# Simplex Method and Reduced Costs, **Duality and Marginal Costs**

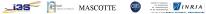
Frédéric Giroire







\*\* Simplex Method and Reduced Costs, Strong Duality Theorem \*\*







#### Simplex - Reminder

Start with a problem written under the standard form.

Maximize  $5x_1 + 4x_2 + 3x_3$ Subject to :



#### Simplex - Reminder

#### Write the Dictionary:

<i>x</i> <sub>4</sub>	=	5	_	2 <u>x</u> 1	_	3 <mark>x</mark> 2	_	<i>x</i> 3
<i>x</i> 5	=	11	_	4 <u>x</u> 1	_	<i>x</i> <sub>2</sub>	_	2 <u>x</u> 3
<i>x</i> 6	=	8	_	3 <mark>x</mark> 1	_	4 <u>x</u> 2	_	2 <u>x</u> 3
Ζ	=			5 <mark>x</mark> 1	+	4 <u>x</u> 2	+	3 <mark>x</mark> 3.

Basic variables:  $x_4$ ,  $x_5$ ,  $x_6$ , variables on the left. Non-basic variable:  $x_1$ ,  $x_2$ ,  $x_3$ , variables on the right.

A dictionary is feasible if a feasible solution is obtained by setting all non-basic variables to 0.





#### Write the **Dictionary**:

<i>x</i> 4	=	5	_	2 <mark>x</mark> 1	_	3 <mark>x</mark> 2	_	<i>x</i> 3
<i>x</i> 5	=	11	_	4 <mark>x</mark> 1	—	<i>x</i> <sub>2</sub>	_	2 <u>x</u> 3
<i>x</i> 6	=	8	_	3 <mark>x</mark> 1	_	4 <u>x</u> 2	_	2 <mark>x</mark> 3
Ζ	=			5 x <sub>1</sub>	+	<b>4</b> <i>x</i> <sub>2</sub>	+	<b>3</b> <i>x</i> <sub>3</sub> .

We call Reduced Costs the coefficients of *z*. The reduced cost of  $x_1$  is 5, of  $x_2$  is 4 and of  $x_3$  is 3.

Reminder: If all reduced cost are non-positive, the solution is optimal and the simplex algorithm stops.





Relationship between reduced costs,  $\overline{c} = (\overline{c}_1, ..., \overline{c}_n)$  and optimal solution of the dual problem  $\pi = (\pi_1, ..., \pi_m)$ .

If we consider a general LP:

Lemma: When the simplex algorithm finishes, we have:

$$\overline{c}_j = c_j - \sum_{i=1}^m \pi_i A_{ij}$$

We consider a general LP:

$$\begin{array}{rll} \text{Maximize} & \sum_{j=1}^{n} c_j x_j \\ \text{Subject to:} & \sum_{j=1}^{n} a_{ij} x_j &\leq b_i \quad (i=1,2,\cdots,m) \\ & x_j &\geq 0 \quad (j=1,2,\cdots,n) \end{array} \tag{1}$$

We introduce the following notations, A and B.

Maximize 
$$c^T x$$
  
Subject to:  $Ax = b$   
 $x \ge 0$ 

The method of the simplex finishes with an optimal solution x and an associated basis. Let  $B(1), \ldots, B(m)$  be the indices of basic variables.

We define  $B = [A_{B(1)}...A_{B(m)}]$  the matrix associated to the basis. We have  $x_B = B^{-1}b$ 

Maximize 
$$c^T x$$
  
Subject to:  $Ax = b$   
 $x \ge 0$ 

By studying what happens during a step of the simplex method, we can get the following expression for the reduced cost of variable  $x_i$ 

MASCOTTE

125

$$\overline{c}_j = c_j - c_B^T B^{-1} A_j.$$



When the method of the simplex finishes, the reduced costs are non-positive.

$$c^T - c_B^T B^{-1} A \leq 0^T.$$

Let  $\pi$  be such that

$$\pi^T = c_B^T B^{-1}$$



We get

$$egin{aligned} & c^T - c_B^T B^{-1} A \leq 0^T. \ & c^T - \pi^T A \leq 0^T. \ & \pi^T A \geq c^T. \ & A^T \pi \geq c. \end{aligned}$$

MASCOTTE

 $\Rightarrow \pi$  is a feasible solution of the dual problem:

\_i25

$$\begin{array}{ll} \text{Minimize} & \pi^T b\\ \text{Subject to:} & A^T \pi \geq c\\ & \pi \geq 0 \end{array} \tag{2}$$



Moreover, the value of *p* equals the value of the optimal value of the primal:

$$\pi^T b = c_B^T B^{-1} b = c_B^T x_B = c^T x$$

 $\Rightarrow \pi$  is an optimal solution of the dual problem (by the weak duality theorem).

Theorem [Strong Duality]: If the primal problem has an optimal solution,

$$x^* = (x_1^*, ..., X_n^*),$$

then the dual also has an optimal solution,

$$y^* = (y_1^*, ..., y_n^*),$$

and



Moreover, the value of *p* equals the value of the optimal value of the primal:

$$\pi^T b = c_B^T B^{-1} b = c_B^T x_B = c^T x$$

 $\Rightarrow \pi$  is an optimal solution of the dual problem (by the weak duality theorem).

Theorem [Strong Duality]: If the primal problem has an optimal solution,

$$x^* = (x_1^*, ..., X_n^*),$$

then the dual also has an optimal solution,

$$y^* = (y_1^*, ..., y_n^*),$$

and



Moreover, the value of *p* equals the value of the optimal value of the primal:

$$\pi^T b = c_B^T B^{-1} b = c_B^T x_B = c^T x$$

 $\Rightarrow \pi$  is an optimal solution of the dual problem (by the weak duality theorem).

Theorem [Strong Duality]: If the primal problem has an optimal solution,

$$x^* = (x_1^*, ..., X_n^*),$$

then the dual also has an optimal solution,

$$y^* = (y_1^*, ..., y_n^*),$$

and

$$\sum_j c_j x_j^* = \sum_i b_i y_i^*.$$



#### The Reduced Cost is

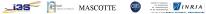
- the amount by which an objective function coefficient would have to improve before it would be possible for a corresponding variable to assume a positive value in the optimal solution.







\*\* Dual Variables and Marginal Costs \*\*







Max	$\sum_{j=1}^{n} c_j x_j$			$(i=1,\cdots,m)$	Min	$\sum_{i=1}^{m} b_i y_i$			
S. t.:	$\sum_{i=1}^{n} a_{ij} x_j$	$\leq$	bi	$(i = 1, \cdots, m)$	S. t.:	$\sum_{i=1}^{m} a_{ij} y_i$	$\geq$	cj	$(j=1,\cdots,n)$ $(i=1,\cdots,m)$
	, xj	$\geq$	0	$(j = 1, \cdots, n)$		Уi	$\geq$	0	$(i=1,\cdots,m)$

Signification can be given to variables of the dual problem:

"The optimal values of the dual variables can be interpreted as the marginal costs of a small perturbation of the right member b."







Maximize

Dimension analysis for a factory problem:

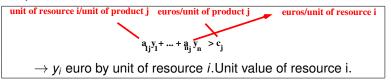
- x<sub>i</sub>: production of a product *i* (chair, ...)
- b<sub>i</sub>: available guantity of resource i (wood, metal, ...)
- a<sub>ii</sub>: unit of resource i per unit of product j
- c<sub>i</sub>: net benefit of the production of a unit of product j





Dimension analysis for a factory problem:

- x<sub>j</sub>: production of a product j (chair, ...)
- b<sub>i</sub>: available quantity of resource i (wood, metal, ...)
- a<sub>ij</sub>: unit of resource i per unit of product j
- c<sub>i</sub>: net benefit of the production of a unit of product j



JISS MASCOTTE P INRIA

**Theorem:** If the LP admits at least one optimal solution, then there exists  $\varepsilon > 0$ , with the property: If  $|t_i| \le \varepsilon \quad \forall i = 1, 2, \dots, m$ , then the LP

$$\begin{array}{rcl} & \text{Max} & \sum_{j=1}^{n} c_{j} x_{j} \\ \text{Subject to:} & \sum_{j=1}^{n} a_{ij} x_{j} & \leq & b_{i} + t_{i} & (i = 1, 2, \cdots, m) \\ & x_{j} & \geq & 0 & (j = 1, 2, \cdots, n). \end{array}$$
(3)

has an optimal solution and the optimal value of the objective is

$$z^* + \sum_{i=1}^m y_i^* t_i$$

with  $z^*$  the optimal solution of the initial LP and  $(y_1^*, y_2^*, \dots, y_m^*)$  the optimal solution of its dual.



# To be remembered

Definition of the Reduced Costs

<i>x</i> <sub>4</sub>	=	5	—	2 <mark>x</mark> 1	_	3 <mark>x</mark> 2	—	<i>x</i> 3
<i>x</i> 5	=	11	—	4 <b>x</b> <sub>1</sub>	—	<i>x</i> <sub>2</sub>	_	2 <i>x</i> <sub>3</sub>
<i>x</i> 6	=	8	—	3 <u>x</u> 1	—	4 <u>x</u> 2	_	2 <u>x</u> 3
Ζ	=			5 x <sub>1</sub>	+	<b>4</b> <i>x</i> <sub>2</sub>	+	<b>3</b> <i>x</i> <sub>3</sub> .

- If all reduced cost are non-positive, the solution is optimal and the simplex algorithm stops.
- Relationship between reduced costs, c

   (c
   (π
   (π
   (π
   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

   (π

$$\overline{c} = c^T - \pi^T A$$