Master 2 Graph Algorithms and Combinatorial Optimization
Final Exam, November 2021
3 hours
No documents are allowed. No computers, cellphones.

*Instruction and comments: the points awarded for your answer will be based on the correctness of your answer as well as the clarity of the main steps in your reasoning. All proposed solutions must be proved. The points are indicated so you may adapt your effort (***there are 23 points in total, the mark will be on 20, i.e., you may have 23/20***). The total estimated time is 2h55 minutes, meaning that you will have 5 minutes to be sure that you have indicated your name on every sheet of paper!** *Also, you must give us back* **two separated groups of papers: one for the problem, the second one for the exercises 1, 2, and 3 on linear programming!** *(we may remove points if you do not follow the instructions.)*

1 Problem: Approximation algorithm for minimum Hamiltonian cycle (12 points)

1.1 Introduction

The goal of this problem is to design and analyze an approximation algorithm for computing a minimum Hamiltonian cycle in weighted graphs.

Question 1 (1 point) *Let Π be any minimization problem. What are the properties that must be satisfied by a 2-approximation algorithm that solves Π.*

Given a graph $G = (V, E)$, a *trail* in G is any sequence $C = (v_0, v_1, \cdots, v_r)$ such that $\{v_i, v_{i+1}\} \in E$ for every $0 \leq i < r$ and $\{v_0, v_r\} \in E$. That is, a trail is a sequence of vertices such that two consecutive vertices are adjacent and the first and last vertices are also adjacent. Note that a trail may contain a same edge (and so a same vertex) several times.

Let $V(C) \subseteq V$ be the set of vertices that appears at least once in C, i.e., $V(C) = V \cap \bigcup_{0 \leq i \leq r} \{v_i\}$. A trail is *spanning* if $V(C) = V$.

A *cycle* in G is a trail with no repeated vertex, i.e., it is a trail where every vertex of V appears at most once. An *Hamiltonian cycle* is a spanning cycle, i.e., a trail that contains every vertex of V exactly once. A graph G is *Hamiltonian* if it admits an Hamiltonian cycle.

Question 2 (0.5 point) *For each of the two graphs depicted in Figure 1, say whether they are Hamiltonian or not. Your answers must be proved.*

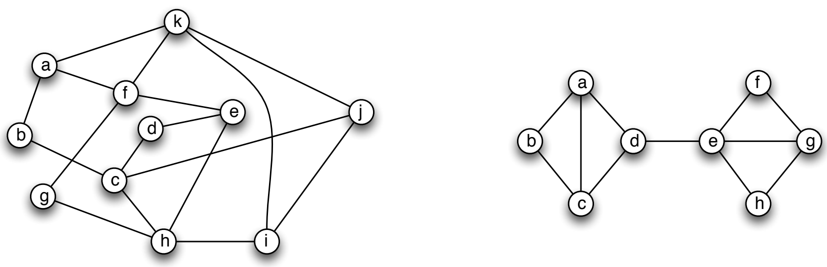

Figure 1: Two graphs: G (left) and H (right).

So, all graphs are not Hamiltonian. To avoid this difficulty, in what follows, we will only consider one particular class of graphs, the complete graphs. The graph K_n is the *complete* graph on n vertices whose vertex-set is $V(K_n) = \{w_1, \cdots, w_n\}$ and such that every two vertices are adjacent, i.e., $E(K_n) = \{\{w_i, w_j\} \mid 1 \leq i < j \leq n\}$.

Question 3 (0.5 **point)** *Draw the graphs K_3, K_4 and K_5.*

Question 4 (0.5 **point)** *Prove that, for every $n \geq 3$, K_n is Hamiltonian.*

A *weighted* graph (G, w) consists of a graph $G = (V, E)$ together with a weight function $w : E \to \mathbb{R}^+$ that assigns to each edge a non-negative real. The weight $w(C)$ of a trail $C = (v_0, \cdots, v_r)$ in the weighted graph (G, w) is the sum of the weights of its edges, that is, $w(C) = w(v_0, v_r) + \sum_{0 \leq i < r} w(v_i, v_{i+1})$.

We are interested in the following problem. Given the graph K_n, $n \geq 3$, together with a weight function $w : E(K_n) \to \mathbb{R}^+$, what is the minimum weight of an Hamiltonian cycle in (K_n, w)?

Unfortunately, this problem is NP-complete and does not even admit any approximation algorithm (unless $P = NP$). Therefore, we need to do a last simplification.

A weight function $w : E(K_n) \to \mathbb{R}^+$ satisfies the *triangular inequality* if and only if, for every three vertices $a, b, c \in V(K_n)$, $w(a, b) \leq w(a, c) + w(c, b)$.

Question 5 (0.5 **point)** *For each of the two weighted complete graphs depicted in Figure 2, say whether their weight function satisfies the triangular inequality or not. Your answers must be proved.*

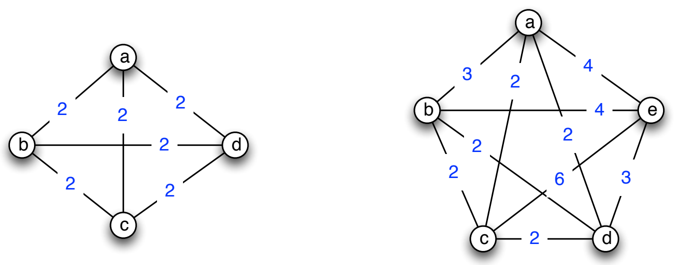

Figure 2: Two weighted complete graphs: K_4 (left) and K_5 (right). The weight of an edge is depicted by the blue integer on this edge.

Question 6 (1.5 **points)** *Let (K_n, w) be a weighted complete graph with w satisfying the triangular inequality. Let $C = (v_0, v_1, \cdots, v_r)$ be a trail in (K_n, w) such that there exist $0 \leq i < j \leq r$ with $v_i = v_j$. Show that*

- *$C' = (v_0, \cdots, v_{j-1}, v_{j+1}, \cdots, v_r)$ is a trail in (K_n, w);*
- *$V(C) = V(C')$;*
- *and $w(C') \leq w(C)$.*

Question 7 (2 points) *Prove that, if there exists a spanning trail C in (K_n, w) with w satisfying the triangular inequality, then (K_n, w) admits an Hamiltonian cycle of weight at most $w(C)$.*

1.2 Mimimum Hamiltonian cycle in K_n with triangular inequality

From now on, we study the following problem (minimum Hamiltonian cycle Problem): Given the graph K_n, $n \geq 3$, together with a weight function $w : E \to \mathbb{R}^+$ that satisfies the triangular inequality, what is the minimum weight, denoted by $OPT(K_n, w)$, of an Hamiltonian cycle in (K_n, w)?

Recall that, given a connected graph $G = (V, E)$, a *spanning tree* of G is a subgraph $T = (V, E')$ (i.e., $E' \subseteq E$) which is a tree (connected and acyclic). The weight of a spanning tree $T = (V, E')$ in a weighted connected graph (G, w) equals $w(T) = \sum_{e \in E'} w(e)$.

Question 8 (1 point) *Describe an algorithm that, given a weighted connected graph (G, w), computes a spanning tree with minimum weight. Give the time-complexity of your algorithm as a function of $|V(G)|$.*

Given a weighted connected graph (G, w), let $w^*(G)$ denote the minimum weight of a spanning tree in G. From now on, let us assume that a spanning tree of G with weight $w^*(G)$ can be computed in time polynomial in $|V(G)|$.

Question 9 (1 point) *Let (G, w) be any weighted connected n-node graph that admits an Hamiltonian cycle $C = (v_0, \cdots, v_{n-1})$. Show that $w^*(G) \leq w(C)$.*

Question 10 (1.5 points) *Let (G, w) be any weighted connected graph with a spanning tree T. Show that G admits a spanning trail of weight at most $2 \cdot w(T)$.*

Question 11 (2 points) *From previous questions, design a 2-approximation algorithm for the minimum Hamiltonian cycle problem in complete weighted graphs (K_n, w) with w satisfying the triangular inequality. You must prove that the algorithm you propose is actually a 2-approximation algorithm.*

Master 2 Graph Algorithms and Combinatorial Optimization
Final Exam, November 2021
3 hours
No documents are allowed. No computers, cellphones.

Instruction and comments: the points awarded for your answer will be based on the correctness of your answer as well as the clarity of the main steps in your reasoning. All proposed solutions must be proved. **All the exercises are independent.** *The points are indicated so you may adapt your effort (***there are 23 points in total, the mark will be on 20, i.e., you may have 23/20***).* **The total estimated time is 2h55 minutes, meaning that you will have 5 minutes to be sure that you have indicated your name on every sheet of paper!** *Also, you must give us back* **two separated groups of papers: one for the problem, the second one for the exercises 1, 2, and 3 on linear programming!** *(we may remove points if you do not follow the instructions.)*

1 Linear Programming

Exercise 1 (Standard form and duality (2 points)) We consider the following optimization program.

$$\begin{array}{llrcrcrcl}
\text{Minimize} & & 2x_1 & + & 3x_2 & - & 5x_3 & & \\
\text{Subject to:} & & & & & & & & \\
& & |2x_1 & + & 3x_2 & - & x_3| & \leq & 1 \\
& & 4x_1 & - & 2x_2 & & & = & 3 \\
& & -3x_1 & + & 4x_2 & - & 1x_3 & \geq & 6 \\
& & & & & & x_1, x_3 & \geq & 0
\end{array}$$

a) Write an equivalent linear program, P, under the standard form.

b) Then, write the dual of the linear program P.

Exercise 2 (Modeling, Graphical Method, and Simplex (6 points)) A farmer has to choose the crop for 200 hectares of his fields for the coming year. He can grow maize and rapeseed. The maize yields more than the rapeseed (he expects to earn 600 euros per hectare for the maize against 500 euros per hectare for the rapeseed). But he is committed to respecting various environmental constraints:
- He must limit his phosphate inputs. In concrete terms, this means that he cannot use more than 30 tons of fertilizer in all.
- He must limit his water consumption. He must not draw more than 200,000m3 of water for the watering of his crops.
For a normal year :
- he must draw 2,000 m3 of water and use 100 kg of fertilizer per hectare of corn.
- In a normal year: he must draw 1,000 m3 of water and use 250 kg of fertilizer per hectare of rapeseed.

How should he divide his crops to expect the maximum gain? What is this gain?

a) Formulate the problem as a linear program.

b) Solve the problem using the graphical method.

c) Solve the problem using the simplex (stop after two steps of simplex even if you do not get the optimal solution).

d) Does this solution require him to use the 200 hectares of land at his disposal? Which variable directly gives the answer to this question?

Exercise 3 (Optimality certificates (3 points)) Is $x_1 = 2$, $x_2 = 0$, $x_3 = 1$ an optimal solution of the linear program below? To answer, use the method to provide optimality certificates seen during the class. Detail all steps.

$$
\begin{array}{llllllll}
\text{Maximize} & 5x_1 & + & 4x_2 & + & 3x_3 & & \\
\text{Subject to:} & & & & & & & \\
 & 2x_1 & + & 3x_2 & + & x_3 & \leq & 5 \\
 & 4x_1 & + & x_2 & + & 2x_3 & \leq & 11 \\
 & 3x_1 & + & 4x_2 & + & 2x_3 & \leq & 8 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$