Master 2
Graph Algorithms and Combinatorial Optimization
First Exam, October 2020
1 hour
No documents nor electronic devices allowed.

Instruction and comments: the points awarded for your answer will be based on the correctness of your answer as well as the clarity of the main steps in your reasoning. All proposed solutions must be proved. **All the exercises are independent.** *The points are indicated so you may adapt your effort.*

Exercise 1 (Standard form and duality (3 points, 10 minutes)) We consider the following linear program.

$$\begin{array}{lrcrcrcl}
\text{Minimize} & x_1 & - & 2x_2 & + & 4x_3 & & \\
\text{Subject to:} & & & & & & & \\
& 3x_1 & + & x_2 & + & x_3 & \geq & 4 \\
& -5x_1 & + & 3x_2 & & & \leq & 2 \\
& 2x_1 & - & x_2 & - & 3x_3 & = & 5 \\
& & & & & x_1, x_2 & \geq & 0
\end{array}$$

a) Write an equivalent linear program, P, under the standard form.

b) Then, write the dual of the linear program P.

Exercise 2 (Modeling and Graphical Method (6 points, 20 minutes)) A food for cows is made of corn, soya, and herb. We want to determine its composition of minimum cost, that is the number of kilograms of corn, soy and herb in one ton (1000 kilograms) of food. The food has to include at most 0.5 % of calcium, at most 5 % of fibers and at least 30 % of proteins to satisfy the clients. In the table below is indicated the percentage of calcium, fibers and proteins contained in corn and soya, as well as their cost per ton. We suppose that the price of herb is 0 and that its content in calcium, fibers and proteins is negligeable.

Raw Product	Pourcentage of calcium	Pourcentage of fibers	Pourcentage of proteins	Price of a ton (€)
Corn	0.1 %	2 %	9 %	400
Soya	0.2 %	6 %	60 %	1200
Required percentage	$\leq 0.5\%$	$\leq 5\%$	$\geq 30\%$	

a) Formulate the problem of finding the composition of minimum cost of one ton of the food as a linear program.

b) Solve the problem using the graphical method (*hint: use here a linear program with* two *variables, and determine later the value of the other variables, if any*) and give the optimal composition of the mixing along with its cost.

Exercise 3 (Optimality certificates (5 points, 15 minutes)) Is $x_1 = \frac{5}{2}$, $x_2 = 3/2$, $x_3 = 0$ an optimal solution of the linear program below? To answer, use the method to provide optimality certificates seen during the class. Detail all steps.

$$\begin{array}{lrcrcrcl}
\text{Maximize} & 3x_1 & + & 3x_2 & + & 4x_3 & & \\
\text{Subject to:} & & & & & & & \\
& x_1 & + & x_2 & + & 2x_3 & \leq & 4 \\
& 2x_1 & & & + & 3x_3 & \leq & 5 \\
& 2x_1 & + & x_2 & + & 3x_3 & \leq & 7 \\
& & & & & x_1, x_2, x_3 & \geq & 0
\end{array}$$

Exercise 4 (Simplex. (5 points, 15 minutes)) Solve the following linear program using the simplex method.

$$
\begin{array}{lrcrcl}
\text{Maximize} & x_1 & & & & \\
\text{Subject to:} & & & & & \\
& x_1 & - & x_2 & \leq & 1 \\
& 2x_1 & - & x_2 & \leq & 2 \\
& x_1 & + & x_2 & \leq & 7 \\
& & & x_1, x_2 & \geq & 0
\end{array}
$$