Modelling Graph Problems using Linear Programmes: An Example

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Maximum Cardinality Matching Problem

Definition. Let $G = (V, E)$ be a graph.

- A matching $M \subseteq E$ is a collection of edges such that every vertex of V is incident to at most one edge of M.
- The maximum cardinality matching problem is to find a matching M of maximum size.

Reminder: The problem is polynomial

- for a bipartite graph (augmenting paths or applications of flows)
- for a general graph (Edmund's algorithm)

Maximum Cardinality Matching Problem

An example:

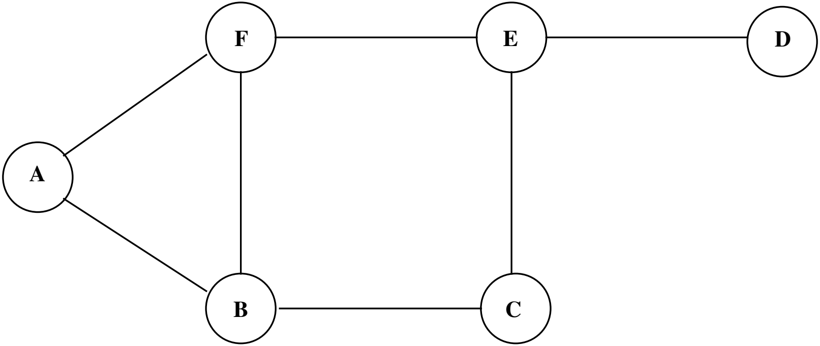

Figure: A graph with 6 vertices

Maximum Cardinality Matching Problem

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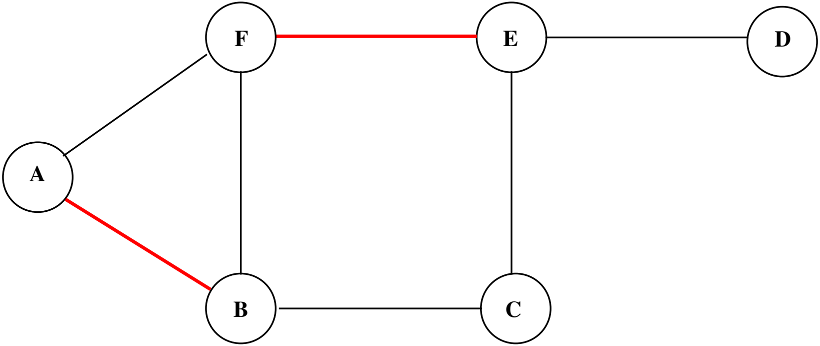

Figure: A matching with 2 edges

Maximum Cardinality Matching Problem

An example:

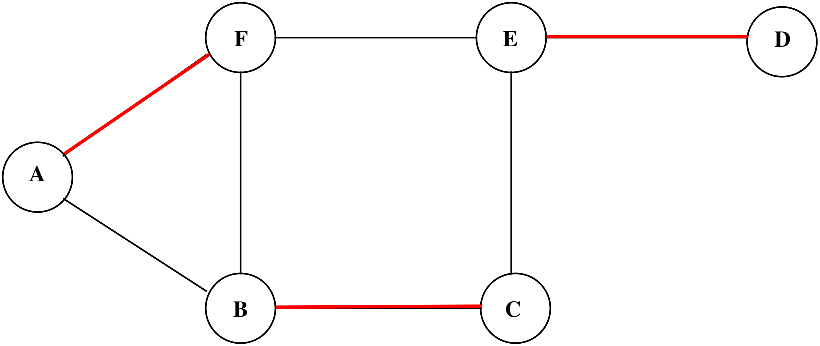

Figure: The maximum matching with 3 edges

Maximum Cardinality Matching Problem

Question: How to write the Maximum Cardinality Matching Problem as a linear programme?

Maximum Cardinality Matching Problem

First step: define the variables. Not always easy, most of the time good idea to think of the objective function.

Goal here: find a maximum subset of edges → variables on the edges seem useful.

Maximum Cardinality Matching Problem

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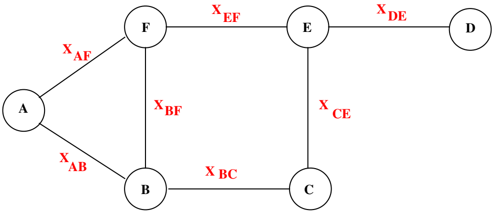
Variables: one variable per edge, x_{AB} for edge AB.



x_{AB} binary variable $x_{AB} = 1$ if AB in the matching $x_{AB} = 0$ otherwise

Maximum Cardinality Matching Problem

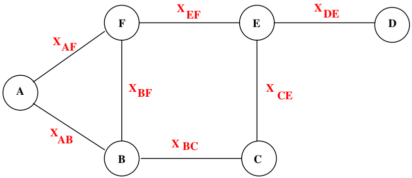
Second step: write the objective function.

$$\max \quad x_{AB} + x_{BC} + \ldots + x_{AF}$$

Maximum Cardinality Matching Problem

Third step: write the contraints.

- A matching $M \subseteq E$ is a collection of edges such that every vertex of V is incident to at most one edge of M.



Constraint on vertex A: $x_{AB} + x_{AF} \leq 1$.

Constraint on vertex B: $x_{AB} + x_{BC} + x_{BF} \leq 1$.

One constraint per vertex.

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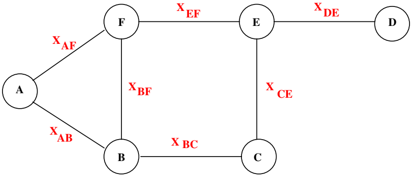

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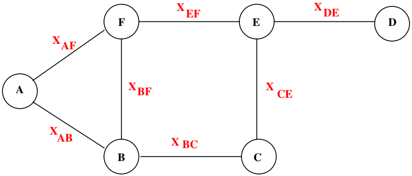

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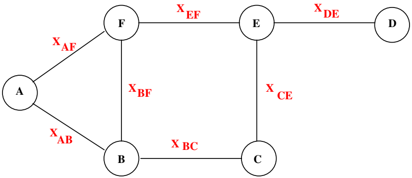

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One constraint per vertex.

Maximum Cardinality Matching Problem

Variables: $x_{AB} = 1$ if edge AB is in the matching and $x_{AB} = 0$ otherwise.

$$\begin{array}{rr} \max & x_{AB} + x_{BC} + x_{CE} + x_{DE} + x_{EF} + x_{AF} + x_{BF} \\ \text{subject to} & \\ & x_{AB} + x_{AF} \leq 1 \\ & x_{AB} + x_{BC} + x_{BF} \leq 1 \\ & x_{BC} + x_{CE} \leq 1 \\ & x_{DE} \leq 1 \\ & x_{CE} + x_{EF} + x_{DE} \leq 1 \\ & x_{BF} + x_{EF} + x_{AF} \leq 1 \\ & x_{AB}, x_{BC}, x_{CE}, x_{DE}, x_{EF}, x_{AF}, x_{BF} \geq 0 \\ & x_{AB}, x_{BC}, x_{CE}, x_{DE}, x_{EF}, x_{AF}, x_{BF} \in \mathbb{N} \end{array}$$

Maximum Cardinality Matching Problem

Can be written in a more concise form and more generally for any graph.

$$
\begin{array}{ll}
\text{Var.:} & x_{AB} = 1 \text{ if } AB \in M, \\
 & x_{AB} = 0 \text{ otherwise} \\
\\
\max & x_{AB} + x_{BC} + x_{CE} \\
 & + x_{DE} + x_{EF} + x_{AF} + x_{BF} \\
\text{s.t.} & \\
 & x_{AB} + x_{AF} \leq 1 \\
 & x_{AB} + x_{BC} + x_{BF} \leq 1 \\
 & x_{BC} + x_{CE} \leq 1 \\
 & x_{DE} \leq 1 \\
 & x_{CE} + x_{EF} + x_{DE} \leq 1 \\
 & x_{BF} + x_{EF} + x_{AF} \leq 1 \\
 & x_{AB}, x_{BC}, x_{CE}, x_{DE}, x_{EF}, x_{AF}, x_{BF} \geq 0 \\
 & x_{AB}, x_{BC}, x_{CE}, x_{DE}, x_{EF}, x_{AF}, x_{BF} \in \mathbb{N}
\end{array}
$$

$$
\begin{array}{lll}
\text{Var.:} & x_{ij} = 1 \text{ if } ij \in M, & \\
 & x_{ij} = 0 \text{ otherwise} & \\
\\
\max & \sum_{(i,j) \in E} x_{ij} & \\
\text{s. t.} & & \\
 & \sum_{ij \in E} x_{ij} \leq 1 & (\forall i \in V) \\
\\
 & x_{ij} \geq 0 & (\forall (i,j) \in E) \\
 & x_{ij} \in \mathbb{N} & (\forall (i,j) \in E)
\end{array}
$$