UBINET, Master 2 IFI Algorithms for telecommunications

First Exam, October 2012

1 hour 1/2

Course and manuscript notes allowed. Not computers, cellphones, books.

Instruction and comments: the points awarded for your answer will be based on the correctness of your answer as well as the clarity of the main steps in your reasoning. All proposed solutions must be proved. **All the exercises are independent.** *The points are indicated so you may adapt your effort.*

Reminders

Let G be a (weighted) graph and $r \in V(G)$. We recall that a **shortest-path tree T rooted in r** is a spanning tree of G such that for any $v \in V(G)$, the (weighted) distance between r and v in G is equal to the length of the unique path from r to v in T.

1 Algorithmics on graphs

Exercise 1 (4 points) Apply the Dijkstra Algorithm to the graph depicted in Figure 1 to compute a shortest-path tree rooted in a and the distance between any vertex and vertex a. The first three steps of the algorithm must be detailed (**at most three or four lines per steps**). Moreover, indicate the order in which vertices are considered during the execution of the algorithm.

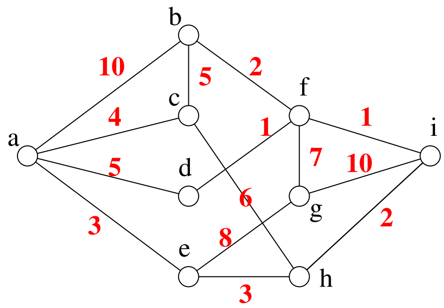

Figure 1: A graph with 9 vertices. A number indicates the length of the arc it is close to.

Exercise 2 (12,5 points) Let $d > 9$.

Let us consider the following graph $G = (V, E)$ with $d + 1$ nodes $V = \{v_0, v_1, \cdots, v_d\}$ such that, for any $1 \leq i \leq d$, there is an edge with weight/length d from v_0 to v_i, and, for any $1 \leq i < d$, there is an edge with weight/length i between v_i and v_{i+1}. Such a graph is depicted in Figure 2.

1. (2 points) Appy the Kruskal Algorithm on G. Explain how you proceed and what is the result that you obtain.
2. (1.5 points) Give d different minimum spanning trees of G.

   (*hint: consider the choices you had when applying the Kruskal Algorithm*)
3. (3 points) Let T be a spanning tree of G and let us assume that there are $1 \leq i < j \leq d$ such that $\{v_0, v_i\} \in E(T)$, $\{v_0, v_j\} \in E(T)$ and, for any $i < k < j$, $\{v_0, v_k\} \notin E(T)$.

   Show that T is not a minimum spanning tree of G.
4. (2 points) Prove that there are exactly d distinct minimum spanning trees in G.

   (*hint: use 2) and 3)*)

5. (*) (4 points) Show that no spanning tree of G is a shortest-path tree. That is, for any minimum spanning-tree T of G and for any $v \in V(G)$, T is not a shortest-path tree rooted in v.

(*hint: the fact that* $d > 9$ *is important here*).

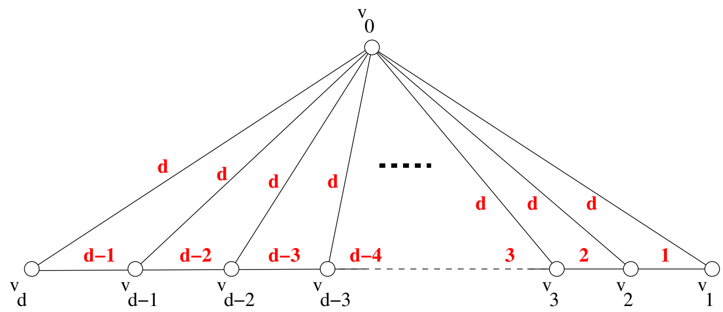

Figure 2: A graph with $d+1$ vertices ($d > 9$). A number indicates the weight/length of the arc it is close to.

2 Linear Programming

Exercise 3 (2.5 points) We consider the following linear programme.

$$\begin{array}{lrcrcrcl}
\text{Minimize} & x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{Subject to:} & & & & & & & \\
 & 2x_1 & - & x_2 & + & x_3 & \leq & 4 \\
 & -4x_1 & + & 3x_2 & & & \geq & 2 \\
 & 3x_1 & - & 2x_2 & - & x_3 & \geq & 5 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}$$

a) Write this linear programme under the standard form.

b) Write the dual of the linear programme.

Exercise 4 (8 points) A food for cows is made of corn, soya, and herb. We want to determine its composition of minimum cost, that is the number of kilograms of corn, soy and herb in one ton (1000 kilograms) of food. The food has to include at most 0.5 % of calcium, at most 5 % of fibers and at least 30 % of proteins to satisfy the clients. In the table below is indicated the percentage of calcium, fibers and proteins contained in corn and soya, as well as their cost per ton. We suppose that the price of herb is 0 and that its content in calcium, fibers and proteins is negligeable.

Raw Product	Pourcentage of calcium	Pourcentage of fibers	Pourcentage of proteins	Price (€)
Corn	0.1 %	2 %	9 %	400
Soya	0.2 %	6 %	60 %	1200
Required percentage	$\leq 0.5\%$	$\leq 5\%$	$\geq 30\%$	

a) Formulate the problem of finding the composition of minimum cost of one ton of the food as a linear program.

b) Solve the problem using the graphical method (*hint: use here a linear program with* two *variables, and determine later the value of the other variables, if any*) and give the optimal composition of the mixing along with its cost.

Exercise 5 (6 points) Let $D(V, A)$ be a digraph. We distinguish two vertices $s, t \in V$. Write a linear program finding two edge-disjoint paths between s and t, if such paths exist (two edge-disjoint paths do not share any edge).