Chapter 1

Basic concepts

All the definitions given in this section are mostly standard and may be found in several books on graph theory like [1, 2, 3].

1.1 Graphs

A *graph* G is a pair (V,E) of sets satisfying $E \subset [V]^2$, where $[V]^2$ denotes the set of all 2-element subsets of V. We also assume tacitly that $V \cap E = \emptyset$. The elements of V are the *vertices* of the graph G and the elements of E are its *edges*. The vertex set of a graph G is referred to as $V(G)$ and its edge set as $E(G)$. An edge $\{x,y\}$ is usually written as xy. A vertex v is *incident* with an edge e if $v \in e$. The two vertices incident with an edge are its *endvertices*. An edge is said to *join* or *link* its two endvertices. Note that in our definition of graphs, there is no *loops* (edges whose endvertices are equal) nor *multiple edges* (two edges with the same endvertices).

Sometimes we will need to allow multiple edges. So we need the notion of *multigraph* which generalises the one of graph. A *multigraph* G is a pair (V,E) where V is the vertex set and E is a collection of elements of $[V]^2$. In a multigraph G, we say that xy is an edge of *multiplicity* m if there are m edges with endvertices x and y. We write $\mu(x,y)$ for the multiplicity of xy, and write $\mu(G)$ for the maximum of the edges multiplicities in G.

The *complement* of a graph $G = (V,E)$ is the graph $\overline{G}$ with vertex set V and edge set $[V]^2 \setminus E$. A graph is *empty* if it has no edges. A graph is *complete* of for all pair of distinct vertices u,v, $\{u,v\}$ is an edge. The complete graph on n vertices is denoted K_n. Trivially, the complement of an empty graph is a complete graph.

A *subgraph* of a graph G is a graph H such that $V(H) \subset V(G)$ and $E(H) \subset E(G)$. Note that since H is a graph we have $E(H) \subset E(G) \cap [V(H)]^2$. If H contains all the edges of G between vertices of $V(H)$, that is $E(H) = E(G) \cap [V(H)]^2$, then H is the subgraph *induced* by $V(H) = S$. It is denoted $G\langle S\rangle$. The notion of *submultigraph* and *induced submultigraph* are defined similarly. If S is a set of vertices, we denote by $G - S$ the (multi)graph induced by $V(G) \setminus S$. For simplicity, we write $G - v$ rather than $G - \{v\}$. For a collection F of elements of $[V^2]$, we write $G \setminus F = (V(G), E(G) \setminus F)$ and $G \cup F = (V(G), E(G) \cup F)$. As above $G \setminus \{e\}$ and $G \cup \{e\}$ are abbreviated to $G \setminus e$ and $G \cup e$ respectively. If H is a subgraph of G, we say that G is a *supergraph* of H.

Let G be a multigraph. When two vertices are the endvertices of an edge, they are *adjacent* and are *neighbours*. The set of all neighbours of a vertex v in G is the *neighbourhood* of G and is denoted $N_G(v)$, or simply $N(v)$. The *degree* $d_G(v) = d(v)$ of a vertex is the number of edges to which it is incident. If G is a graph, then this is equal to the number of neighbours of v.

Proposition 1.1. *Let $G = (V,E)$ be a multigraph. Then*

$$\sum_{v\in V} d(v) = 2|E|.$$

Proof. By counting *inc* the number of edge-vertex incidence in G. On the one hand, every edge has exactly two endvertices, so $inc = 2|E|$. On the other hand, every vertex v is an ednvertex of $d(v)$ edges, so $inc = \sum_{v\in V} d(v)$. □

The *maximum degree* of G is $\Delta(G) = \max\{d_G(v) \mid v \in V(G)\}$. The *minimum degree* of G is $\delta(G) = \max\{d_G(v) \mid v \in V(G)\}$. If the graph G is clearly understood, we often write Δ and δ instead of $\Delta(G)$ and $\delta(G)$. A graph is *k-regular* if every vertex has degree k. The *average degree* of G is $Ad(G) = \frac{1}{|V(G)|}\sum_{v\in V(G)} d(v) = \frac{2|E(G)|}{|V(G)|}$. The *maximum average degree* of G is $Mad(G) = \max\{Ad(H) \mid H \text{ is a subgraph of } G\}$.

Let G be a graph. A *stable set* or *independent set* in G is a set of pairwise non-adjacent vertices. In other words, a set S is stable if $G\langle S\rangle$ is empty. The *stability number* of G, denoted $\alpha(G)$ is the maximum cardinality of a stable set in G. Conversely, a *clique* in G is a set of pairwise adjacent vertices. In other words, a set S is a clique if $G\langle S\rangle$ is a complete graph. The *clique number* of G, denoted $\omega(G)$ is the maximum cardinality of a stable set in G.

1.2 Digraphs

A *multidigraph* D is a pair $(V(D),E(D))$ of disjoint sets (of *vertices* and *arcs*) together with two maps $tail : E(D) \to V(D)$ and $head : E(D) \to V(D)$ assigning to every arc e a *tail*, $tail(e)$, and a *head*, $head(e)$. The tail and the head of an arc are its *endvertices*. An arc with tail u and head v is denoted by uv and is said to *leave* u and to *enter* v; we say that u *dominates* v and write $u \to v$; we also say that u and v are *adjacent*. Note that a directed multidigraph may have several arcs with same tail and same head. Such arcs are called *multiple arcs*. A *multidigraph* without multiple arcs is a *digraph*. It can be seen as a pair (V,E) with a $E \subset V^2$. An arc whose head and tail are equal is a *loop*. All the digraphs we will consider in this monograph have no loops.

The multigraph G *underlying* a multidigraph D is the multigraph obtained from D by replacing each arc by an edge. Note that the multigraph underlying a digraph may not be a graph: there are edges uv of multiplicity 2 whenever uv and vu are arcs of D. *Subdigraphs* and *submultidigraphs* are defined similarly to subgraphs and submultigraphs.

Let D be a multidigraph. If uv is an arc, we say that u is an *inneighbour* of v and that v is an *outneighbour* of u. The *outneighbourhood* of v in D, is the set $N_D^+(v) = N^+(v)$ of outneighbours of v in G. Similarly, the *inneighbourhood* of v in D, is the set $N_D^-(v) = N^-(v)$ of inneighbours of v in G. The *outdegree* of a vertex v is the number $d_D^+(v) = d^+(v)$ of arcs leaving v and the *indegree* of v is the number $d_D^-(v) = d^-(v)$ of arcs entering v. Note that if D is a digraph then $d^+(v) = |N^+(v)|$ and $d^-(v) = |N^-(v)|$. The *degree* of a vertex v is $d(v) = d^-(v) + d^+(v)$. It corresponds to the degree of the vertex in the underlying multigraph.

Proposition 1.2. *Let $D = (V,E)$ be a digraph. Then*

$$\sum_{v \in V} d^+(v) = \sum_{v \in V} d^-(v) = |E|.$$

The *maximum outdegree* of D is $\Delta^+(D) = \max\{d^+(v), v \in V(D)\}$, the *maximum indegree* of D is $\Delta^-(D) = \max\{d^-(v), v \in V(D)\}$, and the *maximum degree* of D is $\Delta(D) = \max\{d(v), v \in V(D)\}$. When D is clearly understood from the context, we often write Δ^+, Δ^- and Δ instead of $\Delta^+(D)$, $\Delta^-(D)$ and $\Delta(D)$ respectively.

The *converse* of the digraph $D = (V,E)$ is the digraph $\overline{D} = (V(D), \overline{E})$ where $\overline{E} = \{(v,u) \mid (u,v) \in E\}$. A digraph D is *symmetric* if $D = \overline{D}$.

An *orientation* of a graph G is a digraph D obtained by substituting each edge $\{x,y\}$ by exactly one of the two arcs (x,y) and (y,x). An *oriented graph* is an orientation of graph.

1.3 Walks, paths, cycles

Let G be a multigraph. A *walk* in G is a finite (non-empty) sequence $W = v_0e_1v_1e_2v_2 \dots e_kv_k$ alternating vertices and edges such that, for $1 \le i \le k$, v_{i-1} and v_i are the endvertices of e_i. The vertex v_0 is called *start* of W and v_k *terminus* of W. They both are *endvertices* of W. The vertices $v_i, 1 \le i \le k-1$ are the *internal vertices*. One says that W *links* v_0 to v_k and that W is a (v_0, v_k)*-walk*.

Let A and B be to set of vertices. An (A,B)*-path* is a path whose start is in A, whose end is in B and whose internal vertices are not in $A \cup B$. We usually abbreviate $(\{a\},B)$-path to (a,B)-path, $(A,\{b\})$-path to (A,b)-path and $(\{a\},\{b\})$-path to (a,b)-path.

If $W_1 = u_0e_1u_1e_2u_2 \dots e_pu_p$ and $W_2 = v_0f_1v_1f_2v_2 \dots f_qv_q$ are two walks such that $u_p = v_0$, the *concatenation* of W_1 and W_2 is the walk $u_0e_1u_1e_2u_2 \dots e_pu_pf_1v_1f_2v_2 \dots f_qv_q$. The *concatenation* of k walks $W_1, \dots, W_k$ such that for all $1 \le i \le k-1$ the terminus of W_i is the start of W_{i+1} is then defined inductively as the concatenation of W_1 and the concatenation of $W_2, \dots, W_k$.

If G is a graph, then the walk W is entirely determined by the sequence of its vertices. Very often, we will then denote $W = (v_0, v_1, \dots, v_k)$. The *length* of W is k, which is its number of edges (with repetitions). A walk is said to be *even* (resp. *odd*) if its length is even (resp. odd).

A walk whose start and terminus are the same vertex is *closed*. A walk whose edges are all distinct is a *trail*. A closed trail is a *tour*. A walk whose vertices are all distinct is a *path* and a walk whose vertices are all distinct except the start and the terminus is a *cycle*. Observe that a path is necessarily a trail and a cycle is a tour.

A path may also be seen as a non-empty graph $P = (V,E)$ of the form $V = \{x_0, x_1, \dots, x_k\}$ and $E = \{x_0x_1, x_1x_2, \dots, x_{k-1}x_k\}$ where the vertices x_i are all distinct. Similarly, a cycle may be seen as a non-empty graph $C = (V,E)$ of the form $V = \{x_0, x_1, \dots, x_k\}$ and $E = \{x_0x_1, x_1x_2, \dots, x_{k-1}x_k, x_kx_0\}$ where the x_i are all distinct.

Proposition 1.3. *Let G be a multigraph.*

(i) There is a (u,v)-walk in G if and only if there is a (u,v)-path.

(ii) An edge is in a closed trail if and only if it is in a cycle.

(iii) There is an odd closed walk, if and only if there is an odd cycle.

Proof. (i) Let $P = v_0e_1v_1e_2v_2 \dots e_kv_k$ be a shortest (u,v)-walk. Then P is a path. Indeed suppose for a contradiction that there exists $i < j$ such that $v_i = v_j$. Then $v_0e_1 \dots e_iv_ie_{j+1} \dots e_kv_k$ is a (u,v)-walk shorter than P, a contradiction.

(ii) Let $C = ve_1v_1e_2v_2 \dots e_kv$ be a shortest closed trail. Then C is a cycle. Indeed suppose for a contradiction that there exists $1 \leq i < j < k$ such that $v_i = v_j$. Then $v_0e_1 \dots e_iv_ie_{j+1} \dots e_kv$ is a (u,v)-trail shorter than C, a contradiction.

(iii) Let $C = ve_1v_1e_2v_2 \dots e_kv$ be a shortest odd closed walk. Then C is an odd cycle. Indeed suppose for a contradiction that there exists $1 \leq i < j < k$ such that $v_i = v_j$. Then $v_0e_1 \dots e_iv_ie_{j+1} \dots e_kv$ and $v_ie_{i+1} \dots e_jv_j$ are shorter closed walks than C. But the sum of the lengths of these two walks is the length of C and so is odd. So, one of the lengths is odd, a contradiction.

□

The *distance* between two vertices u and v in a multigraph is the length of a shortest (u,v)-walk or $+\infty$ if such a walk does not exists. It is denoted $dist_G(u,v)$, or simply $dist(u,v)$ if G is clearly understood from the context. The proof of (i) in the above proposition shows that a shortest (u,v)-walk (if one exists) is a (u,v)-path.

The word "distance" is well chosen because *dist* is a distance in the mathematical sense, that is a binary relation which is *symmetric* (for all $u,v \in V(G)$, $dist(u,v) = dist(v,u)$) and which satisfies the *triangle inequality*: for all $u,v,w \in V(G)$, $dist(u,w) \leq dist(u,v) + dist(v,w)$. See Exercise 1.12.

In multidigraphs, a *directed walk* is a finite (non-empty) sequence $W = v_0e_1v_1e_2v_2 \dots e_kv_k$ alternating vertices and arcs such that, for $1 \leq i \leq k$, v_{i-1} is the start and v_i the terminus of e_i. *Directed trail*, *directed tour*, *directed path* and *directed cycle* are then defined similarly to *trail*, *tour*, *path* and *cycle*. Clearly, Proposition 1.3 has its analog for digraphs. Its proof is left in Exercise 1.11.

Proposition 1.4. *Let D be a multidigraph.*

(i) There is a directed (u,v)-walk in D if and only if there is a directed (u,v)-path.

(ii) An edge is in a closed directed trail if and only if it is in a directed cycle.

(iii) There is an odd closed directed walk, if and only if there is an odd directed cycle.

The *distance* between two vertices in a multidigraph is defined analogously to the distance in a mutigraph. However, it is no more a distance in the mathematical sense because it is not symmetric. However it satisfies the triangle inequality. See Exercise 1.12.

1.4 Connectivity and trees

A graph G is *connected* if for any two vertices u,v, there exists a (u,v)-path in G.

Proposition 1.5. *Let G be a graph and x a vertex of G. The graph G is connected if and only if for any vertex u in G, there is a (u,x)-path.*

The *connected components* of a graph are its maximal connected subgraph.

A graph with no cycle is a *forest*. It is also said to be *acyclic*. A connected forest is a *tree*. The *leafs* of a tree T are the vertices of degree at most 1.

Proposition 1.6. *Let G be a graph. If $\delta(G) \geq 2$ then G has a cycle.*

Proof. Let $P = (v_1, v_2, \dots, v_k)$ be a path of maximal length. Since v_1 has degree 2 it is adjacent to a vertex $w \neq v_2$. The vertex w is in P otherwise $(w, v_1, v_2, \dots, v_k)$ would be a longer path than P. Thus $w = v_j$ for some $j > 2$ and so $(v_1, v_2, \dots, v_j, v_1)$ is a cycle. $\square$

Proposition 1.6 implies that every forest has at least one leaf. In fact, it implies that every forest has at least two leaves.

Corollary 1.7. *Every forest on at least two vertices has at least two leaves.*

Proof. By induction on the number of vertices, the result holding trivially for the two forests on two vertices.

Let F be a tree on n vertices, with $n \geq 3$. By Proposition 1.6, F has at least one leaf x. The graph $F - x$ is a forest on $n-1$ vertices. By the induction hypothesis, it has two leaves y_1 and y_2. One of these two vertices, say y, is not adjacent to x since $d(x) \leq 1$. Hence y is also a leaf of F. $\square$

Corollary 1.8. *For every tree T we have $|E(T)| = |V(T)| - 1$.*

Proof. By induction on the number of vertices of T, the result holding trivially if T is the unique tree on one vertex (K_1).

Let T be a tree on at least two vertices. By Corollary 1.7, T has a leaf x. Since T is connected, x has degree at least one, so $d(x) = 1$. Thus, $|E(T - x)| = |E(T)| - 1$. By the induction hypothesis, $|E(T - x)| = |V(T - x)| - 1 = |V(T)| - 2$. Hence, $|E(T)| = |V(T)| - 1$. $\square$

Proposition 1.9. *Let T be a graph. Then the following four statements are equivalent:*

(i) T is a tree;

(ii) for any two vertices u,v of T, there exists a unique (u,v)-path;

(iii) T is connected-minimal, i.e. T is connected and $T \setminus e$ is not connected for all $e \in E(T)$;

(iv) T is acyclic-maximal, i.e. T is acyclic but $T \cup xy$ has a cycle for any pair $\{x,y\}$ of non-adjacent vertices in T.

Proof. (i)⇒(ii): By the contrapositive. Suppose that there exist two distinct (u,v)-paths $P = (p_1, p_2, \dots, p_k)$ and $Q = (q_1, q_2, \dots, q_l)$. Let i be the smallest index such that $p_{i+1} \neq q_{i+1}$ and let j be the smallest integer greater than i such that $p_j \in \{q_{i+1}, q_{i+2}, \dots, q_l\}$. Let j' be the index for which $q_{j'} = p_j$. Then $(p_i, p_{i+1}, \dots, p_j, q_{j'-1}, q_{j'-2}, \dots, q_i)$ is a cycle.

(ii)⇒(iii): If there exists a unique path between any two vertices, then T is connected. Let $e = xy$ be an edge. Then (x,y) is the unique (x,y)-path in T. Thus $T \setminus e$ contains no (x,y)-path and so T is not connected. Hence T is connected-minimal.

(iii)⇒(i): By the contrapositive. Suppose that T is not tree. If T is not connected then it is not connected-minimal. Thus we may assume that T is connected and so T contains a cycle C. Let e be an edge of C. Let us show that $T \setminus e$ is connected which implies that T is not connected-minimal. Let x and y be two vertices. Since T is connected there is an (x,y)-path P in T. If P does not contain e then it is also a path in $T - e$. If P contains e then replacing e by $C \setminus e$ in P, we obtain an (x,y)-walk in $T \setminus e$. By Proposition 1.3, there is an (x,y)-path in $T \setminus e$.

(i)⇒(iv): If T is a tree then it is acyclic. Let us show that it is acyclic-maximal. Let x and y be two non-adjacent vertices. Then in T there is an (x,y)-path P since T is connected. The concatenation of P and (y,x) is a cycle in $T \cup xy$.

(iv)⇒(i): By the contrapositive. Suppose that $T$is not a tree. If it is not acyclic then it is not acyclic-maximal. Thus we may assume that T is not connected. So there are two vertices x and y for which there is no (x,y)-path in T. Let us show that $T \cup xy$ is acyclic which implies that T is not acyclic-maximal. Indeed if there were a cycle C, then it must contain xy because T is acyclic. Then $C \setminus xy$ would be an (x,y)-path in T, a contradiction. □

A subgraph H of a graph G is *spanning* if $V(H) = V(G)$.

Corollary 1.10. *A graph G is connected if and only if it has a spanning tree.*

Proof. By induction on the number of edges of G. If G is connected-minimal, then by Proposition 1.9, G is a tree and thus a spanning tree of itself. If G is not connected-minimal, then by definition there is an edge e such that $G \setminus e$ is connected. By the induction hypothesis, $G \setminus e$ has a spanning tree which is also a spanning tree of G. □

1.5 Strong connectivity and handle decomposition

A digraph is *strongly connected* or *strong* if for any two vertices u, v there is a directed (u,v)-path. Observe that swapping u and v implies that there is also a directed (v,u)-path. The *strongly connected components* of a digraph G are its maximum strongly connected subgraphs.

The following proposition follows easily from the definition.

Proposition 1.11. *Let D be a strongly connected digraph. Then every arc is in a directed cycle.*

Proof. Let uv be an arc. Since D is strongly connected then there is a directed (v,u)-path in D. Its concatenation with (u,v) is a directed cycle containing uv. □

Definition 1.12. The *union* of two digraphs D_1 and D_2 is the digraph $D_1 \cup D_2$ defined by $V(D_1 \cup D_2 = V(D_1) \cup V(D_2)$ and $E(D_1 \cup D_2) = E(D_1) \cup E(D_2)$.

Let D be a digraph and H be a subdigraph of D. A *H-handle* is a directed path or cycle (all vertices are distinct except possibly the two endvertices) such that its endvertices are in $V(H)$ and its internal vertices are in $V(D) \setminus V(H)$. A *handle decomposition* of D is a sequence $(C, P_1, \dots, P_k)$ such that:

- $C = D_0$ is a directed cycle;
- for all $1 \leq i \leq k$, P_i is a D_{i-1}-handle and $D_i = D_{i-1} \cup P_i$;
- $D_k = D$.

The following proposition follows easily from the definitions.

Proposition 1.13. *Let H be a strongly connected subdigraph of D. For any H-handle P, then $H \cup P$ is strongly connected.*

Proof. Left in Exercise 1.25 □

Since every strongly connected digraph contains a directed cycle (Proposition 1.11), an easy induction immediately yields the following.

Corollary 1.14. *Every digraph admitting a handle decomposition is strongly connected.*

The converse is also true: every strongly connected digraph admits a handle decomposition. In addition, it has a handle decomposition starting at any directed cycle.

Theorem 1.15. *Let D be a strongly connected digraph and C a directed cycle in D. Then D has a handle decomposition $(C, P_1, \dots, P_k)$.*

Proof. Let H be the subdigraph of D that admits a handle decomposition $(C, P_1, \dots P_k)$ with the maximum number of arcs. Since every arc xy in $E(D) \setminus E(H)$ with both endvertices in $V(H)$ is a H-handle, H is an induced subdigraph of D. Assume for a contradiction that $H \neq D$. Then $V(H) \neq V(D)$. Since D is strongly connected, there is an arc vw with $v \in V(D)$ and $w \in V(D) \setminus V(H)$. Since D is strongly connected, D contains a (w, H)-path P. Then, (v, w, P) is a H-handle in D, contradicting the maximality of H. □

1.6 Eulerian graphs

A trail in a graph G is *eulerian G* if it goes exactly once through every edge of G. A graph is *eulerian* if it has an eulerian tour.

Theorem 1.16 (Euler 1736)**.** *A connected graph is eulerian if and only if all its vertices have even degree.*

Proof. The condition can easily seen to be necessary. Indeed if a vertex appears k times (or $k+1$ if it appears as start and terminus) of an eulerian tour, it is incident to exactly $2k$ edges in the tour and so it has degree $2k$.

Let us now show that the condition is sufficient. The proof follows the lines of the following algorithm.

Algorithm 1.1.

1. Initialise $W := v$ for an arbitrary vertex v.
2. If all the edges of G are in W then return W.
3. If not an edge is not in $W = v_0 e_1 v_1 \dots e_l v_l$,
4. If an edge incident to v_l, say $e = v_l v_{l+1}$ is not in W, then $W := v_0 e_1 v_1 \dots e_l v_l e v_{l+1}$; go to 2.
5. If not all the edges incident to v_l are in W. Since there is an even number of them, $v_0 = v_l$. Then G has an edge $e \notin W$ incident to a vertex v_i in W, for it is connected. Let $e = v_i u$ be this edge.
   $W := v_i e_{i+1} v_{i+1} \dots e_l v_l e_1 v_1 \dots e_i v_i e u$; go to 2.

□

1.7 Exercises

Exercise 1.1. Show that K_n, the complete graph on n vertices, has $\binom{n}{2}$ edges.

Exercise 1.2. Build a cubic graph with 11 vertices. (*cubic:* $d(v) = 3$ for all vertex v.)

Exercise 1.3. Show that every graph has two vertices of same degree.

Exercise 1.4. Let G be a graph on at least 4 vertices such that for every vertex v, $G - v$ is regular. Show that G is either a complete graph or an empty graph.

Exercise 1.5. Let n and k be two integers such that $n > k$ and H be a graph on n vertices. Show that if $|E(H)| > (k-1)(n-k/2)$ then H has a subgraph of minimum degree at least k.

Exercise 1.6 (Jealous husbands)**.**
Three jealous husbands and their wives want to cross a river. But they just have a small boat in which at most two persons can fit. None of the husbands would allow his wife to be with another man unless he is present. Draw the graph of all the possible distributions across the river and advice the walkers on the method to cross the river.

Exercise 1.7 (Dog, goat, cabbage)**.**
A man wants to cross a river with his dog, his goat and his (huge) cabbage. Unfortunately, the

man can cross the river with at most one of them. Furthermore, for obvious reasons, the man cannot leave alone on one bank neither the goat and the dog nor the cabbage and the goat. Draw the bipartite graph of all permissible situations. How does the man do to cross the river?

Exercise 1.8. Let u and v be two vertices of a graph G. Show that, if u and v have odd degree and all the other vertices have even degree, then there is a (u,v)-path in G.

Exercise 1.9. Show that in a graph two paths of maximum length have a vertex in common.

Exercise 1.10. Find what is wrong in the following statement: *An edge is in a closed trail if and only if it is in a cycle.*

Exercise 1.11. Show Proposition 1.4.

Exercise 1.12. 1) Show that of G is a multigraph then $dist_G$ is symmetric and satisfies the triangle inequality.

2) Show that of D is a multidigraph then $dist_D$ satisfies the triangle inequality but may be non-symmetric.

Exercise 1.13. Let D be a digraph without directed cycles. Show that D has a vertex with indegree zero.

Exercise 1.14. Let $G = (V,E)$ be a graph. Show the following.

(1) If $|E| \geq |V|$ then G contains a cycle.

(2) If $|E| \geq |V| + 4$ then G contains two edge-disjoint cycles.

Exercise 1.15. Let G be a graph of minimum degree at least 3. Show that G contains an even cycle.

Exercise 1.16. Let G be a connected graph. Show that there exists an orientation of G such that the outdegree of every vertex is even if and only if G has an even number of edges.

Exercise 1.17. Let G be a graph. Its *diameter* is the maximum distance between two vertices.

1) Show that if G has a diameter at least 3 then its complement $\overline{G}$ has diameter at most 3.

2) Deduce that every self-complementary graph ($G = \overline{G}$) has diameter at most 3.

3) For $k = 1,2,3$, give an example of a self-complementary graph with de diameter k.

Exercise 1.18. Let T be a tree on at least two vertices. Show that if T has no vertex of degree 2 then T has at least $|V(T)|/2+1$ leaves.

Exercise 1.19 (Helly property for trees)**.** Let $T_1,\dots,T_k$ be subtrees if a tree T. Show that if $T_i \cap T_j \neq \emptyset$ for all i,j, then $\bigcap_{i=1}^{k} T_i \neq \emptyset$.

Exercise 1.20. Is the complement of a non-connected graph always connected?
Is the complement of a connected graph always non-connected?

Exercise 1.21. 1) Show that every connected graph G has a vertex x such that $G-x$ is connected.
2) Does the same hold for strongly connected digraphs?

Exercise 1.22. A graph is *cherry-free* if every vertex has at most one neighbour of degree 1. Prove that a connected cherry-free graph has two adjacent vertices u and v such that $G-\{u,v\}$ is connected.
Hint: Consider a path of maximum length.

Exercise 1.23. Let G be a connected graph and $(V_1,V_2,\dots,V_n)$ a partition of $V(G)$ such that $G\langle V_i\rangle$ is connected for all $1\le i\le n$. Show that there exists two indices i and j such that $G-V_i$ and $G-V_j$ are connected.

Exercise 1.24. The aim of this exercise is to prove that if a graph has n vertices, m edges and k connected components then $n-k\le m\le \frac{1}{2}(n-k)(n-k+1)$.
1) Let G be a graph on n vertices with m edges and k connected components.

a) Show that if G is connected then $m\ge n-1$.

b) Deduce that if G has k connected components then $m\ge n-k$.

2) Suppose now that G is a graph on n vertices and k connected components with the maximum number of edges.

a) Show that all the connected components of G are complete graphs.

b) Show that if G has (at least) two connected components then one of them has a unique vertex.

c) Deduce that G has $\frac{1}{2}(n-k)(n-k+1)$ edges.

Exercise 1.25. Prove Proposition 1.13.

Exercise 1.26. Let D be a strongly connected digraph and D' a strongly connected subdigraph of D. Show that any handle decomposition $(C,P_1,\dots,P_k)$ of D' may be extended into a handle decomposition $(C,P_1,\dots,P_k,\dots,P_l)$ of D.

Exercise 1.27. Let D be a strongly connected digraph of minimum outdegree 2. Prove that there exists a vertex v such that $D-v$ is strongly connected.

Exercise 1.28. Show that a graph has an eulerian trail if and only if it has zero or two vertices of odd degree.

Exercise 1.29. Let $G=(V,E)$ be a graph such that every vertex has even degree and $|E|\equiv 0[3]$. Prove that E can be partitionned into $l=\frac{|E|}{3}$ sets $E_1,\dots,E_l$ such that for all $1\le i\le l$, the graph induced by E_i is either a path of length 3 or cycle of length 3.

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