Radio channel assignment and (weighted) colouring

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1 Modelling the channel assignment problem

We may think of the radio channel assignment problem as the final stage in the design of a cellular radio communications system. The general idea of such a system is that many low-powered transmitters (base stations) each serve the customers in their local cell, and thus the same radio channel can be used simultaneously in many different cells, as long as these cells are sufficiently well separated. Since the radio spectrum is a finite resource which is heavily in demand, we want to assign the channels to the transmitters carefully in order to take maximum advantage of this re-use possibility.

Suppose then that transmitters are located at various sites in a geographical region, perhaps a city, with power levels set. Engineers often aim to spread the transmitters out to form roughly a part of a triangular lattice, since it gives the best "coverage", that is, it minimises the maximum distance to a transmitter. Sometimes the transmitters may be spread out very differently, for example along a major road. The service region is divided into cells around each transmitter. A cell around transmitter v may be seen as the potential receiver sites which are closer to v than to any other transmitter, at least in the case when each transmitter as the same power. When such transmitters are spread out like part of the triangular lattice, the cells are hexagonal.

For each cell, there is an estimate of the (peak period) expected demand. Using these demand estimates and some (simple) queuing model, an appropriate number $p(v)$ of channels is chosen for each transmitter v. (Note that we are considering a static model : there is interest also in dynamic models, where the demand levels change over the time, and the focus is on the method for re-assigning channels.) The aim is to find an assignment of $p(v)$ channels to each transmitter v, such that the corresponding interference is acceptable, and the span of channel used is minimised.

So, when will interference be acceptable. Typically a "protection ratio" θ is set, depending on engineering considerations involving the selectivity of the equipment used and the width of the channel. We say that the interference arising from some channel assignment is acceptable if the signal-to-interference ratio is at least θ at each potential receiver site.

We assume that the power received does not depend on the frequency used (which is realistic since the range of frequencies involved is usually small). Another simplifying assumption that seems reasonable from the physics of interference is that only the difference between two channels matters. Typically the smaller the difference the greater the interference. But this is not always the case as there may for example be "intermodulation products", in particular at transmitters on the same site.

Consider a pair of transmitters u and v, and suppose that they transmit on channels differing by c. If there is a potential receiver in the cell around u such that the ratio of the received power from u to that from v is less than the protection ratio θ, then we make c a "forbidden distance" for u and v; similarly with u and v interchanged. Hence for each pair (u,v) of distinct transmitters, we have a set T_{uv} of forbidden differences $|i-j|$ for channels i at u and channels j at v. In the case where all the transmitters are identical (and under the above assumptions) if c is forbidden for u and v it is also forbidden for v and u, so $T_{uv} = T_{vu}$. Similarly, for each vertex v, we have a set T_{vv} of forbidden differences between two channels at v.

We form the corresponding *interference* or *constraint* graph G. It has a vertex for each transmitter and distinct vertices u and v are adjacent if T_{uv} is non-empty. In particular, at each vertex v, there is a loop of length $l(vv) = T_{vv}$. It is often convenient to think of the problem as being specified by the graph G with a set T_e for each e of G, where always $0 \in T_e$. In the case, when the interference increases with the proximity between two transmitters then T_e is of the form $\{0, \dots, l(e)-1\}$. Hence we may consider that each edge is associated a non-negative integer length $l(e)$ for each edge e.

An assignment is a mapping $\phi : V \to \mathcal{P}(\{1, \dots, t\})$ such that $|\phi(v)| = p(v)$ for each $v \in V(G)$. It is *feasible* if the following conditions hold:

- (i) for any two distinct adjacent vertices u and v and each $i \in \phi(u)$ and each $j \in \phi(v)$ we have $|i-j| \geq l(uv)$.
- (ii) for each $v \in V(G)$ and any two distincts integers i and j in $\phi(v)$, we have $|i-j| \geq l(vv)$.

The *span* of the problem, $\text{span}(G, l, p)$, is the least integer t such that there is a feasible assignment. (Some authors call $t-1$ the span.)

We want to determine or approximate the span, and find corresponding assignments.

Examples

1. If G is a triangle with each edge of length 3 and the demand of each vertex is 1 then the span is 7.
2. If G is a 4-cycle with each edge length 3 and the demand of each vertex is 1 the span is 4.
3. Let G be the 5-cycle plus the loops on all the vertices, such that every edge has length 1 except the loops which have length 2. If each vertex has demand 2, then the span is 5.

2 General results

In this section we give various results, some introductory, about the span in the channel assignment problem. We restrict our attention here to the case of unit demands. In this case, the co-site constraints (=loops) are irrelevant and we can see an assignment as a mapping $\phi : V \to \{1, \dots, t\}$ which is *feasible* if $|\phi(u) - \phi(v)| \geq l(uv)$ for every edge uv. For convenience, we denote $\text{span}(G, l, \mathbf{1})$ by $\text{span}(G, l)$ where $\mathbf{1}$ the appropriate all 1's function.

Note that a general channel assignment problem can always be transformed into a unit demand problem by blowing up each vertex v into a clique of $p(v)$ vertices with edge lengths equal to $l(vv)$.

2.1 All equal edge lengths

Observe that $\text{span}(G, \mathbf{1})$ equals the chromatic number $\chi(G)$. Note also that, with any positive edge lengths, the least *number* of integers required is just $\chi(G)$, but it is the span that is of interest to us.

When the edge lengths are all the same, we are almost back to colouring. Let $\mathbf{k}$ denote the appropriate all k's function.

Proposition 2.1. *If each edge length is k then*

$$\mathrm{span}(G, \mathbf{k}) = k(\chi(G) - 1) + 1.$$

Proof. Observe that the span is at most the right hand side, since we could always first properly colour G with $\chi(G)$ colours and then assign a channel to each colour, using channels $1, k+1, \ldots, k(\chi(G)-1)+1$.

Now let us show that the span is at least the right hand side. Let t be the span, and consider a feasible assignment ϕ using channels $0, 1, \ldots, t-1$ which uses as few as possible channels which are not multiples of k. Then in fact ϕ must use only multiples of k, for otherwise the least channel not a multiple of k could be pushed down to the nearest multiple of k, giving a contradiction. But now if we let $c(v) = \phi(v)/k$ we obtain a proper colouring of G. So $\chi(G) \leq (t-1)/k + 1$, which is the desired inequality. □

2.2 Lower bound for the span

It follows from Proposition 2.1 that if G is the complete graph K_n and all edge lengths are at least k then $\mathrm{span}(G, l) \geq k(n-1)+1$. This result can be extended as follows. A path P is *hamiltonian* in G if its goes through all the vertices, i.e. $V(P) = V(G)$.

Proposition 2.2. *If G is complete, then*

$$\mathrm{span}(G, l) \geq \mathrm{hp}(G, l) + 1,$$

where $\mathrm{hp}(G, l)$ *is the minimum length of a hamiltonian path.*

Proof. Given a feasible assignment ϕ, list the vertices as $v_1, \ldots, v_n$ so that $\phi(v_1) \leq \phi(v_2) \leq \cdots \leq \phi(v_n)$. $P = v_1 v_2 \ldots v_n$ is a hamiltonian path in G and

$$\phi(v_n) - \phi(v_1) = \sum_{i=1}^{n-1} \phi(v_{i+1}) - \phi(v_i) \geq \sum_{i=1}^{n-1} l(v_i v_{i+1}),$$

which is the length of P. Since $\mathrm{span}(G, l) = \phi(v_n) - \phi(v_1) + 1$, we get the result. □

This last result has the drawback that it is NP-hard to calculate $\mathrm{hp}(G, l)$, but there are good lower bounds which may be efficiently calculated, for example the minimum length of a spanning tree. Observe that Proposition 2.2 is tight if the edge lengths satisfy the triangle inequality, but we should not expect this to hold for minimum channel separations.

2.3 Sequential assignment methods

Suppose that we want to colour the vertices of a graph with colours $1, 2, \ldots$... and we have a given ordering on the vertices. Let us consider two variants of the greedy colouring algorithm. In the "one-pass" method, we run through the vertices in order and always assign the smallest available colour. In the "many-passes" method, we run through the vertices assigning colour 1 whenever possible, then repeat with colour 2 and so on. Both methods yield exactly the same colouring, and show that $\chi(G) \leq \Delta(G) + 1$ colours.

Let us now consider the channel assignment problem (G, l). Define the *weighted degree* of a vertex v by $d_{(G,l)}(v) = \sum_{uv \in E} l(uv)$ and define the *maximum weighted degree* by $\Delta(G, l) = \max_{v \in V(G)} d_{(G,l)}(v)$.

Example: Let G be the 4-cycle C_4, with vertices v_1, v_2, v_3, v_4 and edge lengths $l(v_1v_2) = 1$ and $l(v_2v_3) = l(v_3v_4) = l(v_4v_1) = 2$. Note that $\Delta(G,l) = 4$. The one-pass method assigns channels 1,2,4,6 to the vertices v_1, v_2, v_3, v_4 respectively, with span 6. The many-passes method assigns channel 1 to vertices v_1 and v_3, channel 2 to none of the vertices, and channel 3 to vertices v_2 and v_4, with span 3. In fact the many passes method always uses at most the channels $1,\ldots,\Delta(G,l)+1$.

Theorem 2.3 (McDiarmid [7])**.**

$$\mathrm{span}(G,l) \leq \Delta(G,l)+1.$$

Proof. In order to show that the many-passes method needs a span of at most the above size, suppose that it is about to assign channel c to vertex v. Let A be the set of neighbours u of v to which it has already been assigned a channel $\phi(u)$. For each channel $j \in \{1,\ldots,c-1\}$, there must be a vertex $u \in A$ with $\phi(u) \leq j$ and $\phi(u)+l(uv) \geq j+1$. Hence the intervals $\{\phi(u),\ldots,\phi(u)+l(uv)-1\}$ for $u \in A$ cover $\{1,\ldots,c-1\}$. Thus

$$c-1 \leq \sum_{u\in A} l(uv) \leq d_{(G,l)}(v) \leq \Delta(G,l).$$

This completes the proof. □

3 Computing the span

We noted earlier that the special case when all lengths are 1 is essentially the graph colouring problem. Since graph colouring is NP-hard – see for example [3]–, we cannot expect an easy ride. In fact, the general problem seems to be harder than graph colouring.

3.1 Bipartite graphs and odd cycles

Let G be a graph and l an edge-length. Let $L(G,l) = \max\{l(xy)+1 \mid xy \in E(G)\}$. For any (G,l), clearly $\mathrm{span}(G,l) \geq L(G,l)$.

This inequality is tight for bipartite graphs.

Proposition 3.1. *If G is a bipartite graph, then* $\mathrm{span}(G,l) \geq L(G,l)$*, for any l.*

Proof. If we set $\phi(x) = 1$ for x in one part of the bipartition and $\phi(x) = L(G,l)$ for x in the other part, then we obtain a feasible assignment with span $L(G,l)$. □

This proposition implies that the channel assignment problem for bipartite graphs is easy. After bipartite graphs the next thing to consider is odd cycles. Here again it is easy to determine the span.

Proposition 3.2. *If G is an odd cycle then* $\mathrm{span}(G,l) = \max\{L(G,l), M(G,l)\}$*, where $M(G,l) = \min\{l(uv)+l(vw)+1 \mid uv, vw \in E(G)\}$.*

Proof. Since G is an odd cycle, in any feasible assignment ϕ there exist edges uv and vw of G such that $\phi(u) \leq \phi(v) \leq \phi(w)$. Then $|\phi(w)-\phi(u)| \geq l(uv)+l(vw)$ and so the span of (G,l) is at least $M(G,l)$. Hence it is at least $\max\{L(G,l), M(G,l)\}$.

On the other hand, let us choose two edges uv and vw in G with $l(uv)+l(vw) = M(G,l)-1$. Form an even cycle G' by deleting v and adding the edge uw. Consider the length function l' on $E(G')$ which satisfies $l'(uw) = l(uv)+l(vw)$ and agrees with l elsewhere. Since G' is bipartite, (G',l') admits an optimal feasible assignment ϕ with span $L(G',l') = \max\{L(G,l), M(G,l)\}$. Furthermore, $|\phi(u)-\phi(w)| \geq l(uw) \geq l(uv)+l(vw)$. Hence one can choose $\phi(v)$ between $\phi(u)$ and $\phi(w)$ so that ϕ is a feasible assignment of (G,l). □

Let us call a graph 1-*nearly bipartite* if by deleting at most one vertex we may obtain a bipartite graph. It is of course easy to recognise if a graph is 1-nearly bipartite, by simply deleting each vertex in turn. It is also easy to determine the chromatic number of a 1-nearly bipartite graph G, as it is at most 3. However, it is NP-hard to determine $\text{span}(G,l)$, even if we restrict the edge lengths to be 1 or 2, see [**?**].

3.2 A general exponential algorithm

Theorem 3.3 (McDiarmid [7]). *Given (G,l) with maximum edge-length m, we can compute $\text{span}(G,l)$ in $O^*((2m+1)^n)$ steps.*

Proof. Let us describe the method. Let V denote the set of vertices of G. For each $S \subset V$, let $N_i(S) = \{v \in V \setminus S \mid \exists$ an edge uv with $u \in S$ and $l(uv) \geq i\}$. For each nested family $A \supseteq B_1 \supseteq \cdots \supseteq B_{m-1}$ of m subsets of V and each non-negative integer t, let $F(A;B_1,\ldots,B_{m-1};t)$ be the set of all feasible assignments $\phi : A \to \{1,\ldots,t\}$ for the subproblem on A such that $\phi(v) \leq t-i$ whenever $v \in B_i$, for each $i = 1,\ldots,m-1$. Let $f(A;B_1,\ldots,B_{m-1})$ be the least t such that $F(A;B_1,\ldots,B_{m-1};t)$ is non-empty. Thus the span is $f(V;\emptyset,\ldots,\emptyset)$. By definition, if $A=\emptyset$ then $F=\{\emptyset\}$ and $f=0$.

Claim 3.3.1. *For each non-empty $A \subset V$*

$$f(A;B_1,\ldots,B_{m-1}) = 1 + \min_S f(A\setminus S;B'_1,\ldots,B'_{m-1}),$$

where S runs over all stable subsets of $A \setminus B_1$; $B'_{i-1} = B_i \cup (A \cap N_i(S))$ for each $i = 2,\ldots,m-1$; and $B'_{m-1} = A \cap N_m(S)$.
(Note that $A \setminus S \supseteq B'_1 \supseteq \cdots \supseteq B'_{m-1}$, as required for the domain of f.)

The method to calculate the span is brutal: we use the claim to tabulate all the values $f(A;B_1,\ldots,B_{m-1})$ in increasing order of the size of A. For a given set A of size a, there are m^a points in the domain of f: for each point we have m choices for the smallest element of the nested family that contains a . The additional time to compute f for a given point with set A of size a is at most $O^*(2^a)$ since they are at most 2^a (stable) sets in A. Hence the total time taken is at most $O^*\left(\sum_{a=0}^{n}\binom{n}{a}m^a 2^a\right) = O^*\left((2m+1)^n\right)$. It remains only to prove the claim.

Proof Claim 3.3.1. We show first that the left side is at most the right. Let S be a stable subset of $A \setminus B_1$, and let $f(A\setminus S;B'_1,\ldots,B'_{m-1}) = t-1$. We want to show that $f(A;B_1,\ldots,B_{m-1}) \leq t$. Let $\phi \in F(A\setminus S;B'_1,\ldots,B'_{m-1})$ and extend ϕ to $\hat{\phi} : A \to \{1,\ldots,t\}$ by setting $\hat{\phi}(v) = \phi(v)$ for each $v \in A \setminus S$ and $\hat{\phi}(v) = t$ for each $v \in S$.

We must check that $\hat{\phi} \in F(A;B_1,\ldots,B_{m-1})$. Let uv be an edge with $u \in S$ and $v \in A \setminus S$. Thus $\hat{\phi}(u) = t$ and $\hat{\phi}(v) \leq t-1$. If $l(uv) = i \in \{2,\ldots,m\}$, then $v \in N_i(S) \subseteq B'_{i-1}$, and so $\hat{\phi}(v) = \phi(v) \leq (t-1)-(i-1) = t-i$. Thus in each case $\hat{\phi}(u) - \hat{\phi}(v) \geq l(uv)$. Since S is stable and ϕ is feasible for the subproblem on $A \setminus S$, it now follows easily that $\hat{\phi}$ is feasible for the subproblem on A. If $v \in B_1$ then $\hat{\phi}(v) = \phi(v) \leq t-1$ since $S \subseteq A \setminus B_1$; and if $v \in B_i$ for some $i \in \{2,\ldots,m-1\}$ then $v \in B'_{i-1}$ and so $\hat{\phi}(v) = \phi(v) \leq t-i$ by our choice of ϕ. Hence $\hat{\phi} \in F(A;B_1,\ldots,B_{m-1})$.

Conversely, let us show that the right side is at most the left. Let $f(A;B_1,\ldots,B_{m-1}) = t$ and let $\phi \in F(A;B_1,\ldots,B_{m-1})$. Let S be the stable set $\phi^{-1}(t)$. Then S must be non-empty by the minimality of t, and $S \subseteq A \setminus B_1$ since $\phi(v) \leq t-1$ for each $v \in B_1$. If $t=1$ then $S=A$ and the result holds, so let us assume that $t \geq 2$. Define $\phi' : A\setminus S \to \{1,\ldots,t-1\}$ by setting $\phi'(v) = \phi(v)$ for each $v \in A \setminus S$.

Let us check that $\phi' \in F(A \setminus S; B'_1, \ldots, B'_{m-1})$. Clearly ϕ' is feasible for the subproblem on $A \setminus S$. Let $i \in \{2, \ldots, m-1\}$ and let $v \in B'_{i-1} = B_i \cup (A \cap N_i(S))$. If $v \in B_i$ then $\phi'(v) = \phi(v) \leq t - i = (t-1) - (i-1)$ by the condition on ϕ, and if $v \in N_i(S)$ then the same inequality holds, since S is non-empty and ϕ is feasible for A. Finally, if $v \in B'_{m-1} = A \cap N_m(S)$, then as before $\phi'(v) = \phi(v) \leq t - m = (t-1) - (m-1)$. Thus $\phi' \in F(A \setminus S; B'_1, \ldots, B'_{m-1})$. □

The proof of Claim 3.3.1 completes the proof of Theorem 3.3. □

3.2.1 An Integer Programme model

The following integer programme (IP) gives a simple reformulation of the channel assignment model, though other formulations may be better suited to computation for particular types of problem, see also [10].

Choose an upper limit f_{max}, and let $F = \{1, \ldots, f_{max}\}$be the set of available channels. We introduce a binary variable $y_{u,i}$ for each transmitter u and channel i : setting $y_{u,i} = 1$ will correspond to assigning channel i as one of the channels at transmitters u. Then $\mathrm{span}(G, l, p)$ is given by the following integer programme.

Minimise z subject to :

$$\begin{array}{rcll} z & \geq & j \times y_{v,j} & \forall v \in V(G), j \in F \\ \sum_{j \in F} y_{v,j} & = & p(v) & \forall v \in V(G) \\ y_{u,i} + y_{v,j} & \leq & 1 & \forall u, v \in V(G) \text{ and } i, j \in F \text{ such that } (u,i) \neq (v,j) \text{ and } |i-j| < l(uv) \\ y_{v,j} & \in & \{0,1\} & \forall v \in V(G), j \in F \end{array}$$

To see that this IP formulation is correct, consider an optimal assignment $\phi : V \to F$. Set $y_{v,j} = 1$ if $j \in \phi(v)$ and $y_{v,j} = 0$ otherwise; and set z to be the maximum channel used. It is easy to see that this gives a feasible solution to the IP, wiht $z = \mathrm{span}(G, l, p)$. Conversely, given a feasible solution to the IP with value t, we may obtain in a similar way a feasible assignment ϕ.

4 Channel assignment in the plane

It is natural to specialise the channel assignment problem to the case where the transmitter sites are located in the plane, and the minimum channel separation for a pair of sites depends on the distance between them.

In this section, we will consider only co-channel interference, which corresponds to each minimum channel separation being 0 (if there is no edge) or 1 (at the same site or between adjacent sites). Hence we have to determine $\mathrm{span}(G, \mathbf{1}, p)$. Then we are left with a colouring problem of a weighted graph. A *weighted graph* is a pair (G, p), where G is a graph and p a weight function on the vertex set of G. A *t-colouring* of a weighted graph (G, p) is a mapping $C : V(G) \to \mathcal{P}(\{1, \ldots, t\})$ such that for every vertex $v \in V(G)$, $|C(v)| = p(v)$ and for all edge $uv \in E(G)$, $C(u) \cap C(v) = \emptyset$. The *chromatic number* of a weighted graph (G, p), denoted $\chi(G, p)$, is the least integer t such that (G, p) admits a t-colouring. This is a natural generalisation of the chromatic number of a graph since $\chi(G, \mathbf{1}) = \chi(G)$. Moreover, $\chi(G, p) = \mathrm{span}(G, \mathbf{1}, p)$.

The *clique number* of a weighted graph (G, p), denoted $\omega(G, p)$, is the maximum weight of a clique, that is $\max\{p(C) \mid C \text{ clique of } G\}$, where $p(C) = \sum_{v \in C} p(v)$.

Generalising Proposition **??**, for any weighted graph (G,p), we have

$$\chi(G,p) \geq \omega(G,p).$$

4.1 Disk graphs

Suppose that we are given a threshold distance d_0, such that interference will be acceptable as long as no channel is re-used at sites less than distance d_0 apart. Given a set of V points in the plane and given $d_0 > 0$, let $G(V,d_0)$ denote the graph with vertex set V in which distinct vertices u and v are adjacent whenever the euclidean distance between them is less than d_0. Equivalently, we may centre an open disk of diameter d_0 at each point v, and then two vertices are adjacent when their disks meet. Such a graph is called a *unit disk* (or *proximity*) graph.

Observe that if G is a unit disk graph, the graph G_p obtained from the weighted graph (G,p) by replacing each vertex v by a complete graph of size $p(v)$ is also a unit disk graph. Hence our basic version of the channel assignment problem is equivalent to colouring unit disk graphs. The following result shows that the clique and chromatic numbers of such graphs are not too far apart.

Proposition 4.1 (Clark, Colbourn and Johnson [2])**.** *For a unit disk graph G,*

$$\chi(G) \leq 3\omega(G) - 2.$$

Proof. In a realisation of G with diameter d_0, consider the "bottom left" point v. All its neighbours lie within an angle of less than 180 degrees at v. Thus we can cover all the neighbours with three sectors, each with radius less than d_0 and angle less than 60 degrees. But the points in each sector together with v form a clique, and so the degree of v is at most $3(\omega(G)-1)$. It follows that the degeneracy of G is at most $3\omega(G)-3$. The result follows from Proposition **??**. □

It would be nice to improve this result: perhaps the factor 3 could be replaced by 3/2? It is shown in [1] that it is NP-hard to recognise unit disk graphs. Many problems are NP-hard for unit disk graphs, even given a realisation in the plane, see [2]: for example finding $\chi(G)$ or $\alpha(G)$. However there is a polynomial time algorithm to find $\omega(G)$, given a realisation in the plane.

Theorem 4.2 (Clark, Colbourn and Johnson [2])**.** *Given a realisation of a unit disk graph G, there is a polynomial time algorithm to find* $\omega(G)$.

Proof. Suppose we are given a representation with diameter 1. The algorithm relies on the following fact. For every clique K, there are points u and v in K at euclidean distance $d < 1$ such that all points of K is contained in the region R_{uv} of points in the plane within distance at most d of both u and v. See Figure 1.

Hence we consider in turn each edge uv of G and the corresponding region R_{uv} and find a maximum clique in R_{uv}. The line uv cuts R_{uv} into two halves: let A and B be the sets of points in the two halves. It is easy to verify that A and B are cliques.

Let H be the bipartite graph with bipartition (A,B) where $a \in A$ and $b \in B$ are adjacent if they are at euclidean distance at least 1. Then we obtain a maximum clique within L by forming $(A \cup B) \setminus C$, where C is a minimum cover in H. □

If the transmitters can have different powers, we are led to consider disk graphs, which are defined as for unit disk graphs except that the diameters may be different.

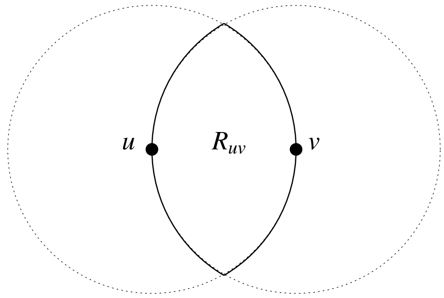

Figure 1: The region R_{uv}.

Proposition 4.3. *For a disk graph G,*

$$\chi(G) \leq 6\omega(G) - 5.$$

Proof. Consider a vertex v with disk of smallest diameter, and proceed as in the proof of Proposition 4.1 to show that the degeneracy is at most $6(\omega(G) - 1)$. □

4.2 Triangular lattice

The triangular lattice TL crops up naturally in radio channel assignment. It is sensible to aim to spread the transmitters out to form roughly a part of a triangular lattice, with hexagonal cells, since that will give the best "coverage", that is, for a given number of transmitters in a given area this pattern minimises the maximum distance to a transmitter.

The triangular lattice graph TL may be described as follows. The vertices are all integer linear combinations $a\mathbf{e_1} + b\mathbf{e_2}$ of the two vectors $\mathbf{e_1} = (1,0)$ and $\mathbf{e_2} = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. Thus we may identify the vertices with the pairs (a,b) of integers. Two vertices are adjacent when the Euclidean distance between them is 1. Therefore, each vertex $x = (a,b)$ has six neighbours: its *left neighbour* $(a-1,b)$, its *right neighbour* $(a+1,b)$, its *leftup neighbour* $(a-1,b+1)$, its *rightup neighbour* $(a,b+1)$, its *leftdown neighbour* $(a,b-1)$ and its *rightdown neighbour* $(a+1,b-1)$. See Figure 2.

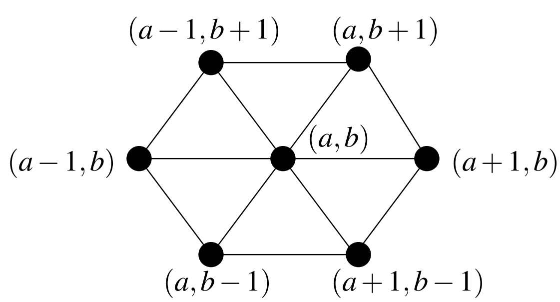

Figure 2: The vertex (a,b) and its six neighbours.

A *hexagonal graph* is an induced subgraph of the triangular lattice.

The triangular lattice has a unique (up to colours permutations) 3-colouring c_T, defined by $c_T((a,b)) = a - b \bmod 3$. See Figure 3. It follows immediately that for any weighted hexagonal graph (G,p),

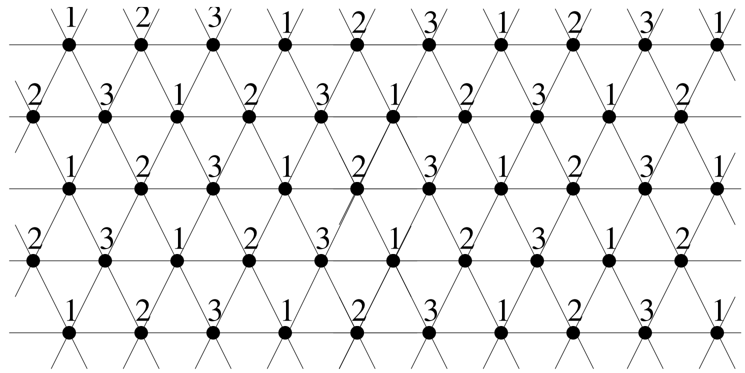

Figure 3: The unique 3-colouring of the triangular lattice.

$$\chi(G,p) \leq 3\max\{p(v) \mid v \in V(G)\} \leq 3\omega(G,p).$$

In the rest of this section, we will improve this upper bound. It is fairly easy to improve it for bipartite graphs.

Lemma 4.4. *Let (G,p) be a weighted bipartite graph. Then $\chi(G,p) = \omega(G,p)$.*

Proof. Let (A,B) be a bipartition of G. Let v be a vertex of G. Assign it the colour set $\{1,2,\ldots,p(v)\}$ if it is in A and the colour set $\{\omega(G,p)-p(v)+1,\ldots,\omega(G,p)\}$ if it is in B. This give us an optimal colouring of (G,p). Indeed, if uv is an edge with $u \in A$ and $v \in B$ then $p(u)+p(v) \leq \omega(G,p)$. □

Remark 4.5. This lemma gives us a polynomial time algorithm to compute the chromatic number of a bipartite weighted graph. Indeed, calculating $\omega(G,p)$ is easy because it is the maximum of $\max\{p(v) \mid v \in V(G)\}$ and $\max\{p(u)+p(v) \mid uv \in E(G)\}$ since a clique has size at most 2 in a bipartite graph.

McDiarmid and Reed [8] showed that it is NP-complete to decide whether the chromatic number of a weighted hexagonal graph is 3 or 4. Hence, there is no polynomial time algorithm for finding the chromatic number of weighted hexagonal graphs (unless P=NP). Therefore, one has to find approximate algorithms. The better known so far has approximation ratio $4/3$ and is based on the following result:

Theorem 4.6 (McDiarmid and Reed [8])**.** *For any weigthed hexagonal graph G,*

$$\chi(G,p) \leq \frac{4\omega(G,p)+1}{3}.$$

Proof. The weighted colouring of (G,p) with $\frac{4\omega_p(G)+1}{3}$ colours is given by the following algorithm : We first calculate the 3-colouring c_T of G. We then compute $\omega(G,p)$ and set $k = \lfloor\frac{\omega(G,p)+1}{3}\rfloor$. The algorithm proceeds in two stages.

In the first stage, we use $3k$ colours (i,j) for $i=1,2,3$ and $j=1,\ldots,k$. For each vertex v, we compute the value $m(v)$ which is the maximum of $p(u)$ over its neighbours u such that $c_T(u) = c_T(v)+1 \bmod 3$. (These are its right, leftup and leftdown neighbours if they are present.) If v has no such neighbours then $m(v)=0$. Let $r(v) = \min\{p(v)-k, k-m(v)\}$. We assign to v the colours $(c_T(v),1),\ldots,(c_T(v),\min\{k,p(v)\})$. Moreover, if $r(v)>0$ we assign to v the colours $(c_T(v)+1, k-r(v)+1),\ldots,(c_T(v)+1,k)$. By definition of $r(v)$ those colours are not assigned to the neighbours of v, so the colouring is proper.

Now let U be the set of vertices whose demand is not yet fulfilled after the first stage. Then $v \in U$ if and only if $p(v) > \max\{k, 2k - m(v)\}$, and in this case, the number of colours that remain to be assigned to v is $p'(v) = p(v) - \max\{k, 2k - m(v)\}$.

Let H be the graph induced by the vertices of U. H is triangle-free because $p(v) \geq k+1$ for every vertex $v \in U$. Hence $\omega(H) \leq 2$. In addition, for all vertex $v \in U$, $p'(v) \leq p(v) + m(v) - 2k \leq \omega(G,p) - 2k$ and for any two neighbours u and v in H, $p'(u) + p'(v) \leq \omega(G,p) - 2k$. Hence $\omega(H,p') \leq \omega(G,p) - 2k$.

We shall prove that H is acyclic and thus bipartite. Hence by Lemma 4.4, one can assign $p'(v)$ colours to every vertex v of U using $\omega(H,p')$ colours. Hence in total, we have used at most

$$3k + \omega(H,p') \leq \omega(G,p) + k \leq \frac{4\omega(G,p)+1}{3}$$

colours.

Remains to prove that H is acyclic. To do so, we shall prove that in H a vertex $v = (a,b)$ has at most one neighbour to its right, that is among its rightup neighbour $x = (a, b+1)$, its right neighbour $y = (a+1, b)$ and its rightdown neighbour $z = (a+1, b-1)$. Since H is triangle-free, it suffices to prove that v, x and z cannot be all three in U (and y is not in U). Suppose for a contradiction that these three vertices are in U. Set $s = \min\{p(x), p(z)\}$. Then $s \geq k+1$. Furthermore, if $m(v) > 0$ and u is the neighbour of v in which $m(v)$ is attained, then v, u and either x or z form a triangle, so $p(v) + m(v) + s \leq \omega(G,p)$. Notice that this inequality is also true if $m(v) = 0$ because $p(v) + s \leq \omega(G,p)$ for v is adjacent to x and z. Thus we have

$$1 \leq p'(v) \leq p(v) + m(v) - 2k \leq \omega(G,p) - s - 2k \leq \omega(G,p) - 3k - 1 \leq 0$$

which is a contradiction. □

A distributed algorithm which guarantees the $\frac{4}{3}\omega(G,p)$ bound is reported by Narayanan and Schende [11]. However, one expects to have approximate algorithms with ratios better than $4/3$. In particular, Reed and McDiarmid conjecture that, for big weights, the ratio may be decreased to almost $9/8$.

Conjecture 4.7 (McDiarmid and Reed [8])**.** *There exists a constant c such that for any weigthed hexagonal graph (G,p),*

$$\chi(G,p) \leq \frac{9}{8}\omega(G,p) + c.$$

Note that the ratio $9/8$ in the above conjecture is the best possible. Indeed, consider a 9-cycle C_9 with constant weight k. A colour can be assigned to at most 4 vertices, so $\chi(C_9, \mathbf{k}) \geq \frac{9k}{4}$. Clearly, $\omega(C_9, \mathbf{k}) = 2k$. So $\chi(C_9, k) \geq \frac{9}{8}\omega(C_9, \mathbf{k})$. An evidence for this conjecture has been given by Havet [5], who proved that if a hexagonal graph G is triangle-free (i.e. has no K_3) then $\chi(G,p) \leq \frac{7}{6}\omega(G,p) + 5$. See also [12] for an alternative proof and [6] for a distributed algorithm for colouring triangle-free hexagonal graphs with $\frac{5}{4}\omega(G,p) + 3$ colours.

Exercises

Exercise 1. Let (G,l) be a unit demand instance of the channel assignment problem and m a non-negative integer. A subset U of vertices is *m-assignable* if $\mathrm{span}(G\langle U\rangle, l) \leq m$. Let α^m denote the maximum size of an m-assignable set. Show that

$$\mathrm{span}(G,l) \geq \frac{m \times |V(G)|}{\alpha^m} - m - 1.$$

Exercise 2. Let G be a bipartite graph and p a demand function. Let $p_{\max} = \max\{p(v) \mid v \in V(G)\}$ be the maximum demand.

1. Let l be the length defined by $l(uv) = 1$ for all $uv \in E(G)$ and $l(vv) = 3$ for all $v \in V(G)$.

   a. Show that $3p_{\max} - 2 \leq \text{span}(G,l,p) \leq 3p_{\max} - 1$.

   b. Show that $\text{span}(G,l,p) \leq 3p_{\max} - 1$ if and only if there are two adjacent vertices with demand $p_{\max}$.

2. Let l be the length defined by $l(uv) = 1$ for all $uv \in E(G)$ and $l(vv) = 2$ for all $v \in V(G)$.

   a. Show that $2p_{\max} - 1 \leq \text{span}(G,l,p) \leq 2p_{\max}$.

   b. A path in G is *critical* if it has an even number of vertices, the endvertices have demand $p_{\max}$ and the internal vertices have demand $< p_{\max}$.
   Show that $\text{span}(G,l,p) = 2p_{\max} - 1$ if and only if for every critical path P, $p(P) \leq (|V(P)| - 1)p_{\max} - (|V(P)| - 2)/2$. (Gerke [4]).

Exercise 3. A *bicolouring* of a graph G is a colouring of $(G, \mathbf{2})$, where $\mathbf{2}$ is the appropriate all 2's function. A t-bicolouring is a bicolouring in $\{1, \ldots, t\}$.

1 Let $P = (x_0, x_1, \ldots, x_m)$ be a path of length $m \geq 4$. Show that for any 2-subsets c_0 and c_m of $\{1, \ldots, 5\}$, there is a 5-bicolouring such that $C(x_0) = c_0$ and $C(x_m) = c_m$.

2 a) Show that every triangle-free hexagonal graph is 5-bicolourable.
b) Deduce that for any weighted hexagonal graph (G,p), $\chi(G,p) \leq \frac{5}{4}\omega(G,p) + 3$. (Havet [5])

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