Modelling Graph Problems using Linear Programmes: An Example

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Definition. Let $G = (V, E)$ be a graph.

- A matching $M \subseteq E$ is a collection of edges such that every vertex of $V$ is incident to at most one edge of $M$.
- The maximum cardinality matching problem is to find a matching $M$ of maximum size.

Reminder: The problem is polynomial
- for a bipartite graph (augmenting paths or applications of flows)
- for a general graph (Edmund’s algorithm)
Maximum Cardinality Matching Problem

An example:

Figure: A graph with 6 vertices
Maximum Cardinality Matching Problem

An example:

Figure: A matching with 2 edges
Maximum Cardinality Matching Problem

An example:

Figure: The maximum matching with 3 edges
Maximum Cardinality Matching Problem

Question: How to write the Maximum Cardinality Matching Problem as a linear programme?
Maximum Cardinality Matching Problem

First step: define the variables. Not always easy, most of the time good idea to think of the objective function.

Goal here: find a maximum subset of edges → variables on the edges seem useful.
Maximum Cardinality Matching Problem

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Variables: one variable per edge, $x_{AB}$ for edge $AB$.

$x_{AB}$ binary variable $x_{AB} = 1$ if $AB$ in the matching $x_{AB} = 0$ otherwise
Maximum Cardinality Matching Problem

Second step: write the objective function.

\[
\text{max} \quad x_{AB} + x_{BC} + \ldots + x_{AF}
\]
Maximum Cardinality Matching Problem

Third step: write the constraints.

- A matching $M \subseteq E$ is a collection of edges such that every vertex of $V$ is incident to at most one edge of $M$.

Constraint on vertex $A$: $x_{AB} + x_{AF} \leq 1$.
Constraint on vertex $B$: $x_{AB} + x_{BC} + x_{BF} \leq 1$.
One constraint per vertex.
Maximum Cardinality Matching Problem

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Maximum Cardinality Matching Problem

Variables: \( x_{AB} = 1 \) if edge \( AB \) is in the matching and \( x_{AB} = 0 \) otherwise.

\[
\text{maximize} \quad x_{AB} + x_{BC} + x_{CE} + x_{DE} + x_{EF} + x_{AF} + x_{BF}
\]

subject to

\[
\begin{align*}
    x_{AB} + x_{AF} & \leq 1 \\
    x_{AB} + x_{BC} + x_{BF} & \leq 1 \\
    x_{BC} + x_{CE} & \leq 1 \\
    x_{DE} & \leq 1 \\
    x_{CE} + x_{EF} + x_{DE} & \leq 1 \\
    x_{BF} + x_{EF} + x_{AF} & \leq 1 \\
    x_{AB}, x_{BC}, x_{CE}, x_{DE}, x_{EF}, x_{AF}, x_{BF} & \geq 0 \\
    x_{AB}, x_{BC}, x_{CE}, x_{DE}, x_{EF}, x_{AF}, x_{BF} & \in \mathbb{N}
\end{align*}
\]
Maximum Cardinality Matching Problem

Can be written in a more concise form and more generally for any graph.

| Var.:                     | \( x_{AB} = 1 \) if \( AB \in M \),  
                          | \( x_{AB} = 0 \) otherwise |
|---------------------------|----------------------------------------|
| max                       | \( x_{AB} + x_{BC} + x_{CE} \)     
                          | + \( x_{DE} + x_{EF} + x_{AF} + x_{BF} \) |
| s.t.                      | \( x_{AB} + x_{AF} \leq 1 \)            
                          | \( x_{AB} + x_{BC} + x_{BF} \leq 1 \)    
                          | \( x_{BC} + x_{CE} \leq 1 \)              
                          | \( x_{DE} \leq 1 \)                      
                          | \( x_{CE} + x_{EF} + x_{DE} \leq 1 \)    
                          | \( x_{BF} + x_{EF} + x_{AF} \leq 1 \)    
                          | \( x_{AB}, x_{BC}, x_{CE}, x_{DE}, x_{EF}, x_{AF}, x_{BF} \geq 0 \) |
                          | \( x_{AB}, x_{BC}, x_{CE}, x_{DE}, x_{EF}, x_{AF}, x_{BF} \in \mathbb{N} \) |

| Var.:                     | \( x_{ij} = 1 \) if \( ij \in M \),  
                          | \( x_{ij} = 0 \) otherwise |
|---------------------------|----------------------------------------|
| max                       | \( \sum_{(i,j) \in E} x_{ij} \)      |
| s. t.                      | \( \sum_{ij \in E} x_{ij} \leq 1 \)  
                          | \( \forall i \in V \)                   |
                          | \( x_{ij} \geq 0 \)                  
                          | \( \forall (i,j) \in E \)             |
                          | \( x_{ij} \in \mathbb{N} \)          
                          | \( \forall (i,j) \in E \)             |