Master 2 Graph Algorithms and Combinatorial Optimization

First Exam, October 2019

1 hour

No documents are allowed. No computers, cellphones.

Instruction and comments: the points awarded for your answer will be based on the correctness of your answer as well as the clarity of the main steps in your reasoning. All proposed solutions must be proved. **All the exercises are independent.** *The points are indicated so you may adapt your effort.*

1 Linear Programming

Exercise 1 (Modeling. (5 points, 15 minutes)) A person concerned about her physical fitness wants to absorb *at least* 36 units of Vitamin A, 28 units of Vitamin C and 32 units of Vitamin D every day. Two brands are likely to provide these contributions. The brand 1 costs 3 euros and provides 2 units of Vitamin A, 2 units of Vitamin C and 8 units of Vitamin D. The brand 2 costs 4 euros and provides 3 units of Vitamin A, 2 units of Vitamin C and 2 units of Vitamin D. The person wants to find the combination of both brands that meets the requirements of daily absorption at the lowest cost.

a) Formulate the problem of as a linear program.

b) Solve the problem using the graphical method.

Exercise 2 (Duality and Optimality certificates. (5 points, 15 minutes)) *We consider the following linear program (P).*

$$
\begin{array}{lrcrcrcrcl}
\textit{Maximize} & x_1 & + & x_2 & + & x_3 & + & x_4 & & \\
\textit{Subject to:} & & & & & & & & & \\
& 3x_1 & + & 6x_2 & + & 3x_3 & + & x_4 & \leq & 2 \\
& 2x_1 & + & 3x_2 & + & 3x_3 & + & x_4 & \leq & 4 \\
& x_1 & + & 2x_2 & + & 5x_3 & + & 2x_4 & \leq & 3 \\
& 4x_1 & + & 3x_2 & + & 4x_3 & + & x_4 & \leq & 5 \\
& & & & & & & x_1, x_2, x_3, x_4 & \geq & 0.
\end{array}
$$

1) Write the dual program of (P).

2) Is the solution $x_1^* = 1/5, x_2^* = 0, x_3^* = 0, x_4^* = 7/5$ optimal for (P)? Use the method presented during the course and explain every step of the resolution.

Exercise 3 (Simplex. (5 points, 15 minutes)) *Solve the following linear program using the simplex method.*

$$
\begin{array}{lrcrcl}
\textit{Maximise} & x_1 & + & 2x_2 & & \\
\textit{Subject to:} & & & & & \\
& x_1 & + & 3x_2 & \leq & 21 \\
& -x_1 & + & 3x_2 & \leq & 18 \\
& x_1 & - & x_2 & \leq & 5 \\
& & & x_1, x_2 & \geq & 0
\end{array}
$$

Exercise 4 (Modeling (5 points, 15 minutes) We consider a first scenario in which n cellular phones would like to connect to an antenna. There are m antennas and each antenna can only serve cellular phones at a distance at most 1km. Additionally, **an antenna can serve only a single phone.** The network operator has to choose which antenna will serve which phone in order to maximize the number of connected cellular phones.

1) What is the classic graph problem modeling this telecommunication problem? Explain.
2) Model the problem with a linear program given the graph obtained at the first question.

We now consider a second scenario in which **an antenna can now serve a maximum of 10 cellular phones.** Moreover, **any antenna can serve any cellular phone**. However, the throughput that the phone p ($p \in 1, ..., n$) can obtain from the antenna a ($a \in 1, ..., m$) depends on their respective position and is given by the constant $t_{p,a}$.

3) Write a linear program maximizing the sum of the throughput of the cellular phones. Explain each constraint.