Simplex Method and Reduced Costs, Duality and Marginal Costs

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** Simplex Method and Reduced Costs, Strong Duality Theorem **
Start with a problem written under the standard form.

Maximize \[ 5x_1 + 4x_2 + 3x_3 \]
Subject to:
\[ 2x_1 + 3x_2 + x_3 \leq 5 \]
\[ 4x_1 + x_2 + 2x_3 \leq 11 \]
\[ 3x_1 + 4x_2 + 2x_3 \leq 8 \]
\[ x_1, x_2, x_3 \geq 0. \]
Simplex - Reminder

Write the Dictionary:

\[
\begin{align*}
    x_4 & = 5 - 2x_1 - 3x_2 - x_3 \\
    x_5 & = 11 - 4x_1 - x_2 - 2x_3 \\
    x_6 & = 8 - 3x_1 - 4x_2 - 2x_3 \\
    z & = 5x_1 + 4x_2 + 3x_3.
\end{align*}
\]

Basic variables: \(x_4, x_5, x_6\), variables on the left.
Non-basic variable: \(x_1, x_2, x_3\), variables on the right.

A dictionary is **feasible** if a feasible solution is obtained by setting all non-basic variables to 0.
Simplex - Reduced Costs

Write the Dictionary:

\[
\begin{align*}
  x_4 &= 5 - 2x_1 - 3x_2 - x_3 \\
  x_5 &= 11 - 4x_1 - x_2 - 2x_3 \\
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  z &= 5x_1 + 4x_2 + 3x_3.
\end{align*}
\]

We call Reduced Costs the coefficients of \(z\). The reduced cost of \(x_1\) is 5, of \(x_2\) is 4 and of \(x_3\) is 3.

**Reminder:** If all reduced cost are non-positive, the solution is optimal and the simplex algorithm stops.
Simplex - Reduced Costs

Relationship between reduced costs, $\bar{c} = (\bar{c}_1, \ldots, \bar{c}_n)$ and optimal solution of the dual problem $\pi = (\pi_1, \ldots, \pi_m)$.

If we consider a general LP:

Maximize $\sum_{j=1}^{n} c_j x_j$
Subject to: $\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad (i = 1, 2, \ldots, m)$
$x_j \geq 0 \quad (j = 1, 2, \ldots, n)$

Lemma: When the simplex algorithm finishes, we have:

$$\bar{c}_j = c_j - \sum_{i=1}^{m} \pi_i A_{ij}$$
We consider a general LP:

\[
\begin{align*}
\text{Maximize} & \quad \sum_{j=1}^{n} c_j x_j \\
\text{Subject to:} & \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad (i = 1, 2, \ldots, m) \\
& \quad x_j \geq 0 \quad (j = 1, 2, \ldots, n)
\end{align*}
\]

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We introduce the following notations, A and B.

\[
\begin{align*}
\text{Maximize} & \quad c^T x \\
\text{Subject to:} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\]

The method of the simplex finishes with an optimal solution \( x \) and an associated basis. Let \( B(1), \ldots, B(m) \) be the indices of basic variables. We define \( B = [A_{B(1)} \ldots A_{B(m)}] \) the matrix associated to the basis.

We have \( x_B = B^{-1} b \)
Simplex - Reduced Costs

Maximize \[ c^T x \]
Subject to: \[ Ax = b \]
\[ x \geq 0 \]

By studying what happens during a step of the simplex method, we can get the following expression for the reduced cost of variable \( x_j \)

\[ \bar{c}_j = c_j - c_B^T B^{-1} A_j. \]
When the method of the simplex finishes, the reduced costs are non-positive.

\[ c^T - c_B^T B^{-1} A \leq 0^T. \]

Let \( \pi \) be such that

\[ \pi^T = c_B^T B^{-1} \]
Simplex - Reduced Costs

We get

\[ c^T - c_B^T B^{-1} A \leq 0^T. \]
\[ c^T - \pi^T A \leq 0^T. \]
\[ \pi^T A \geq c^T. \]
\[ A^T \pi \geq c. \]

⇒ \( \pi \) is a feasible solution of the dual problem:

Minimize \( \pi^T b \)
Subject to: \( A^T \pi \geq c \)
\( \pi \geq 0 \)

(2)
Simplex - Reduced Costs

Moreover, the value of \( p \) equals the value of the optimal value of the primal:

\[
\pi^T b = c_B^T B^{-1} b = c_B^T x_B = c^T x
\]

\( \Rightarrow \) \( \pi \) is an optimal solution of the dual problem (by the weak duality theorem).

Theorem [Strong Duality]: If the primal problem has an optimal solution,

\[
x^* = (x_1^*, \ldots, x_n^*)
\]

then the dual also has an optimal solution,

\[
y^* = (y_1^*, \ldots, y_n^*)
\]

and

\[
\sum_j c_j x_j^* = \sum_i b_i y_i^*.
\]
Simplex - Reduced Costs

Moreover, the value of $\rho$ equals the value of the optimal value of the primal:

$$
\pi^T b = c_B^T B^{-1} b = c_B^T x_B = c^T x
$$

$\Rightarrow \pi$ is an optimal solution of the dual problem (by the weak duality theorem).

Theorem [Strong Duality]: If the primal problem has an optimal solution,

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$$

and

$$
\sum_j c_j x_j^* = \sum_i b_i y_i^*.
$$
Moreover, the value of \( p \) equals the value of the optimal value of the primal:

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\pi^T b = c_B^T B^{-1} b = c_B^T x_B = c^T x
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\( \Rightarrow \pi \) is an optimal solution of the dual problem (by the weak duality theorem).

**Theorem [Strong Duality]:** If the primal problem has an optimal solution,

\[
x^* = (x_1^*, \ldots, X_n^*),
\]

then the dual also has an optimal solution,

\[
y^* = (y_1^*, \ldots, y_n^*),
\]

and

\[
\sum_j c_j x_j^* = \sum_i b_i y_i^*.
\]
The Reduced Cost is

- *the amount by which an objective function coefficient would have to improve before it would be possible for a corresponding variable to assume a positive value in the optimal solution.*
** Dual Variables and Marginal Costs **
Signification of Dual Variables

Max
\[ \sum_{j=1}^{n} c_j x_j \]
S. t.:
\[ \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad (i = 1, \cdots, m), \]
\[ \sum_{j=1}^{n} a_{ij} x_j \geq 0 \quad (j = 1, \cdots, n) \]

Min
\[ \sum_{i=1}^{m} b_i y_i \]
S. t.:
\[ \sum_{i=1}^{m} a_{ij} y_i \geq c_j \quad (j = 1, \cdots, n), \]
\[ y_i \geq 0 \quad (i = 1, \cdots, m) \]

Signification can be given to variables of the dual problem:

“The optimal values of the dual variables can be interpreted as the marginal costs of a small perturbation of the right member \( b \).”
Signification of Dual Variables

Maximize \[ \sum_{j=1}^{n} c_j x_j \]
Subject to: \[ \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad (i = 1, 2, \ldots, m) \]
\[ x_j \geq 0 \quad (j = 1, 2, \ldots, n) \]

Minimize \[ \sum_{i=1}^{m} b_i y_i \]
Subject to: \[ \sum_{i=1}^{m} a_{ij} y_i \geq c_j \quad (j = 1, 2, \ldots, n) \]
\[ y_i \geq 0 \quad (i = 1, 2, \ldots, m) \]

Dimension analysis for a factory problem:
- \( x_j \): production of a product \( j \) (chair, ...)
- \( b_i \): available quantity of resource \( i \) (wood, metal, ...)
- \( a_{ij} \): unit of resource \( i \) per unit of product \( j \)
- \( c_j \): net benefit of the production of a unit of product \( j \)
Signification of Dual Variables

Maximize \[ \sum_{j=1}^{n} c_j x_j \]
Subject to: \[ \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad (i = 1, 2, \ldots, m) \]
\[ x_j \geq 0 \quad (j = 1, 2, \ldots, n) \]

Minimize \[ \sum_{i=1}^{m} b_i y_i \]
Subject to: \[ \sum_{i=1}^{m} a_{ij} y_i \geq c_j \quad (j = 1, 2, \ldots, n) \]
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\[
\frac{a_{1j} y_1 + \ldots + a_{nj} y_n}{\text{unit of resource } i/\text{unit of product } j} > c_j \quad \text{euros/unit of resource } i/\text{euros/unit of product } j
\]

\[ \rightarrow y_i \text{ euro by unit of resource } i. \text{ Unit value of resource } i. \]
Signification of Dual Variables

Theorem: If the LP admits at least one optimal solution, then there exists $\varepsilon > 0$, with the property: If $|t_i| \leq \varepsilon \ \forall i = 1, 2, \cdots, m$, then the LP

\[
\begin{align*}
\text{Max} & \quad \sum_{j=1}^{n} c_j x_j \\
\text{Subject to:} & \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i + t_i \quad (i = 1, 2, \cdots, m) \\
& \quad x_j \geq 0 \quad (j = 1, 2, \cdots, n).
\end{align*}
\]

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has an optimal solution and the optimal value of the objective is

\[
z^* + \sum_{i=1}^{m} y_i^* t_i
\]

with $z^*$ the optimal solution of the initial LP and $(y_1^*, y_2^*, \cdots, y_m^*)$ the optimal solution of its dual.
To be remembered

- **Definition of the Reduced Costs**

\[
\begin{align*}
x_4 &= 5 - 2x_1 - 3x_2 - x_3 \\
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x_6 &= 8 - 3x_1 - 4x_2 - 2x_3 \\
z &= 5x_1 + 4x_2 + 3x_3.
\end{align*}
\]

- If all reduced cost are **non-positive**, the solution is optimal and the simplex algorithm stops.

- Relationship between reduced costs, \( \overline{c} = (\overline{c}_1, \ldots, \overline{c}_n) \) and optimal solution of the dual problem \( \pi = (\pi_1, \ldots, \pi_m) \).

When the simplex algorithm finishes, we have:

\[
\overline{c} = c^T - \pi^T A
\]