F = \{x_1y_1, x_2y_2, \dots, x_ky_k\}$ be an edge-separator of G such that all x_i's are in the same connected component C of $G \setminus F$. If $G - \{x_1, x_2, \dots, x_k\}$ is not connected, then $\kappa(G) \leq k$. Else, $C = \{x_1, x_2, \dots, x_k\}$. Hence, x_1 has at most k neighbours, namely the x_i's for $x_i \neq x_1$ and the y_i's such that $x_i = x_1$. Then the neighbourhood of x_1 is a vertex-separator of size at most k. So $\kappa(G) \leq k \leq \kappa'(G)$. □

Reciprocally, the edge-connectivity of a graph cannot be bounded by its connectivity. See Exercise 5.3.

A natural question is how to add a vertex (computer) to an already existing k-connected graph (network) so that it remains k-connected. Obviously, since the connectivity is at least the minimum degree by Proposition 5.1, one needs to link the new vertex to at least k existing vertices. This easy necessary condition is in fact sufficient.

Lemma 5.2. *Let G be a k-connected graph. If G' is obtained from G by adding a new vertex x adjacent to at least k vertices of G, then G' is k-connected.*

Proof. Let S be a separator S of G'. Let us show that $|S| \geq k$. If S contains x, then $S \setminus \{x\}$ must be a separator of G. Since G is k-connected then $|S \setminus \{x\}| \geq k$ and so $|S| \geq k+1$. Assume now that $x \notin S$. If $N(x) \subseteq S$ then $|S| \geq k$. Else, $N(x) \setminus S \neq \emptyset$ and $N(x) \setminus S$ belongs to a unique connected component of $G' \setminus S$ (the one of x). Hence, S is a separator of G. Thus $|S| \geq k$ because G is k-connected. □

Similarly, adding a new vertex of degree k to a k-edge-connected graph yields a k-edge-connected graph.

Lemma 5.3. *Let G be a k-edge-connected graph. If G' is obtained from G by adding a new vertex x adjacent to at least k vertices of G, then G' is k-edge-connected.*

Proof. Left in Exercise 5.5 □

5.2 2-edge-connected graphs

For 2-edge-connected graphs, there is a structural theorem similar to Theorem 1.15 for strongly connected digraphs. It can be proved in exactly the same way.

The following proposition follows easily from the definition of 2-edge-connectivity.

Proposition 5.4. *Let D be a* 2*-edge connected graph. Then every edge is in a cycle.*

Proof. Let $e = uv$ be an edge. Since G is 2-edge-connected then $G \setminus e$ is connected. Thus there is a (v,u)-path in $G \setminus e$. Its concatenation with (u,v) is a cycle containing e. □

Definition 5.5. Let G be a graph and H be a subgraph of G. A *H-handle* is a path or a cycle (all vertices are distinct except possibly the two endvertices) such that its endvertices are in $V(H)$ and its internal vertices are in $V(G) \setminus V(H)$. A *handle decomposition* of G is a sequence $(C, P_1, \dots, P_k)$ such that:

- $C = G_0$ is a cycle;
- for all $1 \leq i \leq k$, P_i is a G_{i-1}-handle and $G_i = G_{i-1} \cup P_i$;
- $G_k = G$.

It is straightforward to show that if H is a 2-edge-connected subgraph of a graph G, the graph $H \cup P$ is 2-edge-connected for any H-handle P. (See Exercise 5.6.) Hence, an easy induction immediately yields that every graph admitting a handle decomposition is 2-edge-connected. Conversely, every 2-edge-connected graph admits a handle decomposition starting at any cycle.

Theorem 5.6. *Let G be a 2-edge-connected graph and C a cycle. Then G has a handle decomposition $(C, P_1, \ldots, P_k)$.*

Proof. Let H be the largest subgraph of G such that H admits a handle decomposition $(C, P_1, \ldots P_k)$. Since every edge xy in $E(G) \setminus E(H)$ with both endvertices in $V(H)$ is a H-handle, H is an induced subgraph of G. Suppose for a contradiction that $H \neq G$, then $V(H) \neq V(G)$. Since G is 2-edge connected, there is an edge vw with $v \in V(G)$ and $w \in V(G) \setminus V(H)$. Since G is 2-edge connected, G contains a (w, H)-path P. Then, the concatenation of (v, w) and P is a H-handle in G, contradicting the maximality of H. □

Corollary 5.7 (Robbins, 1939)**.** *A graph admits a strongly connected orientation if and only if it is* 2*-edge connected.*

Proof. Necessity: If a graph G is not connected, then there is no directed path between any two vertices in distinct components whatever be the orientation. Let us assume that G has an edge uv such that $G \setminus uv$ is not connected. Let C_u and C_v be the connected components of u and v in $G \setminus uv$. Then, if uv is oriented from u to v, there is no directed (v, u)-path using this orientation.

Sufficientcy: Let us assume that G is 2-edge connected. From Theorem 5.6, G admits a handle decomposition $(C, P_1, \ldots, P_k)$. Orienting C into a directed cycle and each P_i, $1 \leq i \leq k$, into a directed path, we obtain an orientation D of G having a handle decomposition. So, by Theorem 1.15, D is strongly connected. □

5.3 2-connected graphs

For 2-connected graphs, there is a structural theorem similar to Theorems 5.6 and 1.15.

Observe that since a 2-connected graph is also 2-edge-connected by Proposition 5.1, every edge of a 2-connected graph contains is in a cycle. More generally, for any two vertices x and y (not necessarily adjacent) there is a cycle containing x and y. See Exercise 5.7.

Definition 5.8. Let G be a graph and H be a subgraph of G. A *H-ear* is a path whose endvertices are in $V(H)$ and whose internal vertices are in $V(G) \setminus V(H)$. An *ear decomposition* of G is a sequence $(C, P_1, \ldots, P_k)$ such that:

- $C = G_0$ is a cycle;

- for all $1 \leq i \leq k$, P_i is a G_{i-1}-ear and $G_i = G_{i-1} \cup P_i$;
- $G_k = G$.

It is straightforward to show that if H is a 2-connected subgraph of a graph G, the graph $H \cup P$ is 2-connected for any H-ear P. (See Exercise 5.6.) Hence, an easy induction immediately yields that every graph admitting an ear decomposition is 2-connected. Conversely, every 2–connected graph admits an ear decomposition starting at any cycle.

Theorem 5.9. *Let G be a 2-connected graph and C a cycle. Then G has an ear decomposition $(C, P_1, \dots, P_k)$.*

Proof. Let H be the largest subgraph of D such that H admits an ear decomposition $(C, P_1, \dots P_k)$. Since every edge xy in $E(G) \setminus E(H)$ with both endvertices in $V(H)$ is a H-ear, H is an induced subgraph of G. Suppose for a contradiction that $H \neq G$, then $V(H) \neq V(G)$. Since G is connected, there is an edge vw with $v \in V(G)$ and $w \in V(G) \setminus V(H)$. Since G is 2-connected, $G - v$ contains a (w, H)-ear P. The concatenation of (v, w) and P is a H-ear in G, contradicting the maximality of H. □

5.4 Contraction and k-connected graphs

Definition 5.10. Let $e = xy$ be an edge of a graph $G = (V, E)$. Let G/e denote the graph obtained from G by *contracting* the edge e in a new vertex v_e that is adjacent to all neighbours of x and y. Formally, G/e has vertex set $V' = (V \setminus \{x, y\}) \cup \{v_e\}$ and edge set $E' = \{vw \in E \mid \{v, w\} \cap \{x, y\} = \emptyset\} \cup \{v_e w \mid w \in (N_G(x) \cup N_G(y)) \setminus \{x, y\}\}$.

Since a connected graph contains a spanning tree (Corollary 1.10), contracting an edge of the tree incident to one of its leaves, we get a smaller connected graph.

Proposition 5.11. *If G is connected and $|V(G)| > 1$ then there is an edge e of G such that G/e is connected.*

Similarly, any (large enough) 2-connected graph can be contracted into a smaller 2-connected graph.

Proposition 5.12. *If G is 2-connected and $|V(G)| > 3$ then there is an edge e of G such that G/e is 2-connected.*

Proof. Assume G is 2-connected and $|V(G)| > 3$. By Proposition 5.4, G contains a cycle. Let C be a cycle of minimum length. By Theorem 5.9, G admits an ear decomposition $(C, P_1, P_2, \dots, P_r)$. Let m be the greatest index such that P_m is not an edge. Then, $G' = G \setminus \bigcup_{i=m+1}^{r} E(P_i)$ is a spanning 2-connected subgraph of G. Let e be an edge of P_m. If $m = 1$, then $G = C$ is a cycle of length at least 4 and thus G/e is a cycle and so 2-connected. If $m \geq 2$, then G'/e can be obtained from C by adding the H-paths $P_1, P_2, \dots, , P_{m-1}, P_m/e$. Hence, from Theorem 5.9, G'/e is 2-connected. Since G/e is a supergraph of G'/e (obtained by adding the edges corresponding to P_i, $m < i \leq r$), then G/e is 2-connected. □

Lemma 5.13. *If G is 3-connected and $|V(G)| > 4$ then there is an edge e of G such that G/e is 3-connected.*

Proof. Assume for a contradiction that such an edge does not exist. Then, for any $xy \in E(G)$, the graph G/xy contains a separator S with at most two vertices. Since $\kappa(G) \geq 3$, the vertex v_{xy} of G/xy resulting from the contraction is in S and $|S| = 2$. Let z be the vertex in $S \setminus \{v_{xy}\}$. Then, $\{x,y,z\}$ is a separator of G. Since no proper subset of $\{x,y,z\}$ is a separator of G, every vertex in $\{x,y,z\}$ has a neighbour in each component of $G - \{x,y,z\}$.

Consider x, y and z, as above, such that a component C of $G - \{x,y,z\}$ has minimum size. Let v be the neighbour of z in C. By assumption, G/zv is not 3-connected, so there exists a vertex w such that $\{z,v,w\}$ is a separator of G. Again, every vertex in $\{z,v,w\}$ has a neighbour in every component of $G - \{z,v,w\}$.
At least one of x and y, say x is not w and so the connected component of x in $G - \{z,v,w\}$ contains all the connected components of $G - \{x,y,z\}$. Since x and y are adjacent, there is a component D of $G - \{z,v,w\}$ that contains none of x and y. Hence any other one (there is at least one since $G - \{z,v,w\}$ is not connected) must be included in C and thus is smaller than because $v \in C$. This contradicts the minimality of C.

□

Hence, any 3-connected graph can be reduced to K_4 by a succession of edge-contractions. One can show that, reciprocally, any such graph is 3-connected. See Exercise 5.21.

Theorem 5.14 (Tutte, 1961)**.** *A graph G is 3-connected if and only if there is a sequence $G_0, \ldots, G_n$ of graphs such that:*

(i) $G_0 = K_4$ *and* $G_n = G$;

(ii) for any $i < n$, *there is an edge* xy *of* G_{i+1} *such that* $d(x), d(y) \geq 3$ *and* $G_i = G_{i+1}/xy$.

Propositions 5.11 and 5.12 and Lemma 5.13 cannot be generalized to the 4-connected case. Indeed, the square of a cycle, depicted in Figure 5.1, is 4-connected. However, the contraction of any edge creates a vertex with degree three.

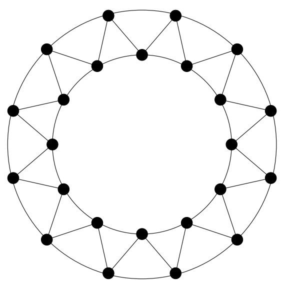


Figure 5.1: 4-connected graph such that the contraction of any edge makes it 3-connected

Any 4-connected graph (except K_5) can be modified into a smaller 4-connected graph by contracting one or two edges. If $k \geq 6$, for any $b \geq 1$, there are k-connected graphs with arbitrary size that cannot be reduced to a k-connected graph by contracting at most b edges (e.g., planar triangulations of a torus). The question is open for $k=5$. For more on this topic, we refer the reader to the survey of M. Kriesell [1].

5.5 Connectivity in digraphs

Similar concepts to k-connectivity and k-edge-connectivity may be defined for digraphs.

Let D be a digraph. Let W be an set of arcs (resp. of vertices). If $D \setminus W$ (resp. $D - W$) is not strongly connected, then W *separates* D, and W is called an *arc-separator* (resp. *vertex-separator*) or simply *separator* of D.

For any $k \geq 1$, D is *k-strongly connected* if if it has order at least $k+1$ and set of $k-1$ vertices is a separator. In particular, the complete symmetric digraph $\vec{K}_{k+1}$ is the only k-strongly connected digraph with $k+1$ vertices. The *strong connectivity* of D, denoted by $\kappa(D)$, is the maximum integer k such that D is k-strongly connected. Similarly, D is *k-arc connected* if it has at least two vertices and no set of $k-1$ arcs is a separator. The *arc-connectivity* of D, denoted by $\kappa'(D)$, is the maximum integer k such that D is k-arc-connected.

Results similar to the one proved in Section 5.1 hold for strong-connectivity and arc-connectivity. For any vertex x, if S is a vertex-separator of $D-x$ then $S \cup \{x\}$ is a vertex-separator of D, hence

$$\kappa(D) \leq \kappa(D-x)+1.$$

However, the strong connectivity of $D-x$ may not be upper bounded by a function of $\kappa(D)$.

For any arc $e \in E(D)$, F is an arc-separator of $D \setminus e$ if and only if $F \cup \{e\}$ is an arc-separator of D. Hence

$$\kappa'(D) - 1 \leq \kappa'(D \setminus e) \leq \kappa'(D).$$

By definition, being 1-strongly connected, 1-arc-connected or strongly connected is equivalent. For larger value of k, k-strong connectivity implies k-arc-connectivity.

Proposition 5.15. *Let D be a graph with at least two vertices.*

$$\kappa(D) \leq \kappa'(D) \leq \min\{\delta^+(D), \delta^-(D)\}.$$

Proof. Removing all arcs leaving a vertex results in a digraph which not strongly connected. Hence, $\kappa'(D) \leq \delta^+(D)$. By directional duality, $\kappa'(D) \leq \delta^+(D)$.

Let us assume that $\kappa'(D) = k \geq 2$. Let $F = \{x_1y_1, x_2y_2, \dots, x_ky_k\}$ be an arc-separator of D. Then $D \setminus F$ has two strongly connected components X and Y such that $x_i \in X$ and $y_i \in Y$, for all $1 \leq i \leq k$ and there is no arc with tail in X and head in Y except those of F (see Exercise 5.4-2)). If $D-$ is not strongly connected, then $\kappa(G) \leq k$. Else $X = \{x_1, \dots, x_k\}$. Then x_1 has at most k outneighbours, namely the x_i's for $x_i \neq x_1$ and the y_i's such that $x_i = x_1$. Then the outneighbourhood of x_1 is a vertex-separator of size at most k. So $\kappa(D) \leq k \leq \kappa'(D)$. □

Reciprocally, the arc-connectivity of a graph cannot be bounded by its strong connectivity.

In fact, as we shall see, all the results on connectivity and edge-connectivity may be seen as particular cases of results on strong connectivity and arc-connectivity for symmetric digraphs. Indeed, for any graph G, its *associated digraph*, denoted $\vec{G}$ is the symmetric digraph obtained from G by replacing each edge xy by the two arcs (x,y) and (y,x).

The following proposition follows directly form the definition of $\vec{G}$.

Proposition 5.16. *Let G be a graph and $\vec{G}$ is associated digraph.*

(i) $(v_1, v_2, \dots, v_p)$ *is a path in G if and only if* $(v_1, v_2, \dots, v_p)$ *is a directed path in $\vec{G}$.*

(ii) G is connected if and only if $\vec{G}$ is strongly connected.

Theorem 5.17. *Let G be a graph and $\vec{G}$ is associated digraph. Then*

(i) $\kappa(G) = \kappa(\vec{G})$.

(ii) $\kappa'(G) = \kappa'(\vec{G})$.

Proof. Let W be a set of vertices. Then the digraph associated to $G - W$ is $\vec{G} - W$. Hence by Proposition 5.16-(ii) the graph $G - W$ is connected if and only if $\vec{G} - W$ is strongly connected. In other words, W is a vertex-separator of G if and only if it is a vertex-separator of $\vec{(}G)$. This implies (i).

Let us now prove (ii). Let F be a minimum edge-separator of G. Then $G \setminus F$ has two components X and Y (see Exercise 5.4-1)). Let $F = \{x_1y_1, x_2y_2, \dots, x_ky_k\}$ with $x_i \in X$ and $y_i \in Y$, for all $1 \le i \le k$. Then $\vec{F} = \{(x_1,y_1), (x_2,y_2), \dots, (x_k,y_k)\}$ is an arc-separator of $\vec{G}$ since there are no arcs with tail in X and head in Y. So $\kappa'(\vec{G}) \le \kappa'(G)$.

Conversely assume that $\vec{F} = \{(x_1,y_1), (x_2,y_2), \dots, (x_k,y_k)\}$ is a minimum arc-separator of $\vec{G}$. Then $\vec{G} \setminus \vec{F}$ has two strongly connected components X and Y such that $x_i \in X$ and $y_i \in Y$, for all $1 \le i \le k$ and there is no arc with tail in X and head in Y except those of $\vec{F}$ (see Exercise 5.4-2)). Thus $F = \{x_1y_1, x_2y_2, \dots, x_ky_k\}$ is an edge-separator of G. □

In view of Theorem 5.17, Proposition 5.1 may be seen as Proposition 5.15 in the case of symmetric digraphs.

5.6 Menger's Theorem

Let W be an edge-separator (resp. vertex-separator) of a graph G. If two vertices u and v are in two distinct connected components of $G \setminus W$ (resp. $G - W$) , then W separates two vertices u and v and is called a (u,v)*-separator*. A separator of a graph is necessarily a (u,v)-separator for some pair of vertices. In addition, if we consider a vertex-separator, these two vertices are not adjacent.

Let u and v be two vertices of a graph G. The *edge-connectivity between u and v* or (u,v)-*edge-connectivity* in G, denoted by $\kappa'_G(u,v)$ or simply $\kappa'(u,v)$, is the minimum cardinality of a (u,v)-edge-separator. If u and v are not adjacent, then the *connectivity between u and v* or (u,v)*-connectivity* in G, denoted by $\kappa_G(u,v)$ or simply $\kappa(u,v)$, the minimum cardinality of a

(u,v)-vertex-separator. Clearly, $\kappa(G) = \min\{\kappa(u,v) \mid u,v \in V(G), uv \notin E(G)\}$ and $\kappa'(G) = \min\{\kappa'(u,v) \mid u,v \in V(G)\}$. So, to compute $\kappa(G)$ (resp. $\kappa'(G)$), it is sufficient to compute $\kappa(u,v)$ (resp. $\kappa'(u,v)$) for every pair of vertices u and v.

Similar concept may be defined in digraphs. Let W be an arc-separator (resp. vertex-separator) of a digraph D. If there is no directed (u,v)-path in $D \setminus W$ (resp. $D - W$) , then W separates u from v and is called a *(u,v)-separator*. Observe that contrary to the undirected case, a (u,v)-separator is not necessarily a (v,u)-separator. A separator of a digraph is necessarily a (u,v)-separator for some pair of vertices. In addition, if we consider a vertex-separator, uv is not an arc (but vu may be an arc).

Let u and v be two vertices of a digraph D. The *arc-connectivity between u and v* or *(u,v)-arc-connectivity* in D, denoted by $\kappa'_D(u,v)$ or simply $\kappa'(u,v)$, is the minimum cardinality of a (u,v)-edge-separator. If uv is not an arc, then the *connectivity between u and v* or *(u,v)-connectivity* in D, denoted by $\kappa_D(u,v)$ or simply $\kappa(u,v)$ the minimum cardinality of a (u,v)-vertex-separator. Clearly, $\kappa(G) = \min\{\kappa(u,v) \mid u,v \in V(G), uv \notin E(G)\}$ and $\kappa'(G) = \min\{\kappa'(u,v) \mid u,v \in V(G)\}$. So, to compute $\kappa(G)$ (resp. $\kappa'(G)$), it is sufficient to compute $\kappa(u,v)$ (resp. $\kappa'(u,v)$) for every pair of vertices u and v.

Two (directed) paths are *independent* if their internal vertices are distinct. In particular, two (directed) (s,t)-paths are independent if their sole common vertices are s and t. The maximum number of pairwise independent (directed) (s,t)-paths is denoted by $\Pi(s,t)$. If W is an (s,t)-vertex-separator of a graph or digraph, then two independent (s,t)-paths intersect W in distinct vertices, so

$$\kappa(s,t) \leq \Pi(s,t). \tag{5.1}$$

Similarly, if F is an (s,t)-edge-separator in a graph, then two edge-disjoint (s,t)-paths intersect W in distinct edges, and if F is an (s,t)-arc separator in a digraph, then two arc-disjoint directed (s,t)-paths intersect W in distinct arcs. Hence denoting by $\Pi'(s,t)$ the maximum number of pairwise edge-disjoint (s,t)-paths (resp. arc-disjoint directed (s,t)-paths), we have

$$\kappa'(s,t) \leq \Pi'(s,t). \tag{5.2}$$

Menger's Theorem shows that Inequalities 5.1 and 5.2 are in fact equalities.

Theorem 5.18 (Menger, 1927)**.** *(i) Let s and t be two distinct vertices of a graph (resp. digraph) such that st is not an edge (resp. arc). Then, the minimum size of an (s,t)-vertex-separator equals the maximum number of independent (s,t)-paths. In symbols,*

$$\kappa(s,t) = \Pi(s,t).$$

(ii) Let s and t be two distinct vertices of a graph (resp. a digraph). Then, the minimum size of an (s,t)-edge-separator (resp. arc-separator) equals the maximum number of pairwise edge-disjoint (s,t)-paths (resp. pairwise arc-disjoint directed (s,t)-paths). In symbols,

$$\kappa'(s,t) = \Pi'(s,t).$$

Proof. Let us first prove (ii) in a digraph D. We will use a recursive algorithmic approach as follows. Suppose we have found a set of k pairwise arc-disjoint directed (s,t)-paths, then we will either construct a set of $k+1$ pairwise arc disjoint paths from s to t or find an (s,t)-arc-separator of size k.

The induction can be started with $k=0$ or $k=1$ by finding a directed (s,t)-path.

Let $\mathcal{P}=\{P_1,\dots,P_k\}$ be a set of k pairwise arc-disjoint directed (s,t)-paths. Let $E(\mathcal{P})=E(P_1)\cup\cdots\cup E(P_k)$. Let us construct a set S of vertices with the following algorithm.

Algorithm 5.1 (Constructing S)**.**

1. Put s in S.
2. If there exist $x\in S$ and an arc xy in $E(D)\setminus E(\mathcal{P})$ (i.e. xy is in none of the paths P_i), then add y to S. Go to 2.
3. If there exist $x\in S$ and an arc yx in $E(\mathcal{P})$, then add y to S. Go to 2.

Observe that the so constructed set S is connected but not necessarily strongly connected. (The graph underlying $D\langle S\rangle$ is connected). Two cases can appear at the end of the algorithm.
Case 1: $t\in S$. In that case we will construct a set of $k+1$ pairwise arc-disjoint directed (s,t)-paths.

Since S is connected there is an *oriented* (not necessarily directed) (s,t)-path P_{k+1}, that is a sequence $x_0e_1x_1\dots x_je_{j+1}x_{j+1}\dots e_px_p$ where $x_0=s$, $x_p=t$ and for all $1\le j\le p$, e_j is either the arc $x_{j-1}x_j$ or the arc x_jx_{j-1}. If $e_j=x_{j-1}x_j$, then e_j is called a *forward arc*; if not it is called a *backward arc*. Observe that by construction of S, if an arc is forward in P_{k+1} if and only if it is not in $E(\mathcal{P})$.

If P_{k+1} does not contain any backward arc, then the set of paths $(P_1,P_2,\dots,P_k,P_{k+1})$ is a set of $k+1$ pairwise arc-disjoint directed (s,t)-paths.

Otherwise starting from the set $\mathcal{P}$ and $P_{k+1})$, we shall construct a set $\mathcal{P}'=(P'_1,\dots,P'_k)$ of k pairwise arc-disjoint directed (s,t)-paths and an oriented (s,t)-path P'_{k+1} having one backward arc fewer than P_{k+1} and such that if an arc is forward in P'_{k+1} if and only if it is not in $E(\mathcal{P}')$. Repeating this construction p times (where$p$ is the number of backward arcs in P_{k+1}), we obtain a set of $k+1$ pairwise arc-disjoint directed (s,t)-paths.

Let j be the smallest index for which e_j is backward and let i_0 be the index such that $e_j\in E(P_{i_0})$. Let us define the following paths.

- $P'_i=P_i$ for $1\le i\ne i_0\le k$;

  P'_{i_0} is the concatenation of the directed (s,x_{j-1})-subpath of P_{k+1} and the directed (x_{j-1},t)-subpath of P_{i_0};

  P'_{k+1} is the concatenation of the directed (s,x_j)-subpath of P_{i_0} and the oriented (x_j,t)-subpath of P_{k+1}.

It is simple matter to check that the paths $P'_1,\dots,P'_k,P'_{k+1}$ satisfy the property describe above.

Case 2: $t \notin S$. In that case we will find an (s,t)-arc-separator of size k.

Let $T = V \setminus S$. Let F be the set of arcs from S to T, i.e. with tail in S and head in T. Then F separates S from T and so form an (s,t)-arc-separator. But every such arc belongs to $E(\mathcal{P})$; otherwise we could have applied Step 2 of Algorithm 5.1 and so this algorithm was not finished. Furthermore, a path P_i cannot contain two arcs from S from T, otherwise there will exist an arc yx from T to S in P_i and this arc should have been added to the set S by Step 3 of Algorithm 5.1. Hence $|F| \le k$.

Let us deduce (i) for digraphs. We outline the proof. Some details are left in Exercise 5.15.

Let D be a digraph and s and t two vertices such that st is not an arc. The *split digraph* $S(D)$ is the digraph D obtained in splitting every vertex into an arc v^-v^+:

$$\begin{aligned} V(S(D)) &= \bigcup_{v\in V(D)} \{v^-,v^+\}, \\ E(S(D)) &= \{v^-v^+ \mid v\in V(D)\} \cup \{u^+v^- \mid uv \in E(D)\}. \end{aligned}$$

An arc of the form v^-v^+ is called an *inner arc* of $S(D)$.

Trivially, if W is an (s,t)-vertex-separator of D then $\{v^-v^+ \mid v \in W\}$ is an (s^+,t^-)-arc-separator of $S(D)$. Conversely, one can show that there is a minimum (s^+,t^-)-arc-separator F in $S(D)$ made of inner arcs and for such an F, the set $\{v \mid v^-v^+ \in F\}$ is an (s,t)-vertex-separator of D. Hence $\kappa_D(s,t) = \kappa'_{S(D)}(s^+,t^-)$.

For any directed path $P = (x_1,\dots,x_p)$, its *split path* $S(P)$ is defined as the directed path $(x_1^+,x_2^-,x_2^+,\dots,x_{p-1}^+,x_p^-)$. One can easily see that every directed (s^+,t^-)-path is the split path of some directed (s,t)-path. Moreover, one shows that $P_1,\dots,P_k$ are pairwise independent directed (s,t)-path in D if and only if $S(P_1),\dots,S(P_k)$ are pairwise arc-disjoint directed (s^+,t^-)-path in $S(D)$. Hence, $\Pi_D(s,t) = \Pi'_{S(D)}(s^+,t^-)$.

Now by (ii) for digraphs, we have $\kappa'_{S(D)}(s^+,t^-) = \Pi'_{S(D)}(s^+,t^-)$. Thus $\kappa_D(s,t) = \Pi_D(s,t)$.

Both assertions (i) and (ii) for graphs may be deduced from themselves for digraphs using the fact that connectivity in graphs corresponds to connectivity in symmetric digraphs. Let G be a graph and $\vec{G}$ its associated digraph. Similarly to Theorem 5.17, one can show that $\kappa_G(s,t) = \kappa_{\vec{G}}(s,t)$ and $\kappa'_G(s,t) = \kappa'_{\vec{G}}(s,t)$. It is also not difficult to show that $\Pi_G(s,t) = \Pi_{\vec{G}}(s,t)$ and $\Pi'_G(s,t) = \Pi'_{\vec{G}}(s,t)$. (See Exercise 5.14). □

We now give a short inductive proof of Theorem 5.18-(i) for graphs.

Alternative proof of Theorem 5.18-(i) for graphs: For sake of contradiction, let $k = \kappa(s,t)$ be the smallest integer contradicting the theorem. Clearly, $k \ge 2$. Let G be a counterexample (for this minimum value of k) that has the minimum number of edges. Then, there are at most $k-1$ independent (s,t)-paths.

There is no vertex x adjacent both to s and t otherwise $G-x$ would be a counterexample for $k-1$, a contradiction to the minimality of k.

Let W be an (s,t)-vertex-separator of size k.

Let us assume first that both s and t are not adjacent to all vertices in W. Then, each of s and t has a neighbour in $V(G)\setminus W$ (otherwise, the neighbourhood of s or t would be a smallest vertex-separator). Let G_s be the graph obtained from G by contracting the component C of $G\setminus W$ containing s in a single vertex s' (i.e. replacing C by a single vertex adjacent to all vertices in W). In G_s, an (s',t)-vertex-separator has size at least k. Since C has at least 2 vertices, G_s has less edges than G. By minimality of G, there are at least k independent (s',t)-paths in G_s. Removing s', we obtain k paths from W to t such that, for any $w \in W$, w is the start of exactly one of these paths. Performing the same operation in G_t (obtained in the same way as G_s), we obtain k independent paths from s to W such that, for any $w \in W$, w is the terminus of exactly one of these paths. For any $w \in W$, let us concatenate the (s,w)-path and the (w,t)-path. We obtain k independent (s,t)-paths in G, a contradiction.

So, we can assume that, for every (s,t)-vertex-separator W of size k, either s or t is adjacent to all vertices of W. Let $P = (s,x_1,x_2,\dots,x_l,t)$ be a shortest (s,t)-path. Then, $l \geq 2$b because s and t have no common neighbours. By minimality of G, in $G\setminus x_1x_2$, there is an (s,t)-vertex-separator W_0 of size $k-1$. Hence, $W_1 = W_0 \cup \{x_1\}$ and $W_2 = W_0 \cup \{x_2\}$ are (s,t)-vertex-separators in G. Since s is not adjacent to x_2 because P is a shortest path, then t is adjacent to all vertices of W_2. Similarly, s is adjacent to all the vertices in W_1. Hence, all vertices of W_0 (which is not empty) are common neighbours of s and t, a contradiction.

$\square$

Corollary 5.19 (Menger, 1927)**.** *Let G be a graph with at least two vertices.*

(i) G is k-connected if and only if any two vertices can be joined by k independent paths.

(ii) G is k-edge-connected if and only if any two vertices can be joined by k edge-disjoint paths.

Proof. (i) If any two vertices can be joined by k independent paths, then G is clearly k-connected. Now, if G is k-connected, by Theorem 5.18-(i), any 2 non-adjacent vertices can be joined by k independent paths. It remains to show that if G is k-connected, then any two adjacent vertices x and y can be joined by k independent paths.

Let G' be obtained from G by adding a vertex x' adjacent to all neighbours of x and a vertex y' adjacent to all neighbours of y. Since $\delta(G) \geq \kappa(G) \geq k$, by Lemma 5.2, G' is k-connected. Since x' and y' are not adjacent, there are k independent (x',y')-paths $P_1,\dots,P_k$ in G'. For any $1 \leq i \leq k$, let P'_i be the path obtained as follows:

- if $\{x,y\} \subset V(P_i)$ then $P'_i = (x,y)$;
- if $\{x,y\} \cap V(P_i) = \{x\}$, take the (x,y')-subpath of P_i and replace y' by y;
- if $\{x,y\} \cap V(P_i) = \{y\}$, take the (x',y)-subpath of P_i and replace x' by x;
- if $\{x,y\} \cap V(P_i) = \emptyset$, P'_i is obtained by replacing x' with x and y' with y.

We get k independent (x,y)-paths.

(ii) Straightforward from Theorem 5.18-(ii). $\square$

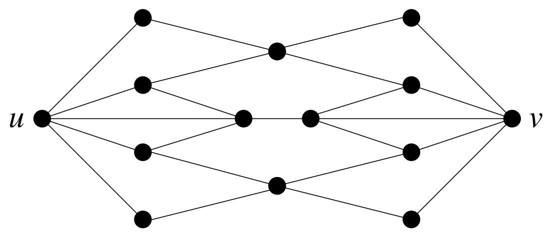
Corollary 5.20. *Let G be a k-connected graph and let A and B be two subsets of $V(G)$. If $|A| \geq k$ and $|B| \geq k$, then there are k disjoint (A,B)-paths*

Proof. Let G' be the graph obtained from G by adding two vertices a and b respectively adjacent to all vertices of A and B. From Lemma 5.2 (applied twice), G' is k-connected and so $\kappa_{G'}(a,b) \geq k$. By Menger's Theorem, there are k independent (a,b)-paths in G'. Removing a and b from these paths, we obtain k disjoint (A,B)-paths.

□

5.7 Exercises

Exercise 5.1. Compute $\kappa(u,v)$ and $\kappa'(u,v)$ in the graph below:



Exercise 5.2. Prove the following assertion or give a counterexample. *If P is a (u,v)-path in a 2-connected graph G, then there exists a (u,v)-path Q independent of P.*

Exercise 5.3. Let k and l be two integers with $1 \leq k < l$. Give graphs G_1, G_2 and G_3 such that :

- (i) $\kappa(G_1) = 1$ and $\kappa'(G_1) = l$;
- (ii) $\kappa(G_2) = k$ and $\kappa(G_2 - x) = l$ for some particular vertex x;
- (iii) $\kappa'(G_3 - x) = k$ and $\kappa'(G_3 \setminus xy) = l$ for some particular edge xy.

Exercise 5.4. 1) Show that if the edge-connectivity of a graph is $k \geq 1$, then when removing at most k edges then we obtain at most two connected components. Does there exists a similar result for connectivity? arc-connectivity?

2) Let D be a k-arc connected digraph and $F = \{x_1y_1, \ldots, x_ky_k\}$ a minimum arc-separator. Show that $D \setminus F$ has two strongly connected components X and Y such that $x_i \in X$ and $y_i \in Y$, for all $1 \leq i \leq k$ and there is no arc with tail in X and head in Y except those of F.

Exercise 5.5. Prove Lemma 5.3.

Exercise 5.6. Show that if H is a 2-edge-connected subgraph of a graph G, then for any H-handle P, the graph $H \cup P$ is 2-edge-connected.

Exercise 5.7. Let G be a graph on at least 2 vertices. Show that the following propositions are equivalent:

- (i) G is 2-connected;
- (ii) any two vertices are in a cycle;
- (iii) any two edges are in a cycle and $\delta(G) \geq 2$;
- (iv) for any three vertices x,y et z, there is a (x,z)-path containing y.

Exercise 5.8. Let G be a graph on at least 3 vertices. Show that the following propositions are equivalent:

- (i) G is 2-edge-connected;
- (ii) any edge is in a cycle;
- (iii) any two edges are in a tour and $\delta \geq 1$;
- (iv) any two vertices are in a tour.

Exercise 5.9. Let $d_1 \leq d_2 \leq \cdots \leq d_n$ be the degree sequence of a graph. We assume that $d_j \geq j+k-1$ for $1 \leq j \leq n-1-d_{n-k+1}$.
Show that G is k-connected.

Exercise 5.10. Let G be a regular bipartite graph on at least two vertices. Prove that $\kappa(G) \neq 1$.

Exercise 5.11. Show a graph which is not 2-connected but admits a strongly connected orientation.

Exercise 5.12. Inspired by Algorithm 2.6, give an algorithm in time $O(|E|)$ that computes the 2-connected components of a graph.

Exercise 5.13. Let G be a connected graph, all vertices of which have even degree. Show that G is 2-edge-connected.

Exercise 5.14. Let G be a graph and $\vec{G}$ its associated digraph. Show that

(a) there are k independent (s,t)-paths in G if and only there are k independent directed (s,t)-paths in $\vec{G}$, and

(b) there are k pairwise edge-disjoint (s,t)-paths in G if and only there are k pairwise arc-disjoint directed (s,t)-paths in $\vec{G}$.

Exercise 5.15. Let D be a digraph and $S(D)$ its split digraph. Let s and t be two vertices of D such that st is not an arc.

1) a) Let F be an (s^+,t^-)-arc-separator in $S(D)$. For an non-inner arc $e = u^+v^-$, we define $r(e)$ to be u^-u^+ if $u \neq s$ and v^-v^+ otherwise. Show that if e is non-inner then the set $(F \setminus \{e\}) \cup \{r(e)\}$ is also an (s^+,t^-)-arc-separator.

b) Show that if F is an (s^+,t^-)-arc-separator made of inner arcs, then $\{v \mid v^-v^+ \in F\}$ is a vertex-separator of $S(D)$.

c) Deduce $\kappa_D(s,t) = \kappa'_{S(D)}(s^+,t^-)$.

2) a) Shows that $P_1,\dots,P_k$ are pairwise independent directed (s,t)-paths in D if and only if $S(P_1),\dots,S(P_k)$ are pairwise arc-disjoint directed (s^+,t^-)-paths in $S(D)$. Hence,

b) Deduce $\Pi_D(s,t) = \Pi'_{S(D)}(s^+,t^-)$.

Exercise 5.16.
Let G be a graph on at least three vertices which is not a complete graph.
1) Show that G has three vertices u, v, and w such that $uv \in E(G)$, $vw \in E(G)$ and $uw \notin E(G)$.
2) Show that if G is 2-connected and $\delta(G) \geq 3$ then there exists such a triple u,v,w such that, in addition, $G - \{u,w\}$ is connected.

Exercise 5.17. Let a and b be two vertices of a graph G. Let X and X' be two (a,b)-vertex-separators. Let us denote C_a (resp. C'_a) the connected component of a in $G-X$ (resp. $G-X'$) and C_b (resp. C'_b) the connected component of b in $G-X$ (resp. $G-X'$).

Prove that the two sets $Y_a = (X \cap C'_a) \cup (X \cap X') \cup (X' \cap C_a)$ and $Y_b = (X \cap C'_b) \cup (X \cap X') \cup (X' \cap C_b)$ are (a,b)-separators.

Exercise 5.18 (Dirac, 1960)**.** Let x be a vertex of a graph G and U a set of vertices of G not containing x. An (x,U)-*fan* is a set of (x,U)-paths such that the intersection of any two is $\{x\}$. Prove that G is k-connected if and only if it has at least $k+1$ vertices and for any choice of x and U such that $x \notin U$ and $|U| \geq k$, there is a (x,U)-fan of cardinality k.

Exercise 5.19. Let $k \geq 2$ be an integer. Prove that, if G is k-connected, then any set of k vertices is contained in a cycle. Is the converse also true?

Exercise 5.20. Let G be a cubic graph.
1) Show that if $\kappa'(G) \geq 2$ then $\kappa(G) \geq 2$.
2) Show that if $\kappa'(G) = 3$ and $\kappa(G) \geq 2$ then $\kappa(G) = 3$.

Exercise 5.21. Let x and y be two adjacent vertices of degree at least k in a graph G. Show that if G/xy is k-connected then G is also k-connected.

Exercise 5.22. Let $k \geq 2$ be an integer. Let G be a k-connected graph and xy an edge of G. Show that G/xy is k-connected if and only if $G - \{x,y\}$ is $(k-1)$-connected.

Exercise 5.23. Let G be a 2-connected graph of order at least 4. Prove that for every edge e, $G \setminus e$ or G/e is 2-connected.

Exercise 5.24. Let v be a vertex of a 2-connected graph G. Show that v has a neighbour u such that $G - \{u,v\}$ is connected.

Exercise 5.25. Let xy be an edge of a 2-connected graph G. Show that $G \setminus xy$ is 2-connected if and only if x and y are in a cycle of $G \setminus xy$.

Exercise 5.26 (W. T. Tutte)**.** A *wheel* is a graph obtained from a cycle by adding a vertex adjacent to all vertices of the cycle.

Let G be a 3-connected graph different from a wheel. Show that, for any edge e, either G/e or $G \setminus e$ is also a 3-connected graph.

Bibliography

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