UBINET, Master 2 IFI Algorithms for telecommunications

Final Exam, November 2014

3 hours

Course and manuscript notes NOT allowed. No computers, cellphones, books.

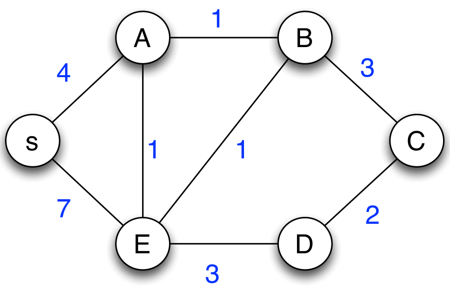
Instruction and comments: the points awarded for your answer will be based on the correctness of your answer as well as the clarity of the main steps in your reasoning. All proposed solutions must be proved. **All the exercises are independent.** *The points are indicated so you may adapt your effort.*

1 Algorithmics on graphs

Definition 1 *Given a connected graph $G = (V, E)$, recall that a* spanning tree *of G is any subgraph of G that is a tree and that spans (i.e., touches) all vertices of G. Note that a graph G admits a spanning tree if and only if G is connected.*

Exercise 1 (4 points) Explain in few sentences (**at most five lines**) how the Kruskal's algorithm proceeds to compute a minimum spanning tree in a graph.

Apply the Kruskal's algorithm on the following graph.



Exercise 2 (8 points) In this exercise, let $G = (V, E)$ be a connected graph and $w : E \to \mathbb{R}$ be a weight function on the edges. Let n be the number of vertices of G.

1. Let T be a spanning tree of G. Give the number of edges of T.
2. Assume that all edges have the same weight. That is, there is $c \in \mathbb{R}$ such that, for any edge $e \in E$, $w(e) = c$. Let T be a minimum spanning tree of G. Give the weight of T.

An *articulation point* in G is any vertex $v \in V$ such that removing v from G disconnects the graph. A connected graph is said 2-*connected* if has no articulation point.

From now on, we assume that G is 2-connected.

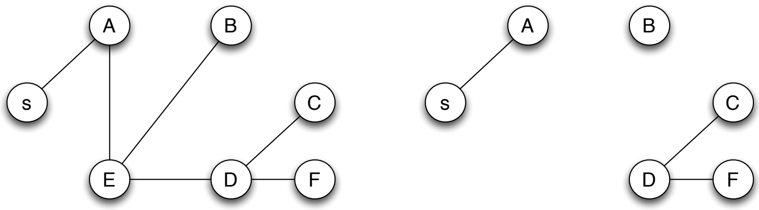
3. Let $v \in V$ and let $G' = G - v$ be the graph obtained from G by removing v. Show that G' admits a spanning tree.
4. Let T' be a spanning tree of G'. Let $\{u, v\} \in E$ be an edge of G incident to v. Show that the graph obtained from T' and adding the vertex v and the edge $\{u, v\}$ is a spanning tree of G.
5. Show that, if all spanning trees of G have same weight, then all edges incident to v have the same weight.
6. Deduce that, if all the spanning trees of G have same weight, then all edges of G have the same weight.
7. Show that, in a 2-connected graph, all the spanning trees have minimum weight if and only if all the edges have the same weight.
8. Give an example of a connected edge-weighted graph for which all the spanning tree have the same weight but whose edges do not all have the same weight.

Definition 2 *Two edges in a graph are* independent *if they don't share a vertex. Recall that a* matching *in a graph G is a subset of edges of G that are pairwise independent. A matching is* perfect *if it spans (touches) all vertices of the graph.*

Exercise 3 (8 points) The goal of this exercise is to characterize the trees (acyclic connected graphs) that have a perfect matching.

1. Show that, if a graph admits a perfect matching, then it has an even number of vertices.
2. Let T be a tree that admits a perfect matching M. Let $x \in V(T)$ be a leaf of T (i.e., x has degree 1) and $e = \{u, v\} \in E(T)$ be the edge of T incident to T. Show that $e \in M$.
3. Give an example of a tree with an even number of vertices and that does not admit a perfect matching.

Let T be a tree and $v \in V(T)$ be a vertex of T. Let $T - v$ denote the subgraph obtained from T by removing v. Let $imp(T - v)$ denote the number of connected components of $T - v$ that have an odd number of vertices. For instance, in Figure below, $imp(T - E) = 2$ since the graph $T - E$ has 2 connected components with an odd number of vertices.



Tree T

Connected components of T - E

3. Let T be a tree and $v \in V(T)$. Show that T admits a perfect matching only if $imp(T - v) > 0$.

   Hint: assume that $imp(T - v) = 0$ and use Question 1.
4. Let T be a tree and $v \in V(T)$ and let C be a connected component of $T - v$ with an odd number of vertices.
   - Show that v has a unique neighbor in C. Let w be this neighbor.
   - (*) Show that any perfect matching of T must contain the edge $\{v, w\}$.
5. Deduce that, if a tree T has a perfect matching, then for any $v \in V(T)$, $imp(T - v) = 1$.

Let T be a tree such that, for any $v \in V(T)$, $imp(T - v) = 1$.

6. Let $x \in V(T)$ be a vertex of T. Let y be a leaf of T that maximizes the distance to x. Let z be the neighbor of y. Show that either w has degree 2, or T has a unique edge.
7. From Questions 1, 2 and 6, prove that, if $imp(T - v) = 1$ for any $v \in V(T)$, then T admits a perfect matching.

   Hint: by induction on the number of vertices of T.
8. Conclusion?

2 Linear Programming

Exercise 4 (Modeling as a linear program (4 points)) *A factory produces two models of machines, Model A and Model B. A machine of Model A requires 2 KG of wood and 30 hours of work and gives a profit of 7 euros. A machine of Model B requires 4 KG of wood, 15 hours of work and gives a profit of 6 euros. The factory has at its disposal 200 Kg of wood and 1200 hours of work. Additionally, a market study tells it is impossible to sell more than 30 machines of Model A and 60 machines of Model B. What should the factory produce to obtain a maximum profit?*

1. *Express the problem as a linear program.*

2. *Solve the linear program using the graphical method. Give the optimum production and the maximum profit.*

Exercise 5 (Duality and Optimality certificates. (4 points)) *Consider the following linear program.*

$$\begin{array}{llllllll} \textit{Maximize} & 6x_1 & + & 6x_2 & + & 8x_3 & & \\ \textit{Subject to:} & & & & & & & \\ & 2x_1 & + & 2x_2 & + & 4x_3 & \leq & 8 \\ & 4x_1 & & & + & 6x_3 & \leq & 10 \\ & 4x_1 & + & 2x_2 & + & 6x_3 & \leq & 14 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

1. *Write the dual program of this linear program.*

2. *Is $x_1 = 5$, $x_2 = 3$, $x_3 = 0$ an optimal solution of the linear programme below? To answer, use the method to provide optimality certificates seen during the class. Detail all steps.*

Exercise 6 (5 points) Let $D(V, A)$ be a digraph. We distinguish four vertices $s_1, s_2, t_1, t_2 \in V$. Write a linear program finding two node-disjoint paths, the first one between s_1 and t_1 and the second one between s_2 and t_2, if such paths exist (two node-disjoint paths do not share any node). *Explain all the lines of your linear program.*

3 Design of Satellite Networks

Exercise 7 (10 points) We consider a generalization of the problem of design of on-board satellite that we studied during the class (with switches of degree 4).

Because of the satellite rotation, all the ports (inputs) and amplifiers (outputs) are not always well oriented to receive and send signals. Thus, we define a (p, λ, k)-network as a network with $p + \lambda$ inputs and $p + k$ outputs. A (p, λ, k)-network is valid, if, for any choice of p inputs and p outputs, there exist p edge-disjoint paths linking all the chosen inputs to the chosen outputs.

By symmetry, we will suppose that $\lambda \leq k$.

Proposition (Cut Criterion) Consider a (p, λ, k)-network and $W \subseteq V$ a subset of vertices. The excess in inputs of W is defined as

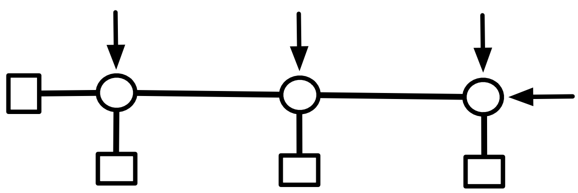
$$\varepsilon_i(W) := \delta(W) + o(W) - \min(k, o(W)) - \min(i(W), p).$$

A (p, λ, k)-network is valid if and only if, for any subset of vertices $W \subseteq V$, the excess of W satisfies

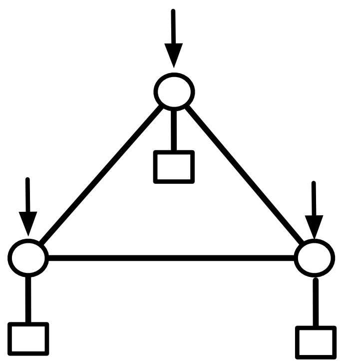
$$\varepsilon_i(W) \geq 0.$$

1. Explain the intuition of this cut criterium with 2 or 3 sentences.

2. Consider the $(3, 1, 1)$-network below 4 inputs, 4 outputs (amplifiers). Is it a valid $(3, 1, 1)$-network? Prove your answer.



3. Consider the $(2, 1, 1)$-network below with 3 inputs, 3 outputs (amplifiers). Is it a valid $(2, 1, 1)$-network? Prove your answer.



4. Prove that in a valid (p, λ, k)-network, with $\lambda \geq 1$ and with $k \geq 1$, there is no switch connected to 1 input and 2 outputs. inputs.

5. Propose a valid $(4, 2, 2)$-network with 6 switches. Show it is minimum.

6. For any $p \geq 1$, propose a valid $(p, 2, 2)$-network with minimum number of switches. Show it is minimum.

7. Propose a valid $(12, 4, 4)$-network.

8. **(Hard question)** Propose a valid $(12, 4, 4)$-network with 15 switches.

9. The excess in outputs of W can defined as

$$\varepsilon_o(W) := \delta(W) + i(W) - \min(\lambda, i(W)) - \min(o(W), p).$$

Let $(V, E), i, o$ be a (p, λ, k)-network. Consider a subset $W \subseteq V$. We note $\bar{W}$ the complementary set of W, $\bar{W} = V \setminus W$.

(a) Show that we have

$$\varepsilon_o(\bar{W}) = \varepsilon_i(W).$$

(b) **(Hard question)** Comment (2-3 sentences).

10. **(Hard question)** Propose a valid $(24, 6, 6)$-network with 42 switches.