Master 2
Graph Algorithms and Combinatorial Optimization
First Exam, October 2022
1 hour 30 minutes
No documents nor electronic devices allowed.

Instruction and comments: the points awarded for your answer will be based on the correctness of your answer as well as the clarity of the main steps in your reasoning. All proposed solutions must be proved. **All the exercises are independent.** *The points are indicated so you may adapt your effort.*

Exercise 1 (Standard form and duality (2 points, 5 minutes)) We consider the following linear program.

$$\begin{array}{llcccccc} \text{Minimize} & 3x_1 & + & 4x_2 & - & 2x_3 & & \\ \text{Subject to:} & & & & & & & \\ & 4x_1 & + & 2x_2 & - & x_3 & \geq & 3 \\ & 3x_1 & - & 5x_2 & & & \leq & 7 \\ & -2x_1 & + & 3x_2 & + & 2x_3 & = & 4 \\ & & & & & x_1, x_3 & \geq & 0 \end{array}$$

a) Write an equivalent linear program, P, under the standard form.

Exercise 2 (Project planning with precedences. (5 points, 20 minutes)) A project consists of the following 7 activities, whose duration in days are given in brackets: A (4), B (3), C (5), D (2), E (10), F (10), G (1). The first activity is A and the last one is G. There are precedence constraints for the tasks. A $\rightarrow$ B means that task B cannot start before task A has been finished. Below is the list of all constraints: A $\rightarrow$ B,C,D,E,F,G (A is the first task); A,B,C,D,E,F $\rightarrow$ G (G is the last task). E $\rightarrow$ F; D $\rightarrow$ C; F $\rightarrow$ C; F $\rightarrow$ B.

Each day of work costs 1000 euros; furthermore a special machinery must be rented from the beginning of activity A to the end of activity B at a daily cost of 5000 euros.

a) Formulate the problem of minimizing the cost of the project as a linear program.

b) Suggest an algorithm for solving such a problem (for any list of precedence constraints).

Exercise 3 (Solving a linear program (5 points, 20 minutes))

$$\begin{array}{llcccc} \text{Maximize} & x_1 & + & 6x_2 & & \\ \text{Subject to:} & & & & & \\ & x_1 & + & x_2 & \leq & 5 \\ & -\frac{1}{2}x_1 & + & x_2 & \leq & 2 \\ & 3x_1 & + & x_2 & \leq & 3 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

a) Solve the linear program using the graphical method.

b) The program can also be solved using the simplex method. Carry out one full step of the simplex method. Explain carefully with sentences.

c) We now consider a slightly different linear program in which the first constraint was changed.

$$\begin{array}{llcccc} \text{Maximize} & x_1 & + & 6x_2 & & \\ \text{Subject to:} & & & & & \\ & x_1 & - & x_2 & \leq & -3 \\ & -\frac{1}{2}x_1 & + & x_2 & \leq & 2 \\ & 3x_1 & + & x_2 & \leq & 3 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

What is the additional difficulty compared to question b)? Explain what should be done to solve this new linear program. Start the resolution by writing a feasible dictionary. Stop there and explain the next steps with sentences.

Exercise 4 (Optimality certificates (5 points, 20 minutes)) We consider the following linear program.

$$
\begin{array}{lrcrcrcrcr}
\text{Minimize} & 8x_1 & + & 6x_2 & + & 10x_3 & + & 2x_4 & & \\
\text{Subject to:} & & & & & & & & & \\
& 2x_1 & + & 1x_2 & + & 3x_3 & + & 1x_4 & \geq & 6 \\
& 6x_1 & + & 4x_2 & + & 8x_3 & + & 2x_4 & \geq & 3 \\
& 10x_1 & - & 4x_2 & + & 8x_3 & + & 4x_4 & \geq & 2 \\
& -4x_1 & + & 2x_2 & - & 4x_3 & - & 2x_4 & \geq & -3 \\
& & & & & & & x_1, x_2, x_3, x_4, x_5 & \geq & 0.
\end{array}
$$

a) Write the dual of this linear program.

b) Is the solution $y_1^* = 4, y_2^* = 0, y_3^* = 0, y_4^* = 1$ optimal for the dual? To answer, use the method to provide optimality certificates seen during the class. Detail all steps.

Exercise 5 (Column Generation. (8 points, 30 minutes)) We consider the following instance of the cutting stock problem. Suppose for the paper company, a big roll of paper is W = 105 cm. The customers of the company want:

- 24 rolls of length 45 cm
- 21 small rolls of length 60 cm
- 2 small rolls of length 50 cm

The company wants to use a minimum number of big rolls to supply all the small rolls demanded by its customer. The goal of the exercise is to solve the problem using column generation.

a) Explain what is the column generation method. (max 10 lines + figures).

b) We recall the linear program formulation of the cutting stock problem used for the CG formulation.

$$
\begin{array}{lll}
\min & \sum_{p \in P} \lambda_p & \\
\text{s.t.} & \sum_{p \in P} a_i^p \lambda_p \geq b_i & i = 1, 2, ..., m \\
& \lambda_p \geq 0 \text{ and integer} & p \in P.
\end{array}
$$

b.1) Explain what are the variables of the problem, the objective function, and the constraints. Each notation should be explained.

b.2) How many variables and constraints are there in this program? Comment.

c) Explain what are the master problem and the restricted master problem.

d) We will use $a^1 = (1, 0, 0)^T, a^2 = (0, 1, 0)^T, a^3 = (0, 0, 1)^T$ as the initial basis. Write the restricted master problem for the specific instance of the problem considered here and for this basis.

e) Give the solution of the restricted master problem with this basis.

f) We recall the pricing problem.

$$
\begin{array}{lll}
\max & \sum_{i=1}^{m} u_i a_i & \\
\text{s.t.} & \sum_{i=1}^{m} w_i a_i \leq W & \\
& a_i \geq 0 & i = 1, 2, ..., m \\
& a_i \in \mathbb{Z}^+ & i = 1, 2, ..., m.
\end{array}
$$

and that $u = c_B \, B^{-1}$.

Explain what are the u_i, c_B and B.

g) Write the pricing problem corresponding to the first step of the CG process.

f) Solve the first pricing problem. Explain carefully with sentences.

g) Write the new restricted master problem and solve it.

h) Finish the resolution of the problem. Stop when the fractional problem is solved. Explain what are the following steps to solve exactly the integral problem.