ABSOLUTE STABILITY AND COMPLETE SYNCHRONIZATION IN A CLASS OF NEURAL FIELDS MODELS

OLIVIER FAUGERAS *, FRANÇOIS GRIMBERT †, AND JEAN-JACQUES SLOTINE ‡

Abstract. Neural fields are an interesting option for modelling macroscopic parts of the cortex involving several populations of neurons, like cortical areas. Two classes of neural field equations are considered: voltage and activity based. The spatio-temporal behaviour of these fields is described by nonlinear integro-differential equations. The integral term, computed over a compact subset of $\mathbb{R}^q$, $q = 1, 2, 3$, involves space and time varying, possibly non-symmetric, intra-cortical connectivity kernels. Contributions from white matter afferents are represented as external input. Sigmoidal nonlinearities arise from the relation between average membrane potentials and instantaneous firing rates. Using methods of functional analysis, we characterize the existence and uniqueness of a solution of these equations for general, homogeneous (i.e. independent of the spatial variable), and spatially locally homogeneous inputs. In all cases we give sufficient conditions on the connectivity functions for the solutions to be absolutely stable, that is to say asymptotically independent of the initial state of the field. These conditions bear on some compact operators defined from the connectivity kernels, the maximal slope of the sigmoids, and the time constants used in describing the temporal shape of the post-synaptic potentials. Numerical experiments are presented to illustrate the theory. To our knowledge this is the first time that such a complete analysis of the problem of the existence and uniqueness of a solution of these equations has been obtained. Another important contribution is the analysis of the absolute stability of these solutions, more difficult but more general than the linear stability analysis which it implies. The reason why we have been able to complete this work programme is our use of the functional analysis framework and the theory of compact operators in a Hilbert space which has allowed us to provide simple mathematical answers to some of the questions raised by modellers in neuroscience.

Key words. neural fields, integro-differential equations, compact operators, Hilbert space, absolute stability, complete synchronization, Lyapunov function, neural masses, cortical columns.

AMS subject classifications. 34G20, 34L30, 47B15, 47G10, 47G20, 47J05, 82C32, 92B20, 92C20

1. Introduction.

We model neural fields as continuous networks of cortical units, and investigate the ability of these units to completely synchronize, i.e. to produce the same output when receiving the same input independently of their initial state. We therefore emphasize the dynamics and the spatio-temporal behaviour of these networks.

Cortical units are built from a local description of the dynamics of a number of interacting neuron populations, called *neural masses* [14], where the spatial structure of the connections is neglected. These "vertically" built units can be thought of as *cortical columns* [28, 29, 3]. Probably the most well-known neural mass based column model is that of Jansen and Rit [20] based on the original work of Lopes Da Silva, Van Rotterdam and colleagues [24, 25, 37]. A complete analysis of the bifurcations of this model can be found in [16]. More realistic models can be derived from experimental connectivity studies, such as the one shown in figure 1.1. This figure, adapted from [18], is based on the work of Alex Thomson and colleagues [36]. It shows the local connectivity graph of six populations of neurons and can be thought of as a model of a column comprising six interacting neural masses.

Such columns are then assembled spatially to form the neural field, which is meant to represent a macroscopic part of the neocortex, e.g. a visual area such as V1. Con-

*INRIA/ENPC/ENS, Odyssée Team, 2004 route des Lucioles, Sophia-Antipolis, France
†INRIA/ENPC/ENS, Odyssée Team, 2004 route des Lucioles, Sophia-Antipolis, France
‡MIT, Nonlinear Systems Laboratory, 77 Massachusetts Avenue, Cambridge, MA, USA

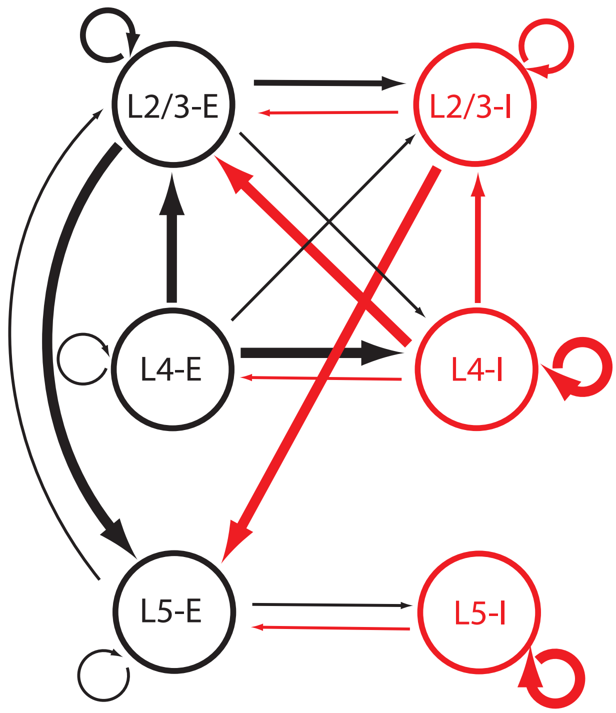


Figure 1.1. *A simplified model of local cortical interactions based on six neuron populations. This local connectivity graph can be seen as a model of a cortical column composed of six interacting neural masses. There are three layers corresponding to cortical layers II/III, IV and V, and two types of neurons (excitatory ones in black and inhibitory ones in red) in each of these layers. The size of the arrows gives an idea of the average strength of the postsynaptic potentials elicited by the presynaptic neurons, see section 2.1.1. This figure is adapted from [18].*

nections between columns are intra-cortical (gray matter) connections. Connections made via white matter with, e.g., such visual areas as the LGN or V2 are also considered in our models, but are treated as input/output quantities.

There are at least three reasons why we think this is the relevant granularity to do modelling

- Realistic modelling of a macroscopic part of the brain at the scale of the neuron is still difficult for obvious complexity reasons. Starting from mesoscopic building blocks like neural masses, described by the average activity of their neurons, is therefore a reasonable choice.
- While MEG and scalp EEG recordings mostly give a bulk signal of a cortical area, multi-electrode recordings, in vitro experiments on pharmacologically treated brain slices and new imaging techniques like extrinsic optical imaging can provide a spatially detailed description of neural masses dynamics in a macroscopic part of the brain like an area.
- The column/area scales correspond to available local connectivity data. Indeed, these are obtained by averaging over local populations of neurons we can think of as neural masses. Besides, local (intracolumnar) connectivity is supposed to be spatially invariant within an area.

We now present a general mathematical framework for neural field modelling that agrees with the ideas of using average descriptions of neuronal activity and spatial invariance of the local connectivity across the field. This framework uses the elegant tools of functional analysis with the advantage of providing simple characterizations of some important properties of neural field equations.

In section 2 we describe the local and spatial models of neural masses and derive the equations that govern their spatio-temporal variations. In section 3 we analyze the problem of the existence and uniqueness of the smooth general and homogeneous solutions of these equations. In section 4 we study the absolute stability of these solutions, i.e. their robustness to arbitrary perturbations caused by changes of the initial conditions. In section 5 we extend this analysis to the absolute stability of the homogeneous, i.e. independent of space, solutions when they exist. A consequence of the absolute stability is the ability of the network to completely synchronize. In section 6 we revisit the functional framework of our analysis and extend our results to non-smooth functions with the effect that we can discuss the existence and absolute stability of locally homogeneous solutions. We also propose another extension of the model by generalizing the previous results to higher order synaptic responses. In section 7 we present a number of numerical experiments to illustrate the theory and conclude in section 8.

2. The models.

We discuss local and spatial models.

2.1. The local models.

We consider n interacting populations of neurons such as those shown in figure 1.1. The following derivation is built after Ermentrout's review [10]. We consider that each neural population i is described by its average membrane potential $V_i(t)$ or by its average instantaneous firing rate $\nu_i(t)$, the relation between the two quantities being of the form $\nu_i(t) = S_i(V_i(t))$ [15, 8], where S_i is sigmoidal. The functions S_i, $i = 1, \cdots, n$ satisfy the following properties introduced in the

DEFINITION 2.1. *For all $i = 1, \cdots, n$, S_i and S'_i are positive and bounded (S'_i is the derivative of the function S_i). We note $S_{im} = \sup_x S_i(x)$, $S_m = \max_i S_{im}$, $S'_{im} = \sup_x S'_i(x)$ and $DS_m = \max_i S'_{im}$. Finally, we define $D\mathbf{S}_m$ as the diagonal matrix $diag(S'_{im})$.*

Neurons in population j are connected to neurons in population i. A single action potential from neurons in population j is seen as a post-synaptic potential $PSP_{ij}(t-s)$ by neurons in population i, where s is the time of the spike hitting the synapse and t the time after the spike. We neglect the delays due to the distance travelled down the axon by the spikes.

Assuming that the post-synaptic potentials sum linearly, the average membrane potential of population i is

$$V_i(t) = \sum_{j,k} PSP_{ij}(t - t_k)$$

where the sum is taken over the arrival times of the spikes produced by the neurons in population j. The number of spikes arriving between t and $t+dt$ is $\nu_j(t)dt$. Therefore we have

$$V_i(t) = \sum_j \int_{t_0}^{t} PSP_{ij}(t - s)\nu_j(s)\, ds = \sum_j \int_{t_0}^{t} PSP_{ij}(t - s)S_j(V_j(s))\, ds,$$

or, equivalently

$$\nu_i(t) = S_i\left(\sum_j \int_{t_0}^{t} PSP_{ij}(t-s)\nu_j(s)\,ds\right) \tag{2.1}$$

The PSP_{ij}s can depend on several variables in order to account for adaptation, learning, etc ...
There are two main simplifying assumptions that appear in the literature [10] and yield two different models.

2.1.1. The voltage-based model. The assumption, made in [19], is that the post-synaptic potential has the same shape no matter which presynaptic population caused it, the sign and amplitude may vary though. This leads to the relation

$$PSP_{ij}(t) = W_{ij}PSP_i(t).$$

PSP_i represents the unweighted shape of the postsynaptic potentials and W_{ij} is the average strength of the postsynaptic potentials elicited by neurons of type j on neurons of type i. In biophysical connectivity models, like the one presented in figure 1.1, the W_{ij}s should be chosen proportional to the number of presynaptic cells, the average amplitude of postsynaptic potentials and the probability of connection between the considered neuron species [17]. In particular, if $W_{ij} > 0$ the population j excites population i whereas it inhibits it when $W_{ij} < 0$.

Finally, if we assume that $PSP_i(t) = e^{-t/\tau_i}Y(t)$ (where Y is the Heaviside distribution), or equivalently that

$$\tau_i \frac{dPSP_i(t)}{dt} + PSP_i(t) = \tau_i\delta(t), \tag{2.2}$$

we end up with the following system of ordinary first order differential equations

$$\frac{dV_i(t)}{dt} + \frac{V_i(t)}{\tau_i} = \sum_j W_{ij}S_j(V_j(t)) + I^i_{\text{ext}}(t), \tag{2.3}$$

that describes the dynamic behaviour of a cortical column. We have added an external current $I_{\text{ext}}(t)$ to model the non-local connections of population i.

The approach developed in this article generalizes easily to the case of more sophisticated postsynaptic potentials models resulting in higher order differential equations, as shown in section 6.3.

We introduce the $n \times n$ matrix $\mathbf{W} = (W_{ij})_{i,j}$, and the function $\mathbf{S}$: $\mathbb{R}^n \to \mathbb{R}^n$ such that $\mathbf{S}(\mathbf{x})$ is the vector of coordinates $S_i(x_i)$. We rewrite (2.3) in vector form and obtain the following system of n ordinary differential equations

$$\mathbf{V}' = -\mathbf{L}\mathbf{V} + \mathbf{W}\mathbf{S}(\mathbf{V}) + \mathbf{I}_{\text{ext}}, \tag{2.4}$$

where $\mathbf{L}$ is the diagonal matrix $\mathbf{L} = \text{diag}(1/\tau_{\text{i}})$.

2.1.2. The activity-based model. The assumption is that the shape of a PSP depends only on the nature of the presynaptic cell, that is

$$PSP_{ij}(t) = W_{ij}PSP_j(t).$$

As above we suppose that $PSP_i(t)$ satisfies the differential equation (2.2) and define the activity to be

$$A_j(t) = \int_{t_0}^{t} PSP_j(t-s)\nu_j(s)\,ds.$$

A similar derivation yields the following set of n ordinary differential equations

$$\frac{dA_i(t)}{dt} + \frac{A_i(t)}{\tau_i} = S_i\left(\sum_j W_{ij}A_j(t) + I^i_{\mathrm{ext}}(t)\right), \quad i = 1, \cdots, n.$$

We rewrite this in vector form as

$$\mathbf{A}' = -\mathbf{L}\mathbf{A} + \mathbf{S}(\mathbf{W}\mathbf{A} + \mathbf{I}_{\mathrm{ext}}), \tag{2.5}$$

We introduce the following

DEFINITION 2.2. *We note $\tau_{\max}$ the maximum of the decay time constants τ_i, $i = 1, \cdots, n$:*

$$\tau_{\max} = \max_i \tau_i.$$

2.2. Neural fields models. We now combine these local models to form a continuum of columns, e.g., in the case of a model of a significant part Ω of the cortex. From now on we consider a compact subset Ω of $\mathbb{R}^q$, $q = 1, 2, 3$. This encompasses several cases of interest.

When $q = 1$ we deal with one-dimensional neural fields. Even though this appears to be of limited biological interest, it is one of the most widely studied cases because of its relative mathematical simplicity and because of the insights one can gain of the more realistic situations.

When $q = 2$ we discuss properties of two-dimensional neural fields. This is perhaps more interesting from a biological point of view since Ω can be viewed as a piece of cortex where the third dimension, its thickness, is neglected. This case has received by far less attention than the previous one, probably because of the increased mathematical difficulty.

Finally $q = 3$ allows us to discuss properties of volumes of neural masses, e.g. cortical sheets where their thickness is taken into account [21, 4].

The results that are presented in this paper are independent of q. Nevertheless, we have a good first approximation of a real cortical area with $q = 2$, and cortical depth given by the index $i = 1, \cdots, n$ of the considered cortical population, following the idea of a field composed of columns, or equivalently, of interconnected cortical layers.

We note $\mathbf{V}(\mathbf{r}, t)$ (respectively $\mathbf{A}(\mathbf{r}, t)$) the n-dimensional state vector at the point $\mathbf{r}$ of the continuum and at time t. We introduce the $n \times n$ matrix function $\mathbf{W}(\mathbf{r}, \mathbf{r}', t)$ which describes how the neural mass at point $\mathbf{r}'$ influences that at point $\mathbf{r}$ at time t. More precisely, $W_{ij}(\mathbf{r}, \mathbf{r}', t)$ describes how population j at point $\mathbf{r}'$ influences population i at point $\mathbf{r}$ at time t. We call $\mathbf{W}$ the connectivity matrix function. Neglecting, as in the local case above, the delays due to the distance between the neural masses, we extend equation (2.4) to

$$\mathbf{V}_t(\mathbf{r},t) = -\mathbf{L}\mathbf{V}(\mathbf{r},t) + \int_\Omega \mathbf{W}(\mathbf{r},\mathbf{r}',t)\mathbf{S}(\mathbf{V}(\mathbf{r}',t))\,d\mathbf{r}' + \mathbf{I}_{\rm ext}(\mathbf{r},t), \tag{2.6}$$

and equation (2.5) to

$$\mathbf{A}_t(\mathbf{r},t) = -\mathbf{L}\mathbf{A}(\mathbf{r},t) + \mathbf{S}\left(\int_\Omega \mathbf{W}(\mathbf{r},\mathbf{r}',t)\mathbf{A}(\mathbf{r}',t))\,d\mathbf{r}' + \mathbf{I}_{\rm ext}(\mathbf{r},t)\right). \tag{2.7}$$

$\mathbf{V}_t$ (resp. $\mathbf{A}_t$) stands for the partial derivative of the multivariate vector $\mathbf{V}$ (resp. $\mathbf{A}$) with respect to the time variable t. A special case which will be considered later is when $\mathbf{W}$ is translation invariant, $\mathbf{W}(\mathbf{r},\mathbf{r}',t) = \mathbf{W}(\mathbf{r}-\mathbf{r}',t)$. We give below sufficient conditions on $\mathbf{W}$ and $\mathbf{I}_{\rm ext}$ for equations (2.6) and (2.7) to be well-defined and study their solutions.

3. Existence and uniqueness of a solution. In this section we deal with the problem of the existence and uniqueness of a solution to (2.6) and (2.7) for a given set of initial conditions. Unlike previous authors [12, 5, 27] we consider the case of a neural field with the effect that we have to use the tools of functional analysis to characterize their properties.

We start with the assumption that the state vectors $\mathbf{V}$ and $\mathbf{A}$ are differentiable (respectively continuous) functions of the time (respectively the space) variable. This is certainly reasonable in terms of the temporal variations because we are essentially modeling large populations of neurons and do not expect to be able to represent time transients. It is far less reasonable in terms of the spatial dependency since one should allow neural masses activity to be spatially distributed in a locally non-smooth fashion with areas of homogeneous cortical activity separated by smooth boundaries. A more general assumption is proposed in section 6. But it turns out that most of the groundwork can be done in the setting of continuous functions.

Let $\mathcal{F}$ be the set $\mathbf{C}_n(\Omega)$ of the continuous functions from Ω to $\mathbb{R}^n$. This is a Banach space for the norm $\|\mathbf{V}\|_{n,\infty} = \max_{1\le i\le n}\sup_{\mathbf{r}\in\Omega}|\mathbf{V}_i(\mathbf{r})|$, see appendix A.1. We denote by J a closed interval of the real line containing 0.

We will several times need the following

LEMMA 3.1. *We have the following inequalities for all* $\mathbf{x}$, $\mathbf{y}\in\mathcal{F}$ *and* $\mathbf{r}'\in\Omega$

$$\|\mathbf{S}(\mathbf{x}(\mathbf{r}'))-\mathbf{S}(\mathbf{y}(\mathbf{r}'))\|_\infty \le DS_m\|\mathbf{x}(\mathbf{r}')-\mathbf{y}(\mathbf{r}')\|_\infty \quad \textit{and} \quad \|\mathbf{S}(\mathbf{x})-\mathbf{S}(\mathbf{y})\|_{n,\infty} \le DS_m\|\mathbf{x}-\mathbf{y}\|_{n,\infty}.$$

Proof. $\mathbf{S}$ is smooth so we can perform a zeroth-order Taylor expansion with integral remainder, [9], and write

$$\mathbf{S}(\mathbf{x}(\mathbf{r}')) - \mathbf{S}(\mathbf{y}(\mathbf{r}')) = \left(\int_0^1 D\mathbf{S}(\mathbf{y}(\mathbf{r}') + \zeta(\mathbf{x}(\mathbf{r}') - \mathbf{y}(\mathbf{r}')))\,d\zeta\right)(\mathbf{x}(\mathbf{r}') - \mathbf{y}(\mathbf{r}')),$$

and, because of lemma A.1 and definition 2.1

$$\|\mathbf{S}(\mathbf{x}(\mathbf{r}'))-\mathbf{S}(\mathbf{y}(\mathbf{r}'))\|_\infty \le \int_0^1 \|D\mathbf{S}(\mathbf{y}(\mathbf{r}')+\zeta(\mathbf{x}(\mathbf{r}')-\mathbf{y}(\mathbf{r}')))\|_\infty\,d\zeta\,\|\mathbf{x}(\mathbf{r}')-\mathbf{y}(\mathbf{r}')\|_\infty \le DS_m\|\mathbf{x}(\mathbf{r}') - \mathbf{y}(\mathbf{r}')\|_\infty.$$

This proves the first inequality. The second follows immediately. ∎

3.1. General solution. A function $\mathbf{V}(t)$ is thought of as a mapping $\mathbf{V} : \mathrm{J} \to \mathcal{F}$. This means that $\mathbf{V}(t)$ is now a function defined in Ω. Equations (2.6) and (2.7) are formally recast as an initial value problem, see, e.g. [11]:

$$\begin{cases} \mathbf{V}'(t) &= f(t, \mathbf{V}(t)) \\ \mathbf{V}(0) &= \mathbf{V}_0 \end{cases} \tag{3.1}$$

where $\mathbf{V}_0$ is an element of $\mathcal{F}$ and the function f from $\mathrm{J} \times \mathcal{F}$ is equal to f_v defined by the righthand side of (2.6):

$$f_v(t, \mathbf{x})(\mathbf{r}) = -\mathbf{L}\mathbf{x}(\mathbf{r}) + \int_\Omega \mathbf{W}(\mathbf{r}, \mathbf{r}', t)\mathbf{S}(\mathbf{x}(\mathbf{r}'))\, d\mathbf{r}' + \mathbf{I}_{\text{ext}}(\mathbf{r}, t) \quad \forall \mathbf{x} \in \mathcal{F}, \tag{3.2}$$

or to f_a defined by the righthand side of (2.7):

$$f_a(t, \mathbf{x})(\mathbf{r}) = -\mathbf{L}\mathbf{x}(\mathbf{r}) + \mathbf{S}\left(\int_\Omega \mathbf{W}(\mathbf{r}, \mathbf{r}', t)\mathbf{x}(\mathbf{r}')\, d\mathbf{r}' + \mathbf{I}_{\text{ext}}(\mathbf{r}, t) \right) \quad \forall \mathbf{x} \in \mathcal{F}. \tag{3.3}$$

We have the

PROPOSITION 3.2. *If the following two hypotheses are satisfied*

1. *The connectivity function* $\mathbf{W}$ *is in* $C(\mathrm{J}; \mathbf{C}_{\mathrm{n\times n}}(\Omega \times \Omega))$ *(see Appendix A.2),*
2. *The external current* $\mathbf{I}_{\text{ext}}$ *is in* $C(\mathrm{J}; \mathbf{C}_{\mathrm{n}}(\Omega))$,

then the mappings f_v *and* f_a *are from* $\mathrm{J}\times\mathcal{F}$ *to* $\mathcal{F}$*, continuous, and Lipschitz continuous with respect to their second argument, uniformly with respect to the first (*$\mathbf{C}_{n\times n}(\Omega\times\Omega)$ *and* $\mathbf{C}_n(\Omega)$ *are defined in Appendix A.1).*

Proof. Let $t \in \mathrm{J}$ and $\mathbf{x} \in \mathcal{F}$. We introduce the mapping

$$F_v : (t, \mathbf{x}) \to F_v(t, \mathbf{x}) \quad \text{such that} \quad F_v(t, \mathbf{x})(\mathbf{r}) = \int_\Omega \mathbf{W}(\mathbf{r}, \mathbf{r}', t)\mathbf{S}(\mathbf{x}(\mathbf{r}'))\, d\mathbf{r}' \tag{3.4}$$

$F_v(t, \mathbf{x})$ is well defined for all $\mathbf{r} \in \Omega$ because, thanks to the first hypothesis, it is the integral of the continuous function $\mathbf{W}(\mathbf{r}, ., t)\mathbf{S}(\mathbf{x}(.))$ on a compact domain. For all $\mathbf{r}' \in \Omega$, $\mathbf{W}(., \mathbf{r}', t)\mathbf{S}(\mathbf{x}(\mathbf{r}'))$ is continuous (first hypothesis again) and we have (lemma A.1)

$$\|\mathbf{W}(\mathbf{r}, \mathbf{r}', t)\mathbf{S}(\mathbf{x}(\mathbf{r}'))\|_\infty \le \|\mathbf{W}(., ., t)\|_{n\times n,\infty} \|\mathbf{S}(\mathbf{x}(\mathbf{r}'))\|_\infty.$$

Since $\|\mathbf{S}(\mathbf{x}(.))\|_\infty$ is bounded, it is integrable in Ω and we conclude that $F_v(t, \mathbf{x})$ is continuous on Ω. Then it is easy to see that $f_v(t, \mathbf{x})$ is well defined and belongs to $\mathcal{F}$.

Let us prove that f_v is continuous.

$$\begin{aligned} f_v(t, \mathbf{x}) - f_v(s, \mathbf{y}) = -\mathbf{L}(\mathbf{x} - \mathbf{y}) &+ \int_\Omega \left(\mathbf{W}(\cdot, \mathbf{r}', t)\mathbf{S}(\mathbf{x}(\mathbf{r}')) - \mathbf{W}(\cdot, \mathbf{r}', s)\mathbf{S}(\mathbf{y}(\mathbf{r}')) \right) d\mathbf{r}' \\ &+ \mathbf{I}_{\text{ext}}(\cdot, t) - \mathbf{I}_{\text{ext}}(\cdot, s) \\ = -\mathbf{L}(\mathbf{x} - \mathbf{y}) &+ \int_\Omega (\mathbf{W}(\cdot, \mathbf{r}', t) - \mathbf{W}(\cdot, \mathbf{r}', s))\mathbf{S}(\mathbf{x}(\mathbf{r}'))\, d\mathbf{r}' \\ &+ \int_\Omega \mathbf{W}(\cdot, \mathbf{r}', s)(\mathbf{S}(\mathbf{x}(\mathbf{r}')) - \mathbf{S}(\mathbf{y}(\mathbf{r}'))\, d\mathbf{r}' + \mathbf{I}_{\text{ext}}(\cdot, t) - \mathbf{I}_{\text{ext}}(\cdot, s) \end{aligned}$$

It follows from lemma 3.1 that

$$\begin{aligned} \|f_v(t, \mathbf{x}) - f_v(s, \mathbf{y})\|_{n,\infty} \le \|\mathbf{L}\|_\infty \|\mathbf{x} - \mathbf{y}\|_{n,\infty} + |\Omega|\, S_m \|\mathbf{W}(\cdot, \cdot, t) - \mathbf{W}(\cdot, \cdot, s)\|_{n\times n,\infty} + \\ |\Omega|\, \|\mathbf{W}(\cdot, \cdot, s)\|_{n\times n,\infty} DS_m \|\mathbf{x} - \mathbf{y}\|_{n,\infty} + \|\mathbf{I}_{\text{ext}}(\cdot, t) - \mathbf{I}_{\text{ext}}(\cdot, s)\|_{n,\infty}. \end{aligned}$$

Because of the hypotheses we can choose $|t-s|$ small enough so that $\|\mathbf{W}(\cdot,\cdot,t)-\mathbf{W}(\cdot,\cdot,s)\|_{n\times n,\infty}$ and $\|\mathbf{I}_{\text{ext}}(\cdot,t)-\mathbf{I}_{\text{ext}}(\cdot,s)\|_{n,\infty}$ are arbitrarily small. Similarly, since $\mathbf{W}$ is continuous on the compact interval J, it is bounded there and $\|\mathbf{W}(\cdot,\cdot,s)\|_{n\times n,\infty} \leq w > 0$ for all $s \in$ J. This proves the continuity of f_v.

It follows from the previous inequality that

$$\|f_v(t,\mathbf{x}) - f_v(t,\mathbf{y})\|_{n,\infty} \leq \|\mathbf{L}\|_\infty \|\mathbf{x}-\mathbf{y}\|_{n,\infty} + |\Omega| \, \|\mathbf{W}(\cdot,\cdot,t)\|_{n\times n,\infty} DS_m \|\mathbf{x}-\mathbf{y}\|_{n,\infty},$$

and because $\|\mathbf{W}(\cdot,\cdot,t)\|_{n\times n,\infty} \leq w > 0$ for all ts in J, this proves the Lipschitz continuity of f_v with respect to its second argument, uniformly with respect to the first.

A very similar proof applies to f_a. ☐

We continue with the proof that there exists a unique solution to the abstract initial value problem (3.1) in the two cases of interest.

Proposition 3.3. *Subject to the hypotheses of proposition 3.2 for any element* $\mathbf{V}_0$ *(resp.* $\mathbf{A}_0$*) of* $\mathcal{F}$ *there is a unique solution* $\mathbf{V}$ *(resp.* $\mathbf{A}$*), defined on a subinterval of* J *containing 0 and continuously differentiable, of the abstract initial value problem (3.1) for* $f = f_v$ *(resp.* $f = f_a$*).*

Proof. All conditions of the Picard-Lindelöf theorem on differential equations in Banach spaces [9, 2] are satisfied, hence the proposition. ☐

This solution, defined on the subinterval J of $\mathbb{R}$ can in fact be extended to the whole real line and we have the

Proposition 3.4. *If the following two hypotheses are satisfied*

1. *The connectivity function* $\mathbf{W}$ *is in* $C(\mathbb{R}; \mathbf{C}_{\mathrm{n\times n}}(\Omega\times\Omega))$,
2. *The external current* $\mathbf{I}_{\text{ext}}$ *is in* $C(\mathbb{R}; \mathbf{C}_{\mathrm{n}}(\Omega))$,

then for any function $\mathbf{V}_0$ *(resp.* $\mathbf{A}_0$*) in* $\mathcal{F}$ *there is a unique solution* $\mathbf{V}$ *(resp.* $\mathbf{A}$*), defined on* $\mathbb{R}$ *and continuously differentiable, of the abstract initial value problem (3.1) for* $f = f_v$ *(resp.* $f = f_a$*).*

Proof. In theorem B.1 of appendix B, we prove the existence of a constant $\tau > 0$ such that for any initial condition $(t_0, \mathbf{V}_0) \in \mathbb{R}\times\mathcal{F}$, there is a unique solution defined on the closed interval $[t_0-\tau, t_0+\tau]$. We can then cover the real line with such intervals and finally obtain the global existence and uniqueness of the solution of the initial value problem. ☐

3.2. Homogeneous solution. A homogeneous solution to (2.6) or (2.7) is a solution $\mathbf{U}$ that does not depend upon the space variable $\mathbf{r}$, for a given homogeneous input $\mathbf{I}_{\text{ext}}(t)$ and a constant initial condition $\mathbf{U}_0$. If such a solution $\mathbf{U}(t)$ exists, then it satisfies the following equation

$$\mathbf{U}'(t) = -\mathbf{L}\mathbf{U}(t) + \int_\Omega \mathbf{W}(\mathbf{r},\mathbf{r}',t)\mathbf{S}(\mathbf{U}(t))\, d\mathbf{r}' + \mathbf{I}_{\text{ext}}(t),$$

in the case of (2.6) and

$$\mathbf{U}'(t) = -\mathbf{L}\mathbf{U}(t) + \mathbf{S}\left(\int_\Omega \mathbf{W}(\mathbf{r},\mathbf{r}',t)\mathbf{U}(t)\, d\mathbf{r}' + \mathbf{I}_{\text{ext}}(t)\right),$$

in the case of (2.7). The integral $\int_\Omega \mathbf{W}(\mathbf{r},\mathbf{r}',t)\mathbf{S}(\mathbf{U}(t))\, d\mathbf{r}'$ is equal to $\left(\int_\Omega \mathbf{W}(\mathbf{r},\mathbf{r}',t)\, d\mathbf{r}'\right) \mathbf{S}(\mathbf{U}(t))$. The integral $\int_\Omega \mathbf{W}(\mathbf{r},\mathbf{r}',t)\mathbf{U}(t)\, d\mathbf{r}'$ is equal to $\left(\int_\Omega \mathbf{W}(\mathbf{r},\mathbf{r}',t)\, d\mathbf{r}'\right) \mathbf{U}(t)$. They must be independent of the position $\mathbf{r}$. Hence a necessary condition for the existence of a homogeneous solution is that

$$\int_\Omega \mathbf{W}(\mathbf{r},\mathbf{r}',t)\, d\mathbf{r}' = \overline{\mathbf{W}}(t), \tag{3.5}$$

where the $n \times n$ matrix $\overline{\mathbf{W}}(t)$ does not depend on the spatial coordinate $\mathbf{r}$.

In the special case where $\mathbf{W}(\mathbf{r}, \mathbf{r}', t)$ is translation invariant, $\mathbf{W}(\mathbf{r}, \mathbf{r}', t) \equiv \mathbf{W}(\mathbf{r} - \mathbf{r}', t)$, the condition is not satisfied in general because of the border of Ω. In all cases, the homogeneous solutions satisfy the differential equation

$$\mathbf{U}'(t) = -\mathbf{L}\mathbf{U}(t) + \overline{\mathbf{W}}(t)\mathbf{S}(\mathbf{U}(t)) + \mathbf{I}_{\text{ext}}(t), \tag{3.6}$$

for (2.6) and

$$\mathbf{U}'(t) = -\mathbf{L}\mathbf{U}(t) + \mathbf{S}\left(\overline{\mathbf{W}}(t)\mathbf{U}(t)) + \mathbf{I}_{\text{ext}}(t)\right), \tag{3.7}$$

for (2.7), with initial condition $\mathbf{U}(0) = \mathbf{U}_0$, a vector of $\mathbb{R}^n$. The following proposition gives a sufficient condition for the existence of a homogeneous solution.

THEOREM 3.5. *If the external current* $\mathbf{I}_{\text{ext}}(t)$ *and the connectivity matrix* $\overline{\mathbf{W}}(t)$ *are continuous on some closed interval* J *containing 0, then for all vector* $\mathbf{U}_0$ *of* $\mathbb{R}^n$*, there exists a unique solution* $\mathbf{U}(t)$ *of (3.6) or (3.7) defined on a subinterval* J_0 *of* J *containing 0 such that* $\mathbf{U}(0) = \mathbf{U}_0$.

Proof. The proof is an application of Cauchy's theorem on differential equations. Consider the mapping $f_{hv} : \mathbb{R}^n \times \mathrm{J} \to \mathbb{R}^\mathrm{n}$ defined by

$$f_{hv}(\mathbf{x}, t) = -\mathbf{L}\mathbf{x} + \overline{\mathbf{W}}(t)\mathbf{S}(\mathbf{x}) + \mathbf{I}_{\text{ext}}(t)$$

We have

$$\|f_{hv}(\mathbf{x}, t) - f_{hv}(\mathbf{y}, t)\|_\infty \leq \|\mathbf{L}\|_\infty \|\mathbf{x} - \mathbf{y}\|_\infty + \|\overline{\mathbf{W}}(t)\|_\infty \|\mathbf{S}(\mathbf{x}) - \mathbf{S}(\mathbf{y})\|_\infty$$

It follows from lemma 3.1 that $\|\mathbf{S}(\mathbf{x}) - \mathbf{S}(\mathbf{y})\|_\infty \leq DS_m \|\mathbf{x} - \mathbf{y}\|_\infty$ and, since $\overline{\mathbf{W}}$ is continuous on the compact interval J, it is bounded there by $w > 0$ and

$$\|f_{hv}(\mathbf{x}, t) - f_{hv}(\mathbf{y}, t)\|_\infty \leq (\|\mathbf{L}\|_\infty + wDS_m)\|\mathbf{x} - \mathbf{y}\|_\infty$$

for all $\mathbf{x}$, $\mathbf{y}$ of $\mathbb{R}^n$ and all $t \in \mathrm{J}$. A similar proof applies to (3.7) and the conclusion of the proposition follows. □

As in proposition 3.4, this existence and uniqueness result extends to the whole time real line if $\mathbf{I}$ and $\overline{\mathbf{W}}$ are continuous on $\mathbb{R}$.

This homogeneous solution can be seen as describing a state where the columns of the continuum are synchronized: they receive the same input $\mathbf{I}_{\text{ext}}(t)$ and produce the same output $\mathbf{U}(t)$.

3.3. Some remarks about the case $\Omega = \mathbb{R}^q$. A significant amount of work has been done on equations of the type (2.6) or (2.7) in the case of a one-dimensional infinite continuum, $\Omega = \mathbb{R}$, or a two-dimensional infinite continuum, $\Omega = \mathbb{R}^2$. The reader is referred to the review papers by Ermentrout [10] and by Coombes [6] as well as to [32, 13, 34].

Beside the fact that an infinite cortex is unrealistic, the case $\Omega = \mathbb{R}^q$ raises some mathematical questions. Indeed, the choice of the functional space $\mathcal{F}$ is problematic. A natural idea would be to choose $\mathcal{F} = \mathbf{L}^2_n(\mathbb{R}^q)$, the space of square-integrable functions with values in $\mathbb{R}^n$, see Appendix A.1. If we make this choice we immediately encounter the problem that the homogeneous solutions (constant with respect to the space variable) do not belong to that space. A further difficulty is that $\mathbf{S}(\mathbf{x})$ does not in general belong to $\mathcal{F}$ if $\mathbf{x}$ does. As shown in this article, these difficulties vanish if Ω is compact.

4. Absolute stability of the general solution. We investigate the absolute stability of a solution to (2.6) and (2.7) for a given input $\mathbf{I}_{\text{ext}}$. Proposition 3.4 guarantees that for a given initial condition there exists a unique solution to (2.6) or (2.7) defined for all times.

In order to investigate its absolute stability we choose a different initial condition, which is a way to perturb the solution, in effect the only way because of the existence uniqueness proposition 3.4, and look for sufficient conditions for the new solution to converge toward the original one. Absolute stability implies linear stability which is studied by perturbing the solution by adding to it a small function, and performing a first-order Taylor expansion of the equations thereby obtaining a perturbed equation. One then usually has to make some assumptions about the spatio-temporal form of the perturbation, e.g. that it is separable in time and space, ending up with a non-trivial eigenvalue problem which has to be solved in order to find sufficient conditions for the perturbation to converge to 0, up to first-order [6, 10, 12, 13, 27, 32, 33, 23, 34]. This is also the case of [1] and [7] who study the convolution case for $n = q = 1$ but incorporate propagation delays. Linear stability is local because it is derived for a particular solution. The functional analysis approach that we use in this paper allows us to find simple sufficient conditions for the absolute stability of the system, hence for all its solutions, regardless of the initial condition or input. In this sense it is a global approach. This is achieved by constructing a Lyapunov function measuring some distance between two state vectors at each time instant. This function has a single minimum corresponding to the equality of the states. One then finds sufficient conditions for the time derivative of this function to be strictly negative thereby guaranteeing the asymptotic equality of the states. This approach has been followed by much fewer people. In [22] the authors study the case where $\mathbf{W}(\mathbf{r}, \mathbf{r}')$ is symmetric in with respect to the space variables $\mathbf{r}$ and $\mathbf{r}'$ for $n = q = 1$ for a finite interval and add the translation invariance assumption when the interval is infinite. They do not study the case of general time-varying input currents.

Absolute stability is a relevant concept for systems of neurons. Indeed, absolutely stable systems forget their initial state exponentially fast, but do not forget their input. Hence such systems can differentiate distinct stimuli by converging to the corresponding states without being influenced by their initial state. This property is desirable for example in modelling visual perception: different forms elicit different percepts but the percepts should not depend on the initial state of the visual system. We first look at the general case then at the convolution case.

4.1. The general case. We define a number of matrices and linear operators that are useful in the sequel

DEFINITION 4.1. *Let*

$$\mathbf{W}_{cm} = \mathbf{W} D \mathbf{S}_m \quad \mathbf{W}_{mc} = D \mathbf{S}_m \mathbf{W}$$

Consider also the linear operators, noted g, g_m, and h_m defined on $\mathcal{F}$:

$$g(\mathbf{x})(\mathbf{r}, t) = \int_\Omega \mathbf{W}(\mathbf{r}, \mathbf{r}', t)\mathbf{x}(\mathbf{r}')\, d\mathbf{r}' \quad \forall \mathbf{x} \in \mathcal{F},$$

$$g_m(\mathbf{x})(\mathbf{r}, t) = \int_\Omega \mathbf{W}_{cm}(\mathbf{r}, \mathbf{r}', t)\mathbf{x}(\mathbf{r}')\, d\mathbf{r}' \quad \forall \mathbf{x} \in \mathcal{F},$$

and

$$h_m(\mathbf{x})(\mathbf{r}, t) = \int_\Omega \mathbf{W}_{mc}(\mathbf{r}, \mathbf{r}', t)\mathbf{x}(\mathbf{r}')\, d\mathbf{r}' \quad \forall \mathbf{x} \in \mathcal{F}.$$

We start with a lemma.

LEMMA 4.2. *With the hypotheses of proposition 3.2, the operators g, g_m, and h_m are compact operators from $\mathcal{F}$ to $\mathcal{F}$ for each time $t \in$ J.*

Proof. This is a direct application of the theory of Fredholm's integral equations [9]. We prove it for g.

Because of the hypothesis 1 in proposition 3.2, at each time instant t in J, $\mathbf{W}$ is continuous on the compact set $\Omega \times \Omega$, therefore it is uniformly continuous. Hence, for each $\varepsilon > 0$ there exists $\eta(t) > 0$ such that $\|\mathbf{r}_1 - \mathbf{r}_2\| \leq \eta(t)$ implies that $\|\mathbf{W}(\mathbf{r}_1, \mathbf{r}', t) - \mathbf{W}(\mathbf{r}_2, \mathbf{r}', t)\|_\infty \leq \varepsilon$ for all $\mathbf{r}' \in \Omega$, and, for all $\mathbf{x} \in \mathcal{F}$

$$\|g(\mathbf{x})(\mathbf{r}_1, t) - g(\mathbf{x})(\mathbf{r}_2, t)\|_\infty \leq \varepsilon|\Omega|\|\mathbf{x}\|_{n,\infty}$$

This shows that the image $g(B)$ of any bounded subset B of $\mathcal{F}$ is equicontinuous.

Similarly, if we set $w(t) = \|\mathbf{W}(.,.,t)\|_{n\times n,\infty}$, we have $\|g(\mathbf{x})(\mathbf{r}, t)\|_\infty \leq w(t)|\Omega|\|\mathbf{x}\|_{n,\infty}$. This shows that for every $\mathbf{r} \in \Omega$, the set $\{\mathbf{y}(\mathbf{r}), \mathbf{y} \in g(B)\}$, is bounded in $\mathbb{R}^n$, hence relatively compact. From the Arzelà-Ascoli theorem, we conclude that the subset $g(B)$ of $\mathcal{F}$ is relatively compact for all $t \in$ J. And so the operator is compact.

The same proof applies to g_m and h_m. ◻

To study the absolute stability of the solutions of (2.6) and (2.7) it is convenient to use an inner product on $\mathcal{F}$. It turns out that the natural inner-product will pave the ground for the generalization in section 6. We therefore consider the pre-Hilbert space $\mathcal{G}$ defined on $\mathcal{F}$ by the usual inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_\Omega \mathbf{x}(\mathbf{r})^T \mathbf{y}(\mathbf{r})\, d\mathbf{r}$$

We note $\|\mathbf{x}\|_{n,2}$ the corresponding norm to distinguish it from $\|\mathbf{x}\|_{n,\infty}$, see Appendix A.1. It is easy to show that all previously defined operators are also compact operators from $\mathcal{G}$ to $\mathcal{G}$. We have the

LEMMA 4.3. *g, g_m and h_m are compact operators from $\mathcal{G}$ to $\mathcal{G}$ for each time $t \in$ J.*

Proof. We give the proof for g.

The identity mapping $\mathbf{x} \to \mathbf{x}$ from $\mathcal{F}$ to $\mathcal{G}$ is continuous since $\|\mathbf{x}\|_{n,2} \leq \sqrt{n|\Omega|}\, \|\mathbf{x}\|_{n,\infty}$. Consider now g as a mapping from $\mathcal{G}$ to $\mathcal{F}$. As in the proof of lemma 4.2, for each $\varepsilon > 0$ there exists $\eta(t) > 0$ such that $\|\mathbf{r}_1 - \mathbf{r}_2\| \leq \eta(t)$ implies $\|\mathbf{W}(\mathbf{r}_1, \mathbf{r}', t) - \mathbf{W}(\mathbf{r}_2, \mathbf{r}', t)\|_\infty \leq \varepsilon$ for all $\mathbf{r}' \in \Omega$. Therefore the ith coordinate $g^i(\mathbf{x})(\mathbf{r}_1, t) - g^i(\mathbf{x})(\mathbf{r}_2, t)$ satisfies (Cauchy-Schwarz' inequalities):

$$|g^i(\mathbf{x})(\mathbf{r}_1, t) - g^i(\mathbf{x})(\mathbf{r}_2, t)| \leq \sum_j \int_\Omega |W_{ij}(\mathbf{r}_1, \mathbf{r}', t) - W_{ij}(\mathbf{r}_2, \mathbf{r}', t)|\, |x_j(\mathbf{r}')|\, d\mathbf{r}' \leq$$

$$\varepsilon \sum_j \int_\Omega |x_j(\mathbf{r}')|\, d\mathbf{r}' \leq \varepsilon\sqrt{|\Omega|} \sum_j \left(\int_\Omega |x_j(\mathbf{r}')|^2\, d\mathbf{r}' \right)^{1/2} \leq \varepsilon\sqrt{n\,|\Omega|}\|\mathbf{x}\|_{n,2},$$

and the image $g(B)$ of any bounded set B of $\mathcal{G}$ is equicontinuous. Similarly, if we set $w(t) = \|\mathbf{W}(.,.,t)\|_{n\times n,\infty}$ in $\Omega \times \Omega$, we have $|g^i(\mathbf{x})(\mathbf{r}, t)| \leq w(t)\sqrt{n\,|\Omega|}\, \|\mathbf{x}\|_{n,2}$. The same reasoning as in lemma 4.2 shows that the operator $\mathbf{x} \to g(\mathbf{x})$ from $\mathcal{G}$ to $\mathcal{F}$ is compact and since the identity from $\mathcal{F}$ to $\mathcal{G}$ is continuous, $\mathbf{x} \to g(\mathbf{x})$ is compact from $\mathcal{G}$ to $\mathcal{G}$.

The same proof applies to g_m and h_m. ◻

We then proceed with the following

LEMMA 4.4. *The adjoint g^* of the operator g of $\mathcal{G}$ is the operator defined by*

$$g^*(\mathbf{x})(\mathbf{r},t) = \int_\Omega \mathbf{W}^T(\mathbf{r}',\mathbf{r},t)\mathbf{x}(\mathbf{r}')\,d\mathbf{r}'$$

It is a compact operator. Similar results apply to g_m^* and h_m^*.

Proof. The adjoint, if it exists, is defined by the condition $\langle g(\mathbf{x}),\,\mathbf{y}\rangle = \langle \mathbf{x},\,g^*(\mathbf{y})\rangle$ for all $\mathbf{x}$, $\mathbf{y}$ in $\mathcal{G}$. We have

$$\langle g(\mathbf{x}),\,\mathbf{y}\rangle = \int_\Omega \mathbf{y}(\mathbf{r})^T\left(\int_\Omega \mathbf{W}(\mathbf{r},\mathbf{r}',t)\mathbf{x}(\mathbf{r}')\,d\mathbf{r}'\right)d\mathbf{r} = \int_\Omega \mathbf{x}(\mathbf{r}')^T\left(\int_\Omega \mathbf{W}^T(\mathbf{r},\mathbf{r}',t)\mathbf{y}(\mathbf{r})\,d\mathbf{r}\right)d\mathbf{r}',$$

from which the conclusion follows. Since $\mathcal{G}$ is not a Hilbert space the adjoint of a compact operator is not necessarily compact. But the proof of compactness of g in lemma 4.3 extends easily to g^*. ◻

We finally prove two useful lemmas that will complete our toolbox for the proof of the main results of this section.

LEMMA 4.5. *Given a diagonal matrix $\mathbf{D} = diag(d_1,\ldots,d_n)$, with $d_1,\ldots,d_n \in \mathbf{L}^\infty(\Omega)$ and a function $\mathbf{x}\in\mathcal{G}$, we have*

$$\|\mathbf{D}\mathbf{x}\|_{n,2} \le \max_i(\|d_i\|_\infty)\|\mathbf{x}\|_{n,2}.$$

Proof.

$$\|\mathbf{D}\mathbf{x}\|_{n,2}^2 = \int_\Omega \mathbf{x}(\mathbf{r})^T\mathbf{D}^2(\mathbf{r})\mathbf{x}(\mathbf{r})\,d\mathbf{r} = \sum_i\int_\Omega d_i^2(\mathbf{r})\mathbf{x}_i^2(\mathbf{r})\,d\mathbf{r} \le \sum_i \|d_i\|_\infty^2\int_\Omega \mathbf{x}_i^2(\mathbf{r})\,d\mathbf{r},$$

from which the result follows. ◻

LEMMA 4.6. *$\|g\|_\mathcal{G}$, $\|g_m\|_\mathcal{G}$, and $\|h_m\|_\mathcal{G}$ satisfy the following inequalities*

$$\|g_m\|_\mathcal{G} \le DS_m\,\|g\|_\mathcal{G} \quad \textit{and} \quad \|h_m\|_\mathcal{G} \le DS_m\,\|g\|_\mathcal{G}\,,$$

where DS_m is defined in definition 2.1.

Proof. By definition

$$\|g_m\|_\mathcal{G} = \sup_{\|\mathbf{x}\|_{n,2}\le 1}\frac{\|g_m(\mathbf{x})\|_{n,2}}{\|\mathbf{x}\|_{n,2}} = \sup_{\|\mathbf{x}\|_{n,2}\le 1}\frac{\|g(D\mathbf{S}_m\mathbf{x})\|_{n,2}}{\|\mathbf{x}\|_{n,2}}.$$

Let $\mathbf{y} = D\mathbf{S}_m\mathbf{x}$. Since $\{\mathbf{x}\in\mathcal{G}, \|\mathbf{x}\|_{n,2}\le 1\} \subset \{\mathbf{x}\in\mathcal{G}, \|D\mathbf{S}_m\mathbf{x}\|_{n,2}\le DS_m\}$ (lemma 4.5),

$$\|g_m\|_\mathcal{G} \le \sup_{\|\mathbf{y}\|_{n,2}\le DS_m}\frac{\|g(\mathbf{y})\|_{n,2}}{\|D\mathbf{S}_m^{-1}\mathbf{y}\|_{n,2}} = \sup_{\|\mathbf{y}\|_{n,2}\le 1}\frac{\|g(\mathbf{y})\|_{n,2}}{\|D\mathbf{S}_m^{-1}\mathbf{y}\|_{n,2}} \le \sup_{\|\mathbf{y}\|_{n,2}\le 1}\frac{\|g(\mathbf{y})\|_{n,2}}{\|\mathbf{y}\|_{n,2}}\cdot\sup_{\|\mathbf{y}\|_{n,2}\le 1}\frac{\|\mathbf{y}\|_{n,2}}{\|D\mathbf{S}_m^{-1}\mathbf{y}\|_{n,2}} \le \|g\|_\mathcal{G}\,DS_m$$

The last inequality is also obtained from lemma 4.5, which is used again to prove the inequality for h_m: $h_m = D\mathbf{S}_m g$ and $\|D\mathbf{S}_m g(\mathbf{x})\|_{n,2} \le DS_m\|g(\mathbf{x})\|_{n,2}$, for all $\mathbf{x}\in\mathcal{G}$, from which the result follows. ◻

We show in appendix D a table summarizing the main notations introduced so far for future reference.
We now state an important result of this section.

THEOREM 4.7. *A sufficient condition for the absolute stability of a solution to (2.6) is*

$$DS_m \tau_{\max} \|g\|_{\mathcal{G}} < 1 \tag{4.1}$$

where $\|.\|_{\mathcal{G}}$ *is the operator norm.*

Proof. Let us note $\underline{\mathbf{S}}$ the function $D\mathbf{S}_m^{-1}\mathbf{S}$ and rewrite equation (2.6) as follows

$$\mathbf{V}_t(\mathbf{r},t) = -\mathbf{L}\mathbf{V}(\mathbf{r},t) + \int_\Omega \mathbf{W}_{cm}(\mathbf{r},\mathbf{r}',t)\,\underline{\mathbf{S}}(\mathbf{V}(\mathbf{r}',t))\,d\mathbf{r}' + \mathbf{I}_{\text{ext}}(\mathbf{r},t).$$

Let $\mathbf{U}$ be its unique solution with initial conditions $\mathbf{U}(0) = \mathbf{U}_0$, an element of $\mathcal{G}$. Let also $\mathbf{V}$ be the unique solution of the same equation with different initial conditions $\mathbf{V}(0) = \mathbf{V}_0$, another element of $\mathcal{G}$. We introduce the new function $\mathbf{X} = \mathbf{V} - \mathbf{U}$ which satisfies

$$\mathbf{X}_t(\mathbf{r},t) = -\mathbf{L}\mathbf{X}(\mathbf{r},t) + \int_\Omega \mathbf{W}_{cm}(\mathbf{r},\mathbf{r}',t)\,\mathbf{H}(\mathbf{X},\mathbf{U})(\mathbf{r}',t)\,d\mathbf{r}' = \\ -\mathbf{L}\mathbf{X}(\mathbf{r},t) + g_m(\mathbf{H}(\mathbf{X},\mathbf{U}))(\mathbf{r},t) \tag{4.2}$$

where the vector $\mathbf{H}(\mathbf{X},\mathbf{U})$ is given by $\mathbf{H}(\mathbf{X},\mathbf{U})(\mathbf{r},t)) = \underline{\mathbf{S}}(\mathbf{V}(\mathbf{r},t)) - \underline{\mathbf{S}}(\mathbf{U}(\mathbf{r},t)) = \underline{\mathbf{S}}(\mathbf{X}(\mathbf{r},t) + \mathbf{U}(\mathbf{r},t)) - \underline{\mathbf{S}}(\mathbf{U}(\mathbf{r},t))$. Consider now the functional (Lyapunov function)

$$V(\mathbf{X}) = \frac{1}{2}\left\langle \mathbf{X},\, \mathbf{L}^{-1}\mathbf{X} \right\rangle,$$

where the symmetric positive definite matrix $\mathbf{L}$ can be seen as defining a metric on the state space. Its time derivative is $\left\langle \mathbf{X},\, \mathbf{L}^{-1}\mathbf{X}_t \right\rangle$. We replace $\mathbf{X}_t$ by its value from (4.2) in this expression to obtain

$$\frac{dV(\mathbf{X})}{dt} = -\left\langle \mathbf{X},\, \mathbf{X} \right\rangle + \left\langle \mathbf{X},\, \mathbf{L}^{-1} g_m(\mathbf{H}(\mathbf{X},\mathbf{U})) \right\rangle$$

We consider the second term in the righthand side of this equation:

$$\left| \left\langle \mathbf{X},\, \mathbf{L}^{-1} g_m(\mathbf{H}(\mathbf{X},\mathbf{U})) \right\rangle \right| \le \|\mathbf{X}\|_{n,2}\, \|\mathbf{L}^{-1} g_m(\mathbf{H}(\mathbf{X},\mathbf{U}))\|_{n,2} \le \\ \tau_{\max}\|\mathbf{X}\|_{n,2}\, \|g_m(\mathbf{H}(\mathbf{X},\mathbf{U}))\|_{n,2} \le \tau_{\max}\|\mathbf{X}\|_{n,2}\, \|g_m\|_{\mathcal{G}} \|\mathbf{H}(\mathbf{X},\mathbf{U})\|_{n,2} \tag{4.3}$$

Using a zeroth-order Taylor expansion with integral remainder, as in the proof of lemma 3.1, we write $\mathbf{H}(\mathbf{X},\mathbf{U}) = \mathcal{D}_m\mathbf{X}$, where $\mathcal{D}_m$ is a diagonal matrix whose diagonal elements are continuous functions with values between 0 and 1:

$$\mathcal{D}_m(\mathbf{r},t) = \int_0^1 D\underline{\mathbf{S}}(\mathbf{U}(\mathbf{r},t) + \zeta(\mathbf{V}(\mathbf{r},t) - \mathbf{U}(\mathbf{r},t)))\,d\zeta.$$

Hence, according to lemma 4.5,

$$\|\mathbf{H}(\mathbf{X},\mathbf{U})\|_{n,2} = \|\mathcal{D}_m\mathbf{X}\|_{n,2} \le \|\mathbf{X}\|_{n,2}$$

We use this result and lemma 4.6 in equation (4.3) to obtain

$$\left| \left\langle \mathbf{X},\, \mathbf{L}^{-1} g_m(\mathbf{H}(\mathbf{X},\mathbf{U})) \right\rangle \right| \le \tau_{\max} DS_m\, \|g\|_{\mathcal{G}}\, \|\mathbf{X}\|_{n,2}^2,$$

and the conclusion follows. □
An identical sufficient condition holds for the stability of a solution to (2.7).

THEOREM 4.8. *A sufficient condition for the absolute stability of a solution to (2.7) is*

$$DS_m \tau_{\max} \|g\|_{\mathcal{G}} < 1$$

Proof. Let $\mathbf{U}$ be the unique solution of (2.7) with an external current $\mathbf{I}_{\text{ext}}(\mathbf{r}, t)$ and initial conditions $\mathbf{U}(0) = \mathbf{U}_0$. As in the proof of theorem 4.7 we introduce the new function $\mathbf{X} = \mathbf{V} - \mathbf{U}$, where $\mathbf{V}$ is the unique solution of the same equation with different initial conditions. We have

$$\mathbf{X}_t(\mathbf{r}, t) = -\mathbf{L}\mathbf{X}(\mathbf{r}, t) + \mathbf{S}\left(\int_\Omega \mathbf{W}(\mathbf{r}, \mathbf{r}', t)\mathbf{V}(\mathbf{r}', t)\, d\mathbf{r}' + \mathbf{I}_{\text{ext}}(\mathbf{r}, t)\right) - \mathbf{S}\left(\int_\Omega \mathbf{W}(\mathbf{r}, \mathbf{r}', t)\mathbf{U}(\mathbf{r}', t)\, d\mathbf{r}' + \mathbf{I}_{\text{ext}}(\mathbf{r}, t)\right) \quad (4.4)$$

Using a zeroth-order Taylor expansion, as in the proof of lemma 3.1, this equation can be rewritten as

$$\mathbf{X}_t(\mathbf{r}, t) = -\mathbf{L}\mathbf{X}(\mathbf{r}, t) + \left(\int_0^1 D\mathbf{S}\Big(\int_\Omega \mathbf{W}(\mathbf{r}, \mathbf{r}', t)\mathbf{U}(\mathbf{r}' t)\, d\mathbf{r}' + \mathbf{I}_{\text{ext}}(\mathbf{r}, t) + \zeta \int_\Omega \mathbf{W}(\mathbf{r}, \mathbf{r}', t)\mathbf{X}(\mathbf{r}', t)\, d\mathbf{r}'\Big)\, d\zeta\right)\left(\int_\Omega \mathbf{W}(\mathbf{r}, \mathbf{r}', t)\mathbf{X}(\mathbf{r}', t)\, d\mathbf{r}'\right)$$

We use the same functional as in the proof of theorem 4.7

$$V(\mathbf{X}) = \frac{1}{2}\left\langle \mathbf{X}, \mathbf{L}^{-1}\mathbf{X} \right\rangle.$$

Its time derivative is readily obtained with the help of equation (4.4)

$$\frac{dV(\mathbf{X})}{dt} = -\left\langle \mathbf{X}, \mathbf{X} \right\rangle + \left\langle \mathbf{X}, \mathbf{L}^{-1}\mathcal{D}_m h_m(\mathbf{X}) \right\rangle, \quad (4.5)$$

where $\mathcal{D}_m$ is defined by

$$\mathcal{D}_m(\mathbf{U}, \mathbf{X}, \mathbf{r}, t) = \int_0^1 D\mathbf{S}\Big(\int_\Omega \mathbf{W}(\mathbf{r}, \mathbf{r}', t)\mathbf{U}(t)\, d\mathbf{r}' + \mathbf{I}_{\text{ext}}(\mathbf{r}, t) + \zeta \int_\Omega \mathbf{W}(\mathbf{r}, \mathbf{r}', t)\mathbf{X}(\mathbf{r}', t)\, d\mathbf{r}'\Big)\, D\mathbf{S}_m^{-1}\, d\zeta,$$

a diagonal matrix whose diagonal elements are continuous functions with values between 0 and 1. We consider the second term in the righthand side of equation (4.5) and use the property of matrix $\mathcal{D}_m$ and lemma 4.6 to obtain

$$\left|\left\langle \mathbf{X}, \mathbf{L}^{-1}\mathcal{D}_m h_m(\mathbf{X}) \right\rangle\right| \le \|\mathbf{X}\|_{n,2}\|\mathbf{L}^{-1}\mathcal{D}_m h_m(\mathbf{X})\|_{n,2} \le \tau_{\max}\|\mathbf{X}\|_{n,2}\, \|h_m(\mathbf{X})\|_{n,2} \le \tau_{\max}\, DS_m\, \|g\|_{\mathcal{G}}\, \|\mathbf{X}\|_{n,2}^2,$$

from which the result follows. □
Note that $\|g\|_{\mathcal{G}} = \|g\|_{\mathbf{L}^2}$ by density of $\mathcal{G}$ in $\mathbf{L}^2$ (see section 6). In appendix A, we show how to compute such operator norms.

4.2. The convolution case. In the case where $\mathbf{W}$ is translation invariant we can obtain a slightly easier to exploit sufficient condition for the stability of the solutions than in the theorems 4.7 and 4.8. We first consider the case of a general compact Ω and then the case where Ω is an interval. Translation invariance means that $\mathbf{W}(\mathbf{r}+\mathbf{a}, \mathbf{r}'+\mathbf{a}, t) = \mathbf{W}(\mathbf{r}, \mathbf{r}', t)$ for all $\mathbf{a}$ such that $\mathbf{a}+\mathbf{r} \in \Omega$ and $\mathbf{a}+\mathbf{r}' \in \Omega$, so we can write $\mathbf{W}(\mathbf{r}, \mathbf{r}', t) = \mathbf{W}(\mathbf{r}-\mathbf{r}', t)$. Hence $\mathbf{W}(\mathbf{r}, t)$ must be defined for all $\mathbf{r} \in \widehat{\Omega} = \{\mathbf{r}-\mathbf{r}',\text{ with } \mathbf{r},\ \mathbf{r}' \in \Omega\}$ and we suppose it continuous on $\widehat{\Omega}$ for each t. $\widehat{\Omega}$ is a symmetric with respect to the origin, compact subset of $\mathbb{R}^q$.

4.2.1. General Ω. We note $\mathbf{1}_A$ the characteristic function of the subset A of $\mathbb{R}^q$ and $\mathbf{M}^* = \overline{\mathbf{M}}^T$ the conjugate transpose of the complex matrix $\mathbf{M}$.

We prove the following

Theorem 4.9. *If the eigenvalues of the Hermitian matrix*

$$\widetilde{\mathbf{W}}^*(\mathbf{f}, t)\,\widetilde{\mathbf{W}}(\mathbf{f}, t) \tag{4.6}$$

are strictly less than $(\tau_{\max} DS_m)^{-2}$ for almost all $\mathbf{f} \in \mathbb{R}^q$ and $t \in \mathrm{J}$, then the system (2.6) is absolutely stable[1]*. $\widetilde{\mathbf{W}}(\mathbf{f}, t)$ is the Fourier transform with respect to the space variable $\mathbf{r}$ of $\mathbf{1}_{\widehat{\Omega}}(\mathbf{r})\,\mathbf{W}(\mathbf{r}, t)$,*

$$\widetilde{\mathbf{W}}(\mathbf{f}, t) = \int_{\widehat{\Omega}} \mathbf{W}(\mathbf{r}, t) e^{-2i\pi \mathbf{r}\cdot\mathbf{f}}\, d\mathbf{r}$$

Proof. We recall that

$$\|g\|_{\mathcal{G}}^2 = \sup_{\|\mathbf{x}\|_{n,2} \leq 1} \frac{\|g(\mathbf{x})\|_{n,2}^2}{\|\mathbf{x}\|_{n,2}^2}.$$

We then note that, by definition

$$\|g(\mathbf{x})\|_{n,2} = \|(\mathbf{1}_{\widehat{\Omega}}\mathbf{W}) \otimes (\mathbf{1}_{\Omega}\mathbf{x})\|_{\mathbb{R}^q,\, n,\, 2},$$

where $\otimes$ indicates the convolution over $\mathbb{R}^q$. Parseval's theorem gives

$$\|(\mathbf{1}_{\widehat{\Omega}}\mathbf{W}) \otimes (\mathbf{1}_{\Omega}\mathbf{x})\|_{\mathbb{R}^q,\, n,\, 2}^2 = \int_{\mathbb{R}^q} \widetilde{\mathbf{x}}^*(\mathbf{f}, t)\widetilde{\mathbf{W}}^*(\mathbf{f}, t)\widetilde{\mathbf{W}}(\mathbf{f}, t)\widetilde{\mathbf{x}}(\mathbf{f}, t)\, d\mathbf{f},$$

where $\widetilde{\mathbf{x}}$ is the Fourier transform of $\mathbf{1}_{\Omega}\mathbf{x}$.
As an Hermitian matrix, $\widetilde{\mathbf{W}}^*(\mathbf{f}, t)\widetilde{\mathbf{W}}(\mathbf{f}, t)$ can be rewritten as $\mathbf{U}^*(\mathbf{f}, t)\mathbf{D}(\mathbf{f}, t)\mathbf{U}(\mathbf{f}, t)$, with $\mathbf{U}^*\mathbf{U} = \mathrm{Id_n}$ and $\mathbf{D}$ real and diagonal. In particular, $\mathbf{U}$ preserves length ($\|\mathbf{U}\mathbf{v}\|_2 = \|\mathbf{v}\|_2$). Besides, $\widetilde{\mathbf{W}}^*\widetilde{\mathbf{W}}$ is positive because for any complex vector $\mathbf{v}$,

$$\mathbf{v}^*\widetilde{\mathbf{W}}^*\widetilde{\mathbf{W}}\mathbf{v} = \|\widetilde{\mathbf{W}}\mathbf{v}\|_2^2 \geq 0.$$

So, all values of $\mathbf{D}$ are positive and if the hypothesis of the theorem is satisfied, lemma 4.5 yields

[1] Remark that since $\widetilde{\mathbf{W}}$ is continuous with respect to $\mathbf{f}$, some eigenvalues of the Hermitian matrix may be equal to $(\tau_{\max} DS_m)^{-2}$ on a zero measure domain.

$$\int_{\mathbb{R}^q} \widetilde{\mathbf{x}}^*(\mathbf{f},t)\widetilde{\mathbf{W}}^*(\mathbf{f},t)\widetilde{\mathbf{W}}(\mathbf{f},t)\widetilde{\mathbf{x}}(\mathbf{f},t)\,d\mathbf{f} = \|\sqrt{\mathbf{D}}\mathbf{U}\widetilde{\mathbf{x}}\|^2_{\mathbb{R}^q,\,n,\,2} \leq$$
$$(\tau_{\max}DS_m)^{-2}\|\mathbf{U}\widetilde{\mathbf{x}}\|^2_{\mathbb{R}^q,\,n,\,2} = (\tau_{\max}DS_m)^{-2}\|\widetilde{\mathbf{x}}\|^2_{\mathbb{R}^q,\,n,\,2} = (\tau_{\max}DS_m)^{-2}\|\mathbf{x}\|^2_{n,2},$$

hence $\|g\|_{\mathcal{G}} < (\tau_{\max}DS_m)^{-1}$ and theorem 4.7 applies. □

Since the sufficient condition for the absolute stability of the solution of the activation-based model is identical, we have the

THEOREM 4.10. *If the eigenvalues of the Hermitian matrix*

$$\widetilde{\mathbf{W}}^*(\mathbf{f},t)\,\widetilde{\mathbf{W}}(\mathbf{f},t)$$

are strictly less than $(\tau_{\max}DS_m)^{-2}$ *for almost all* $\mathbf{f}$ *and* $t \in \mathrm{J}$ *then the system (2.7) is absolutely stable.* $\widetilde{\mathbf{W}}(\mathbf{f},t)$ *is the Fourier transform of* $\mathbf{1}_{\widehat{\Omega}}(\mathbf{r})\mathbf{W}(\mathbf{r},t)$ *with respect to the space variable* $\mathbf{r}$*.*

These two theorems are somewhat unsatisfactory since they replace a condition that must be satisfied over a countable set, the spectrum of a compact operator, as in theorems 4.7 and 4.8, by a condition that must be satisfied over a continuum, i.e. $\mathbb{R}^q$. Nonetheless one may consider that the computation of the Fourier transforms of the matrix $\mathbf{W}$, extended by zeros outside $\widehat{\Omega}$, is easier than that of the spectrum of the operator g, for which a method is given in section A.3.

4.2.2. Ω is an interval. In the case where Ω is an interval, i.e. an interval of $\mathbb{R}$ ($q = 1$), a parallelogram ($q = 2$), or a parallelepiped ($q = 3$), we can state different sufficient conditions. We can always assume that Ω is the q-dimensional interval $[0,1]^q$ by applying an affine change of coordinates. The connectivity matrix $\mathbf{W}$ is defined on $\mathrm{J} \times [-1,1]^{\mathrm{q}}$ and extended to a q-periodic function of periods 2 on $\mathrm{J} \times \mathbb{R}^{\mathrm{q}}$, reflecting periodic boundary conditions. Similarly, the state vectors $\mathbf{V}$ and $\mathbf{A}$ as well as the external current $\mathbf{I}_{\mathrm{ext}}$ defined on $\mathrm{J} \times [0,1]^{\mathrm{q}}$ are extended to q-periodic functions of the same periods over $\mathrm{J} \times \mathbb{R}^{\mathrm{q}}$ by padding them with zeros in the complement in the interval $[-1,1]^q$ of the interval $[0,1]^q$. $\mathcal{G}$ is now the space $\mathbf{L}^2_n(2)$ of the square integrable q-periodic functions of periods 2 with values in $\mathbb{R}^n$.

We define the functions $\psi_{\mathbf{m}}(\mathbf{r}) \equiv e^{-\pi i(r_1 m_1 + \cdots + r_q m_q)}$, for $\mathbf{m} \in \mathbb{Z}^q$ and consider the matrix $\widetilde{\mathbf{W}}(\mathbf{m},t)$ whose elements are given by

$$\widetilde{W}_{ij}(\mathbf{m},t) = \int_{[0,2]^q} W_{ij}(\mathbf{r},t)\psi_{\mathbf{m}}(\mathbf{r})\,d\mathbf{r} \quad 1 \leq i,\, j \leq n.$$

We recall the

DEFINITION 4.11. *The matrix* $\widetilde{\mathbf{W}}(\mathbf{m})$ *is the* $\mathbf{m}$*th element of the Fourier series of the periodic matrix function* $\mathbf{W}(\mathbf{r})$. The theorems 4.9 and 4.10 can be stated in this framework.

THEOREM 4.12. *If the eigenvalues of the Hermitian matrix*

$$\widetilde{\mathbf{W}}^*(\mathbf{m},t)\,\widetilde{\mathbf{W}}(\mathbf{m},t) \tag{4.7}$$

are strictly less than $(\tau_{\max}DS_m)^{-2}$ *for all* $\mathbf{m} \in \mathbb{Z}^q$ *and all* $t \in \mathrm{J}$*, then the system (2.6) (resp. (2.7)) is absolutely stable.* $\widetilde{\mathbf{W}}(\mathbf{m},t)$ *is the* $\mathbf{m}$*th element of the Fourier series of the* q*-periodic matrix function* $\mathbf{W}(\mathbf{r},t)$ *with periods 2 at time* t*.*

5. Absolute stability of the homogeneous solution. We next investigate the absolute stability of a homogeneous solution to (2.6) and (2.7). As in the previous section we distinguish the general and convolution cases.

5.1. The general case. The homogeneous solutions are characterized by the fact that they are spatially constant at each time instant. We consider the subspace $\mathcal{G}_c$ of $\mathcal{G}$ of the constant functions. We have the following

LEMMA 5.1. *$\mathcal{G}_c$ is a complete linear subspace of $\mathcal{G}$. The orthogonal projection operator $\mathcal{P}_{\mathcal{G}_c}$ from $\mathcal{G}$ to $\mathcal{G}_c$ is defined by*

$$\mathcal{P}_{\mathcal{G}_c}(\mathbf{x}) = \overline{\mathbf{x}} = \frac{1}{|\Omega|}\int_\Omega \mathbf{x}(\mathbf{r})\,d\mathbf{r}$$

The orthogonal complement $\mathcal{G}_c^\perp$ of $\mathcal{G}_c$ is the subset of functions of $\mathcal{G}$ that have a zero average. The orthogonal projection[2] *operator $\mathcal{P}_{\mathcal{G}_c^\perp}$ is equal to $\mathrm{Id} - \mathcal{P}_{\mathcal{G}_c}$. We also have*

$$\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{M}\mathbf{x} = \mathbf{M}\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{x} \quad \forall\, \mathbf{x} \in \mathcal{G},\, \mathbf{M} \in \mathcal{M}_{n\times n} \tag{5.1}$$

Proof. The constant functions are clearly in $\mathcal{G}$. Any Cauchy sequence of constants is converging to a constant hence $\mathcal{G}_c$ is closed in the pre-Hilbert space $\mathcal{G}$. Therefore there exists an orthogonal projection operator from $\mathcal{G}$ to $\mathcal{G}_c$ which is linear, continuous, of unit norm, positive and self-adjoint. $\mathcal{P}_{\mathcal{G}_c}(\mathbf{x})$ is the minimum with respect to the constant vector $\mathbf{a}$ of the integral $\int_\Omega \|\mathbf{x}(\mathbf{r}) - \mathbf{a}\|^2\,d\mathbf{r}$. Taking the derivative with respect to $\mathbf{a}$, we obtain the necessary condition

$$\int_\Omega (\mathbf{x}(\mathbf{r}) - \mathbf{a})\,d\mathbf{r} = 0$$

and hence $\mathbf{a}_{min} = \overline{\mathbf{x}}$. Conversely, $\mathbf{x} - \mathbf{a}_{min}$ is orthogonal to $\mathcal{G}_c$ since $\int_\Omega (\mathbf{x}(\mathbf{r}) - \mathbf{a}_{min})\mathbf{b}\,d\mathbf{r} = 0$ for all $\mathbf{b} \in \mathcal{G}_c$.

Let $\mathbf{y} \in \mathcal{G}$, $\int_\Omega \mathbf{x}\,\mathbf{y}(\mathbf{r})\,d\mathbf{r} = \mathbf{x}\int_\Omega \mathbf{y}(\mathbf{r})\,d\mathbf{r} = 0$ for all $\mathbf{x} \in \mathcal{G}_c$ if and only if $\mathbf{y} \in \mathcal{G}_c^\perp$.

Finally

$$\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{M}\mathbf{x} = \mathbf{M}\mathbf{x} - \overline{\mathbf{M}\mathbf{x}} = \mathbf{M}\mathbf{x} - \mathbf{M}\overline{\mathbf{x}} = \mathbf{M}(\mathbf{x} - \overline{\mathbf{x}}) = \mathbf{M}\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{x}$$

□

We are now ready to prove the theorem on the absolute stability of the homogeneous solutions to (2.6).

THEOREM 5.2. *If $\mathbf{W}$ satisfies (3.5), a sufficient condition for the absolute stability of a homogeneous solution to (2.6) is that the norm $\|g^*\|_{\mathcal{G}_c^\perp}$ of the restriction to $\mathcal{G}_c^\perp$ of the compact operator g^* be less than $(\tau_{\max} DS_m)^{-1}$ for all $t \in$ J.*

Proof. This proof is inspired by [30]. Note that $\mathcal{G}_c^\perp$ is invariant by g^* and hence by g_m^*. Indeed, from lemma 4.4 and equation (3.5) we have

$$\overline{g^*(\mathbf{x})} = \overline{\mathbf{W}^T}(t)\overline{\mathbf{x}} = 0 \quad \forall \mathbf{x} \in \mathcal{G}_c^\perp$$

Let $\mathbf{V}_p$ be the unique solution of (2.6) with homogeneous input $\mathbf{I}_{\text{ext}}(t)$ and initial conditions $\mathbf{V}_p(0) = \mathbf{V}_{p0} \in \mathcal{G}$, and consider the initial value problem:

$$\begin{cases} \mathbf{X}'(t) &= \mathcal{P}_{\mathcal{G}_c^\perp}\left(f_v(t, \mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{X} + \mathcal{P}_{\mathcal{G}_c}\mathbf{V}_p)\right) \\ \mathbf{X}(0) &= \mathbf{X}_0 \end{cases} \tag{5.2}$$

[2] To be accurate, this is the projection on the closure of $\mathcal{G}_c^\perp$ in the closure of $\mathcal{G}$ which is the Hilbert space $\mathbf{L}_n^2(\Omega)$.

$\mathbf{X} = \mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{V}_p$ is a solution with initial condition $\mathbf{X}_0 = \mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{V}_{p0}$, since $\mathcal{P}^2_{\mathcal{G}_c^\perp} = \mathcal{P}_{\mathcal{G}_c^\perp}$, and $\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{V}_p + \mathcal{P}_{\mathcal{G}_c}\mathbf{V}_p = \mathbf{V}_p$. But $\mathbf{X} = 0$ is also a solution with initial condition $\mathbf{X}_0 = 0$. Indeed $\mathcal{G}_c$ is flow-invariant because of (3.5), that is $f_v(t, \mathcal{G}_c) \subset \mathcal{G}_c$, and hence $\mathcal{P}_{\mathcal{G}_c^\perp}(f_v(t, \mathcal{G}_c)) = 0$. We therefore look for a sufficient condition for the system (5.2) to be absolutely stable at $\mathbf{X} = 0$.

We consider again the functional $V(\mathbf{X}) = \frac{1}{2}\left\langle \mathbf{X}, \mathbf{L}^{-1}\mathbf{X}\right\rangle$ with time derivative $\frac{dV(\mathbf{X})}{dt} = \left\langle \mathbf{X}, \mathbf{L}^{-1}\mathbf{X}_t \right\rangle$. We substitute $\mathbf{X}_t$ with its value from (5.2) which can be rewritten as

$$\mathbf{X}_t = \mathcal{P}_{\mathcal{G}_c^\perp}\left(-\mathbf{L}(\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{X} + \mathcal{P}_{\mathcal{G}_c}\mathbf{V}_p) + \int_\Omega \mathbf{W}(\mathbf{r}, \mathbf{r}', t)\, \mathbf{S}(\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{X}(\mathbf{r}', t) + \mathcal{P}_{\mathcal{G}_c}\mathbf{V}_p(\mathbf{r}', t))\, d\mathbf{r}' \right)$$

Because of lemma 5.1 this yields

$$\mathbf{X}_t = -\mathbf{L}\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{X} + \mathcal{P}_{\mathcal{G}_c^\perp}\left(\int_\Omega \mathbf{W}_{cm}(\mathbf{r}, \mathbf{r}', t)\, \underline{\mathbf{S}}(\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{X}(\mathbf{r}', t) + \mathcal{P}_{\mathcal{G}_c}\mathbf{V}_p(\mathbf{r}', t))\, d\mathbf{r}' \right)$$

Using a zeroth-order Taylor expansion, as in the proof of lemma 3.1, we write

$$\underline{\mathbf{S}}(\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{X} + \mathcal{P}_{\mathcal{G}_c}\mathbf{V}_p) = \underline{\mathbf{S}}(\mathcal{P}_{\mathcal{G}_c}\mathbf{V}_p) + \left(\int_0^1 D\underline{\mathbf{S}}(\mathcal{P}_{\mathcal{G}_c}\mathbf{V}_p + \zeta\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{X})\, d\zeta \right) \mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{X},$$

and since $\underline{\mathbf{S}}(\mathcal{P}_{\mathcal{G}_c}\mathbf{V}_p) \in \mathcal{G}_c$, and because of (3.5)

$$\mathcal{P}_{\mathcal{G}_c^\perp}\left(\int_\Omega \mathbf{W}_{cm}(\mathbf{r}, \mathbf{r}', t)\, \underline{\mathbf{S}}(\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{X}(\mathbf{r}', t) + \mathcal{P}_{\mathcal{G}_c}\mathbf{V}_p(\mathbf{r}', t))\, d\mathbf{r}' \right) =$$
$$\mathcal{P}_{\mathcal{G}_c^\perp}\left(\int_\Omega \mathbf{W}_{cm}(\mathbf{r}, \mathbf{r}', t) \left(\int_0^1 D\underline{\mathbf{S}}(\mathcal{P}_{\mathcal{G}_c}\mathbf{V}_p(\mathbf{r}', t) + \zeta\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{X}(\mathbf{r}', t))\, d\zeta \right) \mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{X}(\mathbf{r}', t)\, d\mathbf{r}' \right)$$

We use (5.1) and the fact that $\mathcal{P}_{\mathcal{G}_c^\perp}$ is self-adjoint and idempotent to write

$$\frac{dV(\mathbf{X})}{dt} = -\langle \mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{X}, \mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{X}\rangle +$$
$$\left\langle \mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{X}, \mathbf{L}^{-1}\int_\Omega \mathbf{W}_{cm}(\mathbf{r}, \mathbf{r}', t) \left(\int_0^1 D\underline{\mathbf{S}}(\mathcal{P}_{\mathcal{G}_c}\mathbf{V}_p(\mathbf{r}', t) + \zeta\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{X}(\mathbf{r}', t))\, d\zeta \right) \mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{X}(\mathbf{r}', t)\, d\mathbf{r}' \right\rangle$$

Let us denote by $\mathcal{D}_v(\mathbf{r}', t)$ the diagonal matrix $\int_0^1 D\underline{\mathbf{S}}(\mathcal{P}_{\mathcal{G}_c}\mathbf{V}_p(\mathbf{r}', t) + \zeta\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{X}(\mathbf{r}', t))\, d\zeta$. Its diagonal elements are continuous functions with values between 0 and 1. Letting $\mathbf{Y} = \mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{X}$ we rewrite the previous equation in operator form, introducing the operator g_m (definition 4.1), as

$$\frac{dV(\mathbf{X})}{dt} = -\langle \mathbf{Y}, \mathbf{Y}\rangle + \left\langle \mathbf{Y}, \mathbf{L}^{-1}g_m(\mathcal{D}_v\, \mathbf{Y})\right\rangle$$

By definition of the adjoint

$$\left\langle \mathbf{Y}, \mathbf{L}^{-1}g_m(\mathcal{D}_v\, \mathbf{Y})\right\rangle = \left\langle g_m^*\left(\mathbf{L}^{-1}\mathbf{Y}\right), \mathcal{D}_v\, \mathbf{Y}\right\rangle$$

Using the Cauchy-Schwarz' inequality and lemma 4.5

$$\left|\left\langle g_m^*\left(\mathbf{L}^{-1}\mathbf{Y}\right), \mathcal{D}_v\, \mathbf{Y}\right\rangle\right| \le \left\|g_m^*\left(\mathbf{L}^{-1}\mathbf{Y}\right)\right\|_{n,2} \|\mathcal{D}_v\, \mathbf{Y}\|_{n,2} \le \left\|g_m^*\left(\mathbf{L}^{-1}\mathbf{Y}\right)\right\|_{n,2} \|\mathbf{Y}\|_{n,2},$$

and since

$$\left\|g_m^*\left(\mathbf{L}^{-1}\mathbf{Y}\right)\right\|_{n,2} \le \|g_m^*\|_{\mathcal{G}_c^\perp}\left\|\mathbf{L}^{-1}\mathbf{Y}\right\|_{n,2} \le \tau_{\max} DS_m \|g^*\|_{\mathcal{G}_c^\perp}\|\mathbf{Y}\|_{n,2},$$

the conclusion follows. ∎

Note that $\|g^*\|_{\mathcal{G}_c^\perp} = \|g^*\|_{\mathbf{L}_0^2}$ by density of $\mathcal{G}_c^\perp$ in $\mathbf{L}_0^2$, where $\mathbf{L}_0^2$ is the subspace of $\mathbf{L}^2$ of zero mean functions. We show in appendix A how to compute this norm.

We prove a similar theorem in the case of (2.7).

THEOREM 5.3. *If $\mathbf{W}$ satisfies (3.5), a sufficient condition for the stability of a homogeneous solution to (2.7) is that the norm $\|g\|_{\mathcal{G}_c^\perp}$ of the restriction to $\mathcal{G}_c^\perp$ of the compact operator g be less than $(\tau_{\max}DS_m)^{-1}$ for all $t \in$ J.*

Proof. The proof is similar to that of theorem 5.2. We consider $\mathbf{A}_p$ the unique solution to (2.7) with homogeneous input $\mathbf{I}_{\text{ext}}(t)$, initial conditions $\mathbf{A}_p(0) = \mathbf{A}_{p0}$, and consider the initial value problem

$$\begin{cases} \mathbf{A}'(t) &=& \mathcal{P}_{\mathcal{G}_c^\perp}\left(f_a(t, \mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{A} + \mathcal{P}_{\mathcal{G}_c}\mathbf{A}_p)\right) \\ \mathbf{A}(0) &=& \mathbf{A}_0 \end{cases} \tag{5.3}$$

$\mathbf{A} = \mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{A}_p$ is a solution with initial conditions $\mathbf{A}_0 = \mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{A}_{p0}$ since $\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{A}_p + \mathcal{P}_{\mathcal{G}_c}\mathbf{A}_p = \mathbf{A}_p$. But $\mathbf{A} = 0$ is also a solution with initial conditions $\mathbf{A}_0 = 0$. Indeed $\mathcal{G}_c$ is flow-invariant because of (3.5), that is $f_a(t, \mathcal{G}_c) \subset \mathcal{G}_c$, and hence $\mathcal{P}_{\mathcal{G}_c^\perp}(f_a(t, \mathcal{G}_c)) = 0$. We therefore look for a sufficient condition for the system (5.3) to be absolutely stable at $\mathbf{A} = 0$.

Consider again the functional $V(\mathbf{A}) = \frac{1}{2}\left\langle \mathbf{A},\, \mathbf{L}^{-1}\mathbf{A}\right\rangle$ with time derivative $\frac{dV(\mathbf{A})}{dt} = \left\langle \mathbf{A},\, \mathbf{L}^{-1}\mathbf{A}_t \right\rangle$. We substitute $\mathbf{A}_t$ with its value from (5.3) which, using (3.5), can be rewritten as

$$\mathbf{A}_t = $$
$$\mathcal{P}_{\mathcal{G}_c^\perp}\left(-\mathbf{L}(\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{A}+\mathcal{P}_{\mathcal{G}_c}\mathbf{A}_p)+\mathbf{S}\left(\int_\Omega \mathbf{W}(\mathbf{r},\mathbf{r}',t)\,\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{A}(\mathbf{r}',t)\,d\mathbf{r}' + \overline{\mathbf{W}}(t)\mathcal{P}_{\mathcal{G}_c}\mathbf{A}_p + \mathbf{I}_{\text{ext}}(t)\right)\right)$$

We perform a first-order Taylor expansion with integral remainder of the $\mathbf{S}$ term and introduce the operator h_m (definition 4.1):

$$\mathbf{S}\left(\int_\Omega \mathbf{W}(\mathbf{r},\mathbf{r}',t)\,\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{A}(\mathbf{r}',t)\,d\mathbf{r}' + \overline{\mathbf{W}}(t)\mathcal{P}_{\mathcal{G}_c}\mathbf{A}_p + \mathbf{I}_{\text{ext}}(t)\right) = \mathbf{S}\left(\overline{\mathbf{W}}(t)\mathcal{P}_{\mathcal{G}_c}\mathbf{A}_p + \mathbf{I}_{\text{ext}}(t)\right) +$$
$$\left(\int_0^1 D\underline{\mathbf{S}}\Big(\overline{\mathbf{W}}(t)\mathcal{P}_{\mathcal{G}_c}\mathbf{A}_p+\mathbf{I}_{\text{ext}}(t)+\zeta\int_\Omega \mathbf{W}(\mathbf{r},\mathbf{r}',t)\,\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{A}(\mathbf{r}',t)\,d\mathbf{r}'\Big)d\zeta\right) h_m(\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{A})(\mathbf{r},t)$$

Let us define

$$\mathcal{D}_a(\mathbf{r},t) = \int_0^1 D\underline{\mathbf{S}}\Big(\overline{\mathbf{W}}(t)\mathcal{P}_{\mathcal{G}_c}\mathbf{A}_p + \mathbf{I}_{\text{ext}}(t) + \zeta\int_\Omega \mathbf{W}(\mathbf{r},\mathbf{r}',t)\,\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{A}(\mathbf{r}',t)\,d\mathbf{r}'\Big)d\zeta,$$

a diagonal matrix whose diagonal elements are continuous functions with values between 0 and 1. Letting $\mathbf{Y} = \mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{A}$ we write

$$\frac{dV(\mathbf{A})}{dt} = -\langle \mathbf{Y},\, \mathbf{Y}\rangle + \left\langle \mathbf{Y},\, \mathbf{L}^{-1}\mathcal{D}_a\, h_m(\mathbf{Y})\right\rangle$$

and the conclusion follows from the Cauchy-Schwarz' inequality and lemmas 4.5 and 4.6

$$\left|\left\langle \mathbf{Y}, \mathbf{L}^{-1}\mathcal{D}_a\, h_m(\mathbf{Y})\right\rangle\right| \leq \|\mathbf{Y}\|_{n,2} \left\|\mathbf{L}^{-1}\mathcal{D}_a\, h_m(\mathbf{Y})\right\|_{n,2} \leq \\ \tau_{\max} \|\mathbf{Y}\|_{n,2} \|h_m(\mathbf{Y})\|_{n,2} \leq \tau_{\max} D S_m \|g\|_{\mathcal{G}_c^\perp} \|\mathbf{Y}\|_{n,2}^2$$

□

5.2. The convolution case. As $\mathbf{W}$ is translation invariant $\int_\Omega \mathbf{W}(\mathbf{r}-\mathbf{r}', t)\, d\mathbf{r}'$ is in general a function of $\mathbf{r}$, unless Ω has no border. In our framework, this case only occurs as Ω is an interval with periodic conditions and we have the following

THEOREM 5.4. *A sufficient condition for the stability of a homogeneous solution to (2.6) (resp. (2.7)) is that the eigenvalues of the Hermitian matrices*

$$\widetilde{\mathbf{W}}^*(\mathbf{m}, t)\widetilde{\mathbf{W}}(\mathbf{m}, t)$$

are strictly less than $(\tau_{\max} D S_m)^{-2}$ *for all* $\mathbf{m} \neq \mathbf{0} \in \mathbb{Z}^q$ *and all* $t \in \mathrm{J}$. $\widetilde{\mathbf{W}}(\mathbf{m}, t)$ *is the* $\mathbf{m}$*th element of the Fourier series of the q-periodic matrix function* $\mathbf{W}(\mathbf{r}, t)$ *with respect to the space variable* $\mathbf{r}$.

The only difference with theorem 4.12 is that there are no constraints on the Fourier coefficient $\mathbf{m} = 0$. This is due to the fact that we only "look" at the subspace of $\mathcal{G}$ of functions with zero spatial average.

5.3. Complete synchronization. The property of absolute stability of the solution that is characterized in theorems 5.2, 5.3 and 5.4 can be seen as the ability for the neural masses in the continuum to synchronize. By synchronization we mean that the state vectors at all points in the continuum converge to a unique state vector that is a function only of the common input $\mathbf{I}_{\text{ext}}$ and not of the initial states of the neural masses. The state vector is the homogeneous solution of (2.6) and (2.7). This effect is called complete synchronization [31].

6. Extending the theory. We have developed our analysis of (2.6) and (2.7) in the Banach space $\mathcal{F}$ of continuous functions of the spatial coordinate $\mathbf{r}$ even though we have used a structure of pre-Hilbert space $\mathcal{G}$ on top of it. But there remains the fact that the solutions that we have been discussing are smooth, i.e., continuous with respect to the space variable. It may be interesting to also consider non-smooth solutions, e.g., piecewise continuous solutions that can be discontinuous along curves of Ω. A natural setting, given the fact that we are interested in having a structure of Hilbert space, is $\mathbf{L}_n^2(\Omega)$, the space of square-integrable functions from Ω to $\mathbb{R}^n$, see appendix A. It is a Hilbert space and $\mathcal{G}$ is a dense subspace: $\overline{\mathcal{G}} = \mathbf{L}_n^2(\Omega)$, where $\overline{A}$ indicates the topological closure of the set A.

6.1. Existence, uniqueness and stability of a solution. The theory developed in the previous sections can be readily extended to $\mathbf{L}_n^2(\Omega)$: the analysis of the stability of the general and homogeneous solutions has been done using the pre-Hilbert space structure of $\mathcal{G}$ and all the operators that have been shown to be compact in $\mathcal{G}$ are also compact in its closure $\mathbf{L}_n^2(\Omega)$ [9]. The only point that has to be re-worked is the problem of existence and uniqueness of a solution addressed in propositions 3.2 and 3.3. This allows us to *relax* the rather stringent spatial smoothness hypotheses imposed on the connectivity function $\mathbf{W}$ and the external current $\mathbf{I}_{\text{ext}}$, thereby bringing in more flexibility to the model. We have the following

PROPOSITION 6.1. *If the following two hypotheses are satisfied*

1. *The mapping* $\mathbf{W}$ *is in* $C(\mathrm{J}; \mathbf{L}^2_{\mathrm{n}\times\mathrm{n}}(\Omega \times \Omega))$,
2. *The external current* $\mathbf{I}_{\mathrm{ext}}$ *is in* $C(\mathrm{J}; \mathbf{L}^2_{\mathrm{n}}(\Omega))$,

then the mappings f_v *and* f_a *are from* $\mathrm{J} \times \mathbf{L}^2_{\mathrm{n}}(\Omega)$ *to* $\mathbf{L}^2_n(\Omega)$, *continuous, and Lipschitz continuous with respect to their second argument, uniformly with respect to the first.*

Proof. Because of the first hypothesis, the fact that $\mathbf{S}(\mathbf{x})$ is in $\mathbf{L}^2_n(\Omega)$ for all $\mathbf{x} \in \mathbf{L}^2_n(\Omega)$, and lemma A.2, f_v is well-defined. Let us prove that it is continuous. As in the proof of proposition 3.2 we write

$$f_v(t, \mathbf{x}) - f_v(s, \mathbf{y}) = -\mathbf{L}(\mathbf{x} - \mathbf{y}) + \int_\Omega (\mathbf{W}(\cdot, \mathbf{r}', t) - \mathbf{W}(\cdot, \mathbf{r}', s))\mathbf{S}(\mathbf{x}(\mathbf{r}'))\, d\mathbf{r}' + \int_\Omega \mathbf{W}(\cdot, \mathbf{r}', s)(\mathbf{S}(\mathbf{x}(\mathbf{r}')) - \mathbf{S}(\mathbf{y}(\mathbf{r}'))\, d\mathbf{r}' + \mathbf{I}_{\mathrm{ext}}(\cdot, t) - \mathbf{I}_{\mathrm{ext}}(\cdot, s),$$

from which we obtain, using lemma A.2

$$\|f_v(t, \mathbf{x}) - f_v(s, \mathbf{y})\|_{n,2} \leq \|\mathbf{L}\|_F \|\mathbf{x} - \mathbf{y}\|_{n,2} + \sqrt{n|\Omega|} S_m \|\mathbf{W}(\cdot, \cdot, t) - \mathbf{W}(\cdot, \cdot, s)\|_F + DS_m \|\mathbf{W}(\cdot, \cdot, s)\|_F \|\mathbf{x} - \mathbf{y}\|_{n,2} + \|\mathbf{I}_{\mathrm{ext}}(\cdot, t) - \mathbf{I}_{\mathrm{ext}}(\cdot, s)\|_{n,2},$$

and the continuity follows from the hypotheses. $\| \, \|_F$ is the Frobenius norm, see appendix A. Note that since $\mathbf{W}$ is continuous on the compact interval J, it is bounded and $\|\mathbf{W}(\cdot, \cdot, t)\|_F \leq w$ for all $t \in \mathrm{J}$ for some positive constant w. The Lipschitz continuity with respect to the second argument uniformly with respect to the first one follows from the previous inequality by choosing $s = t$.

The proof for f_a is similar. ◻

From this proposition we deduce the existence and uniqueness of a solution over a subinterval of $\mathbb{R}$:

PROPOSITION 6.2. *Subject to the hypotheses of proposition 6.1 for any element* $\mathbf{V}_0$ *of* $\mathbf{L}^2_n(\Omega)$ *there is a unique solution* $\mathbf{V}$, *defined on a subinterval of* J *containing 0 and continuously differentiable, of the abstract initial value problem (3.1) for* $f = f_v$ *and* $f = f_a$ *such that* $\mathbf{V}(0) = \mathbf{V}_0$.

Proof. All conditions of the Picard-Lindelöf theorem on differential equations in Banach spaces (here a Hilbert space) [9, 2] are satisfied, hence the proposition. ◻

We can also prove that this solution exists for all times, as in proposition 3.4:

PROPOSITION 6.3. *If the following two hypotheses are satisfied*

1. *The connectivity function* $\mathbf{W}$ *is in* $C(\mathbb{R}; \mathbf{L}^2_{\mathrm{n}\times\mathrm{n}}(\Omega \times \Omega))$,
2. *The external current* $\mathbf{I}_{\mathrm{ext}}$ *is in* $C(\mathbb{R}; \mathbf{L}^2_{\mathrm{n}}(\Omega))$,

then for any function $\mathbf{V}_0$ *in* $\mathbf{L}^2_n(\Omega)$ *there is a unique solution* $\mathbf{V}$, *defined on* $\mathbb{R}$ *and continuously differentiable, of the abstract initial value problem (3.1) for* $f = f_v$ *and* $f = f_a$.

Proof. The proof is similar to the one of proposition 3.4. ◻

The absolute stability of the solution can be studied exactly as in theorems 4.7 and 4.8. Since $\mathcal{G}$ is dense in $\mathbf{L}^2_n(\Omega)$ we have $\|g\|_{\mathcal{G}} = \|g\|_{\mathbf{L}^2_n(\Omega)}$ and similar relations for all the other operators. We have the following

THEOREM 6.4. *If the compact operator g satisfies the condition of theorem 4.7 the solution of the abstract initial value problem (3.1) for* $f = f_v$ *and* $f = f_a$ *is absolutely stable.*

6.2. Locally homogeneous solutions. An application of the previous extension is the following. Consider a partition of Ω into P subregions Ω_i, $i = 1, \cdots, P$. We assume that the Ω_is are closed, hence compact, subsets of Ω intersecting along finitely many piecewise regular curves. These curves form a set of 0 Lebesgue measure of Ω. We consider locally homogeneous input current functions

$$\mathbf{I}_{\mathrm{ext}}(\mathbf{r}, t) = \sum_{k=1}^{P} \mathbf{1}_{\Omega_k}(\mathbf{r}) \mathbf{I}_{\mathrm{ext}}^k(t), \tag{6.1}$$

where the P functions $\mathbf{I}_{\mathrm{ext}}^k(t)$ are continuous on some closed interval J containing 0. On the border between two adjacent regions the value of $\mathbf{I}_{\mathrm{ext}}(\mathbf{r}, t)$ is undefined. Since this set of borders is of 0 measure, the functions defined by (6.1) are in $\mathbf{L}_n^2(\Omega)$ at each time instant.

6.2.1. Existence and uniqueness. We are interested in the existence of solutions to the abstract initial value problem (3.1) that are homogeneous in each subregion Ω_i, $i = 1, \cdots, P$. We call them locally homogeneous solutions.

We assume that the connectivity matrix $\mathbf{W}$ satisfies the following conditions

$$\int_{\Omega_k} \mathbf{W}(\mathbf{r}, \mathbf{r}', t)\, d\mathbf{r}' = \sum_{i=1}^{P} \mathbf{1}_{\Omega_i}(\mathbf{r}) \mathbf{W}_{ik}(t) \quad k = 1, \cdots, P. \tag{6.2}$$

These conditions are analogous to (3.5). A locally homogeneous solution of (2.6) or (2.7) can be written

$$\mathbf{V}(\mathbf{r}, t) = \sum_{i=1}^{P} \mathbf{1}_{\Omega_i}(\mathbf{r}) \mathbf{V}_i(t),$$

where the functions $\mathbf{V}_i$ satisfy the following system of ordinary differential equations

$$\mathbf{V}_i'(t) = -\mathbf{L}\mathbf{V}_i(t) + \sum_{k=1}^{P} \mathbf{W}_{ik}(t) \mathbf{S}(\mathbf{V}_k(t)) + \mathbf{I}_{\mathrm{ext}}^i(t), \tag{6.3}$$

for the voltage-based model and

$$\mathbf{V}_i'(t) = -\mathbf{L}\mathbf{V}_i(t) + \mathbf{S}\left(\sum_{k=1}^{P} \mathbf{W}_{ik}(t) \mathbf{V}_k(t) + \mathbf{I}_{\mathrm{ext}}^i(t) \right), \tag{6.4}$$

for the activity-based model. The conditions for the existence and uniqueness of a locally homogeneous solution are given in the following theorem, analog to theorem 3.5:

THEOREM 6.5. *If the external currents $\mathbf{I}_{\mathrm{ext}}^k(t)$, $k = 1, \cdots, P$ and the connectivity matrices $\mathbf{W}_{ik}(t)$, $i, k = 1, \cdots, P$ are continuous on some closed interval* J *containing 0, then for all sets of P vectors $\mathbf{U}_0^k$, $k = 1, \cdots, P$ of $\mathbb{R}^n$, there exists a unique solution $(\mathbf{U}_1(t), \cdots, \mathbf{U}_P(t))$ of (6.3) or (6.4) defined on a subinterval* J_0 *of* J *containing 0 such that $\mathbf{U}_k(0) = \mathbf{U}_0^k$, $k = 1, \cdots, P$.*

Proof. The system (6.3) can be written in the form

$$\mathcal{V}'(t) = -\mathcal{L}\mathcal{V}(t) + \mathcal{W}(t)\mathcal{S}(\mathcal{V}(t)) + \mathcal{I}_{\mathrm{ext}}(t), \tag{6.5}$$

where $\mathcal{V}$ is the nP dimensional vector $\begin{pmatrix} \mathbf{V}_1 \\ \vdots \\ \mathbf{V}_P \end{pmatrix}$, $\mathcal{I}_{\mathrm{ext}} = \begin{pmatrix} \mathbf{I}_{\mathrm{ext}}^1 \\ \vdots \\ \mathbf{I}_{\mathrm{ext}}^P \end{pmatrix}$, $\mathcal{S}(\mathcal{X}) = \begin{pmatrix} \mathbf{S}(\mathbf{X}_1) \\ \vdots \\ \mathbf{S}(\mathbf{X}_P) \end{pmatrix}$, $\mathcal{W}$ is the block matrix $(\mathbf{W}_{ik})_{i,k}$ and $\mathcal{L}$ is the block diagonal matrix whose diagonal elements are all equal to $\mathbf{L}$. Then we are dealing with a classical initial value problem of dimension nP and the proof of existence and uniqueness is similar to the one of theorem 3.5. A similar proof can be written in the case of system (6.4). □

Again, if $\mathcal{I}_{\text{ext}}$ and $\mathcal{W}$ are continuous on $\mathbb{R}$, the existence and uniqueness result extends to the whole time line $\mathbb{R}$.

6.2.2. Absolute stability. Having proved the existence and uniqueness of a locally homogeneous solution we consider the problem of characterizing its absolute stability. The method is the same as in section 5. We consider the subset, noted $\mathcal{G}_c^P$, of the functions that are constant in the interior $\overset{\circ}{\Omega}_i$ of each region $\Omega_i, i = 1, \cdots, P$ (the interior $\overset{\circ}{A}$ of a subset A is defined as the biggest open subset included in A). We have the following lemma that echoes lemma 5.1

Lemma 6.6. *$\mathcal{G}_c^P$ is a complete linear subspace of $\mathbf{L}_n^2(\Omega)$. The orthogonal projection operator $\mathcal{P}_{\mathcal{G}_c^P}$ from $\mathbf{L}_n^2(\Omega)$ to $\mathcal{G}_c^P$ is defined by*

$$\mathcal{P}_{\mathcal{G}_c^P}(\mathbf{x})(\mathbf{r}) = \overline{\mathbf{x}}^P = \sum_{k=1}^{P} \mathbf{1}_{\Omega_k}(\mathbf{r}) \frac{1}{|\Omega_k|} \int_{\Omega_k} \mathbf{x}(\mathbf{r}')\, d\mathbf{r}'$$

The orthogonal complement $\mathcal{G}_c^{P\,\perp}$ of $\mathcal{G}_c^P$ is the subset of functions of $\mathbf{L}_n^2(\Omega)$ that have a zero average in each $\Omega_i, i = 1, \cdots, P$. The orthogonal projection operator $\mathcal{P}_{\mathcal{G}_c^{P\,\perp}}$ is equal to $\mathrm{Id} - \mathcal{P}_{\mathcal{G}_c^{\mathrm{P}}}$*. We also have*

$$\mathcal{P}_{\mathcal{G}_c^{P\,\perp}} \mathbf{M}\mathbf{x} = \mathbf{M}\mathcal{P}_{\mathcal{G}_c^{P\,\perp}} \mathbf{x} \quad \forall \mathbf{x} \in \mathbf{L}_n^2(\Omega),\, \mathbf{M} \in \mathcal{M}_{n\times n} \tag{6.6}$$

Proof. The proof of this lemma is similar to the one of lemma 5.1. □
We have the following theorem, corresponding to theorems 5.2 and 5.3.

Theorem 6.7. *If $\mathbf{W}$ satisfies (6.2), a sufficient condition for the absolute stability of a locally homogeneous solution to (2.6) (respectively (2.7)) is that the norm $\|g^*\|_{\mathcal{G}_c^{P\,\perp}}$ (respectively $\|g\|_{\mathcal{G}_c^{P\,\perp}}$) of the restriction to $\mathcal{G}_c^{P\,\perp}$ of the compact operator g^* (respectively g) be less than $(\tau_{\max} DS_m)^{-1}$ for all $t \in$* J.

Proof. The proof strictly follows the lines of the ones of theorems 5.2 and 5.3. □
Note that the condition on the operator norm in theorems 4.7 and 4.8 is stronger than the one of theorems 5.2 and 5.3 which is in turn stronger than the one of theorem 6.7 therefore we have the following

Proposition 6.8. *If the operator g satisfies the condition of theorem 4.7 or if $g*$ (respectively g) satisfies the condition of theorem 5.2 (respectively of theorem 5.3), then for every partition of Ω, corresponding locally homogeneous current, and $\mathbf{W}$ satisfying (6.2), the locally homogeneous solution of (2.6) (respectively (2.7)) is absolutely stable.*

Proof. Since all spaces are contained in $\mathbf{L}_n^2(\Omega)$ the first part of the proposition is proved. Next it is clear that $\mathcal{G}_c \subset \mathcal{G}_c^P$, therefore $\mathcal{G}_c^{P\,\perp} \subset \mathcal{G}_c^{\perp}$ and $\|g^*\|_{\mathcal{G}_c^{P\,\perp}} \le \|g^*\|_{\mathcal{G}_c^{\perp}}$ (respectively $\|g\|_{\mathcal{G}_c^{P\,\perp}} \le \|g\|_{\mathcal{G}_c^{\perp}}$). □
Condition (6.2) depends on the partition of Ω. It is therefore unrealistic since one expects this partition to change over time with the external currents. In this context it is interesting to define the notion of *pseudo locally homogeneous* solution.

Definition 6.9. *A pseudo locally homogeneous solution of equation (2.6) (respectively (2.7)) corresponds to a locally homogeneous input current (verifying (6.1)) when the connectivity function satisfies the condition of proposition 6.3 (existence and uniqueness of a solution) but not necessarily conditions (6.2).*

How much a pseudo locally homogeneous solution differs from a locally homogeneous solution obviously depends upon how poorly the connectivity function satisfies the conditions (6.2). But since pseudo locally homogeneous solutions are solutions, they enjoy the following property.

PROPOSITION 6.10. *If the operator g satisfies the condition of theorem 4.7, the unique pseudo locally homogeneous solution of equations (2.6) (respectively of equations (2.7)) corresponding to a locally homogeneous input current, is absolutely stable.*
A numerical example of pseudo locally homogeneous solution is given in section 7 (figures 7.16 and 7.17).

6.2.3. Complete local synchronization. The property of absolute stability of the solution that is characterized in theorem 6.7 can be seen as the ability for the neural masses in the continuum to synchronize locally within each region Ω_i, $i = 1, \dots, P$. By local synchronization we mean that the state vectors at all points of each region Ω_i converge to a unique state vector that is a function only of the common input $\mathbf{I}^i_{\rm ext}$ within Ω_i and not of the initial states of the neural masses. The state vector is the locally homogeneous solution of (2.6) and (2.7). This effect is called complete local synchronization.

6.3. Higher order PSPs. We now show how the theory developed so far can be extended to accomodate more complicated time variations of the postsynaptic potentials (PSPs) than the decaying exponential that we adopted so far with the advantage that we only had to deal with a first order differential equation. We only show how to proceed in the case of a second order differential equation, going to a higher order does not bring in new difficulties. We also treat only the case of the voltage-based model, the case of the activity-based model being similar.

We therefore assume that, with the notations of section 2.1.1, we have $PSP_i(t) = te^{-t/\tau_i}Y(t)$ or equivalently that

$$\frac{d^2 PSP_i(t)}{dt^2} + \frac{2}{\tau_i}\frac{dPSP_i(t)}{dt} + \frac{1}{\tau_i^2} = \delta(t)$$

The analog of equation (2.4) being

$$\mathbf{V}'' = -2\mathbf{L}\mathbf{V}' - \mathbf{L}^2\mathbf{V} + \mathbf{W}\,\mathbf{S}(\mathbf{V}) + \mathbf{I}_{\rm ext}. \tag{6.7}$$

We rewrite this as a first order system of differential equation by introducing the vector $\boldsymbol{\mathcal{V}} = \begin{bmatrix} \mathbf{V} \\ \mathbf{V}' \end{bmatrix}$:

$$\boldsymbol{\mathcal{V}}' = -\boldsymbol{\mathcal{L}}\boldsymbol{\mathcal{V}} + \begin{bmatrix} \mathbf{0} \\ \mathbf{W}\,\mathbf{S}(\mathbf{V}) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_{\rm ext} \end{bmatrix}, \quad \boldsymbol{\mathcal{L}} = \begin{bmatrix} \mathbf{0} & -\mathrm{Id} \\ \mathbf{L}^2 & 2\mathbf{L} \end{bmatrix}$$

The dynamic system $\boldsymbol{\mathcal{V}}' = -\boldsymbol{\mathcal{L}}\boldsymbol{\mathcal{V}}$ is globally asymptotically stable since all the eigenvalues of the $2n \times 2n$ matrix $\boldsymbol{\mathcal{L}}$ have a strictly positive real part, as can be easily verified[3]. This has the following consequence [35, 26] that is used below and that we cite without proof.

THEOREM 6.11 (Lyapunov). *The symmetric positive definite matrix*

$$\boldsymbol{\mathcal{M}} = \int_0^\infty e^{-\boldsymbol{\mathcal{L}}^T t}\, e^{-\boldsymbol{\mathcal{L}} t}\, dt$$

[3]In fact the eigenvalues of $\boldsymbol{\mathcal{L}}$ are the ones of $\mathbf{L}$, $1/\tau_i$s, with multiplicity 2.

satisfies

$$\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{L}} + \boldsymbol{\mathcal{L}}^T\boldsymbol{\mathcal{M}} = \mathrm{Id}_{2n}, \tag{6.8}$$

where Id_{2n} *is the* $2n \times 2n$ *identity matrix.*

The analog of equation (2.6) is readily found to be

$$\boldsymbol{\mathcal{V}}_t(\mathbf{r}, t) = -\boldsymbol{\mathcal{L}}\boldsymbol{\mathcal{V}}(\mathbf{r}, t) + \begin{bmatrix} \mathbf{0} \\ \int_\Omega \mathbf{W}(\mathbf{r}, \mathbf{r}', t)\mathbf{S}(\mathbf{V}(\mathbf{r}', t))\, d\mathbf{r}' \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_{\mathrm{ext}} \end{bmatrix} \tag{6.9}$$

The state is now $2n$-dimensional, the corresponding functional space is $\mathbf{L}^2_{2n}(\Omega)$ and the operator g is defined on the subspace $\mathbf{L}^2_n(\Omega)$ of $\mathbf{L}^2_{2n}(\Omega)$. It keeps all its previous properties. All proofs of the existence and uniqueness of a solution to (2.6) extend *mutatis mutandis* to this new setting.

Let us now examine the problem of the absolute stability of the solution, the analog of theorem 4.7.

THEOREM 6.12. *A sufficient condition for the solution of (2.6) to be absolutely stable is*

$$2\,\lambda_{\max}\, DS_m\, \|g\|_{\mathbf{L}^2_n(\Omega)} < 1,$$

where $\lambda_{\max}$ *is the largest eigenvalue of the* $2n \times 2n$ *matrix* $\boldsymbol{\mathcal{M}}$ *defined in theorem 6.11.*

Proof. We consider the equation

$$\boldsymbol{\mathcal{V}}_t(\mathbf{r}, t) = -\boldsymbol{\mathcal{L}}\boldsymbol{\mathcal{V}}(\mathbf{r}, t) + \begin{bmatrix} \mathbf{0} \\ \int_\Omega \mathbf{W}_{cm}(\mathbf{r}, \mathbf{r}', t)\underline{\mathbf{S}}(\mathbf{V}(\mathbf{r}', t))\, d\mathbf{r}' \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_{\mathrm{ext}} \end{bmatrix},$$

where $\mathbf{V}$ is the vector composed of the first n components of vector $\boldsymbol{\mathcal{V}}$ (the same convention will be used in the following for subvectors of $\boldsymbol{\mathcal{U}}$ and $\boldsymbol{\mathcal{X}}$). Let $\boldsymbol{\mathcal{U}}$ be its unique solution with initial condition $\boldsymbol{\mathcal{U}}(0) = \boldsymbol{\mathcal{U}}_0$, an element of $\mathbf{L}^2_{2n}(\Omega)$. Let also $\boldsymbol{\mathcal{V}}$ be the unique solution of the same equation with different initial conditions $\boldsymbol{\mathcal{V}}(0) = \boldsymbol{\mathcal{V}}_0$, another element of $\mathbf{L}^2_{2n}(\Omega)$. We introduce the new function $\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{V}} - \boldsymbol{\mathcal{U}}$ which satisfies

$$\begin{aligned} \boldsymbol{\mathcal{X}}_t(\mathbf{r}, t) = -\boldsymbol{\mathcal{L}}\boldsymbol{\mathcal{X}}(\mathbf{r}, t) + \begin{bmatrix} \mathbf{0} \\ \int_\Omega \mathbf{W}_{cm}(\mathbf{r}, \mathbf{r}', t)\, \mathbf{H}(\mathbf{X}, \mathbf{U})(\mathbf{r}', t)\, d\mathbf{r}' \end{bmatrix} = \\ -\boldsymbol{\mathcal{L}}\boldsymbol{\mathcal{X}}(\mathbf{r}, t) + \begin{bmatrix} \mathbf{0} \\ g_m(\mathbf{H}(\mathbf{X}, \mathbf{U}))(\mathbf{r}, t) \end{bmatrix} \end{aligned} \tag{6.10}$$

where the vector $\mathbf{H}(\mathbf{X}, \mathbf{U})$ is given by $\mathbf{H}(\mathbf{X}, \mathbf{U})(\mathbf{r}, t) = \underline{\mathbf{S}}(\mathbf{V}(\mathbf{r}, t)) - \underline{\mathbf{S}}(\mathbf{U}(\mathbf{r}, t)) = \underline{\mathbf{S}}(\mathbf{X}(\mathbf{r}, t) + \mathbf{U}(\mathbf{r}, t)) - \underline{\mathbf{S}}(\mathbf{U}(\mathbf{r}, t))$. Consider now the functional

$$V(\boldsymbol{\mathcal{X}}) = \frac{1}{2}\langle\, \boldsymbol{\mathcal{X}},\, \boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{X}}\,\rangle,$$

where the symmetric positive definite matrix $\boldsymbol{\mathcal{M}}$ can be seen as defining a metric on the state space. Its time derivative is $\langle\, \boldsymbol{\mathcal{X}},\, \boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{X}}_t\,\rangle$. We replace $\boldsymbol{\mathcal{X}}_t$ by its value from (6.10) in this expression to obtain

$$\frac{dV(\boldsymbol{\mathcal{X}})}{dt} = -\frac{1}{2}\left\langle \boldsymbol{\mathcal{X}},\, (\boldsymbol{\mathcal{L}}^T\boldsymbol{\mathcal{M}} + \boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{L}})\boldsymbol{\mathcal{X}} \right\rangle + \left\langle \boldsymbol{\mathcal{X}},\, \boldsymbol{\mathcal{M}} \begin{bmatrix} \mathbf{0} \\ g_m(\mathbf{H}(\mathbf{X}, \mathbf{U})) \end{bmatrix} \right\rangle$$

Using the property (6.8) of $\boldsymbol{\mathcal{M}}$ we obtain

$$\frac{dV(\boldsymbol{\mathcal{X}})}{dt} = -\frac{1}{2}\langle \boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{X}} \rangle + \left\langle \boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{M}} \left[\begin{array}{c} \mathbf{0} \\ g_m(\mathbf{H}(\mathbf{X}, \mathbf{U})) \end{array} \right] \right\rangle$$

We consider the second term in the righthand side of this equation. Since $\boldsymbol{\mathcal{M}}$ is symmetric

$$\begin{aligned} \left| \left\langle \boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{M}} \left[\begin{array}{c} \mathbf{0} \\ g_m(\mathbf{H}(\mathbf{X}, \mathbf{U})) \end{array} \right] \right\rangle \right| &= \left| \left\langle \boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{X}}, \left[\begin{array}{c} \mathbf{0} \\ g_m(\mathbf{H}(\mathbf{X}, \mathbf{U})) \end{array} \right] \right\rangle \right| \\ &\leq \|\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{X}}\|_{2n,2} \, \|g_m(\mathbf{H}(\mathbf{X}, \mathbf{U}))\|_{n,2} \leq \lambda_{\max} \|\boldsymbol{\mathcal{X}}\|_{2n,2} \, \|g_m(\mathbf{H}(\mathbf{X}, \mathbf{U}))\|_{n,2} \\ &\leq \lambda_{\max} \, \|\boldsymbol{\mathcal{X}}\|_{2n,2} \, \|g_m\|_{\mathbf{L}^2_n} \|\mathbf{H}(\mathbf{X}, \mathbf{U})\|_{n,2} \end{aligned} \quad (6.11)$$

The inequality $\|\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{X}}\|_{2n,2} \leq \lambda_{\max} \|\boldsymbol{\mathcal{X}}\|_{2n,2}$ is obtained using the spectral properties of the symmetric positive definite matrix $\boldsymbol{\mathcal{M}}$ and lemma 4.5.
Using the idea in the proof of lemma 3.1, we write $\mathbf{H}(\mathbf{X}, \mathbf{U}) = \mathcal{D}_m \mathbf{X}$, where $\mathcal{D}_m$ is a diagonal matrix whose diagonal elements are continuous functions with values between 0 and 1. Hence, because of lemma 4.5

$$\|\mathbf{H}(\mathbf{X}, \mathbf{U})\|_{n,2} = \|\mathcal{D}_m \mathbf{X}\|_{n,2} \leq \|\mathbf{X}\|_{n,2} \leq \|\boldsymbol{\mathcal{X}}\|_{2n,2}$$

We use this result and lemma 4.6 in (6.11) to obtain

$$\left| \left\langle \boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{M}} \left[\begin{array}{c} \mathbf{0} \\ g_m(\mathbf{H}(\mathbf{X}, \mathbf{U})) \end{array} \right] \right\rangle \right| \leq \lambda_{\max} \, DS_m \, \|g\|_{\mathbf{L}^2_n} \, \|\boldsymbol{\mathcal{X}}\|^2_{2n,2},$$

and the conclusion follows. □
All other theorems in sections 4, 5, 6 and in this section can be similarly extended to this more general setting. Complements on $\boldsymbol{\mathcal{M}}$ and $\lambda_{\max}$ can be found in appendix C.

7. Numerical examples. We consider two ($n = 2$) one-dimensional ($q = 1$) populations of neurons, population 1 being excitatory and population 2 inhibitory. The set Ω is the closed interval $[0, 1]$. We note x the spatial variable and f the spatial frequency variable. We consider Gaussian functions, noted $G_{ij}(x), i, j = 1, 2$, from which we define the connectivity functions. Hence we have $G_{ij} = \mathcal{G}(0, \sigma_{ij})$. We consider three cases. In the first case, section 7.1, we assume that the connectivity matrix is translation invariant (see sections 4.2 and 5.2). In the second case, section 7.2, we relax this assumption and study the stability of the homogeneous solutions. The third case, finally, section 7.3, covers the case of the locally homogeneous solutions and their stability. In this section we have $S_1(x) = S_2(x) = 1/(1 + e^{-x})$. Therefore

$$D\mathbf{S}_m = \left[\begin{array}{cc} 0 & \frac{1}{4} \\ \frac{1}{4} & 0 \end{array} \right],$$

hence $DS_m = 1/4$. We also choose $\tau_1 = \tau_2 = 4$, therefore $\tau_{\max} = 4$, and the product $DS_m \, \tau_{\max}$ is equal to 1.

7.1. The convolution case. We define $W_{ij}(x, x') = \pm\alpha_{ij} \, G_{ij}(x - x')$, where the α_{ij}s are positive weights and the sign determines whether population j excites (+) or inhibits (−) population i. As explained in section 4.2, $\mathbf{W}(\mathbf{r})$ is defined on the closed interval $\widehat{\Omega} = [-1, 1]$. For simplicity we use the approach described in section 4.2.1 and approximate the Fourier transform of $\mathbf{1}_{\widehat{\Omega}}(x)\mathbf{W}(x)$ by that of $\mathbf{W}(x)$ for which we have an analytical formula. This approximation is good as long as the σ_{ij}s are small with respect to 1.

The connectivity functions and their Fourier transforms are then given by

$$W_{ij}(x) = \pm \frac{\alpha_{ij}}{\sqrt{2\pi\sigma_{ij}^2}} e^{-\dfrac{x^2}{2\sigma_{ij}^2}} \quad \widetilde{W}_{ij}(f) = \pm\alpha_{ij} e^{-2\pi^2 f^2 \sigma_{ij}^2}$$

The matrices $\mathbf{W}(x)$ and $\widetilde{\mathbf{W}}(f)$ can be written

$$\mathbf{W}(x) = \left[\begin{array}{cc} \dfrac{\alpha_{11}}{\sqrt{2\pi\sigma_{11}^2}} e^{-\dfrac{x^2}{2\sigma_{11}^2}} & -\dfrac{\alpha_{12}}{\sqrt{2\pi\sigma_{12}^2}} e^{-\dfrac{x^2}{2\sigma_{12}^2}} \\ \dfrac{\alpha_{21}}{\sqrt{2\pi\sigma_{21}^2}} e^{-\dfrac{x^2}{2\sigma_{21}^2}} & -\dfrac{\alpha_{22}}{\sqrt{2\pi\sigma_{22}^2}} e^{-\dfrac{x^2}{2\sigma_{22}^2}} \end{array} \right]$$

$$\widetilde{\mathbf{W}}(f) = \left[\begin{array}{cc} \alpha_{11} e^{-2\pi^2 f^2 \sigma_{11}^2} & -\alpha_{12} e^{-2\pi^2 f^2 \sigma_{12}^2} \\ \alpha_{21} e^{-2\pi^2 f^2 \sigma_{21}^2} & -\alpha_{22} e^{-2\pi^2 f^2 \sigma_{22}^2} \end{array} \right]$$

Therefore we have, with the notations of theorem 4.9

$$\widetilde{\mathbf{W}}^*(f)\widetilde{\mathbf{W}}(f) \stackrel{def}{=} \mathbf{X}(f) = \left[\begin{array}{cc} A & C \\ C & B \end{array} \right].$$

It can be easily verified that

$$A = \tau_1 \left(\alpha_{11}^2 \tau_1 e^{-4\pi^2 \sigma_{11}^2 f^2} + \alpha_{21}^2 \tau_2 e^{-4\pi^2 \sigma_{21}^2 f^2} \right)$$

$$B = \tau_2 \left(\alpha_{22}^2 \tau_2 e^{-4\pi^2 \sigma_{22}^2 f^2} + \alpha_{12}^2 \tau_1 e^{-4\pi^2 \sigma_{12}^2 f^2} \right),$$

and

$$C = -\sqrt{\tau_1\tau_2} \left(\alpha_{21}\alpha_{22}\tau_2 e^{-2\pi^2(\sigma_{21}^2 + \sigma_{22}^2) f^2} + \alpha_{12}\alpha_{11}\tau_1 e^{-2\pi^2(\sigma_{12}^2 + \sigma_{11}^2) f^2} \right)$$

By construction the eigenvalues of the matrix $\mathbf{X}$ are positive (it is Hermitian), the largest one, $\lambda_{\max}$, being given by

$$\lambda_{\max} = \frac{1}{2} \left(A + B + \sqrt{(A-B)^2 + 4C^2} \right)$$

Introducing the parameters $A_1 = (\tau_1\alpha_{11})^2$, $A_2 = (\tau_2\alpha_{22})^2$, $r = \tau_1/\tau_2$, $x_1 = \alpha_{21}/\alpha_{11}$, $x_2 = \alpha_{12}/\alpha_{22}$ we can rewrite A, B and C as follows

$$A = A_1 \left(e^{-4\pi^2 \sigma_{11}^2 f^2} + \frac{x_1^2}{r} e^{-4\pi^2 \sigma_{21}^2 f^2} \right) \quad B = A_2 \left(e^{-4\pi^2 \sigma_{22}^2 f^2} + r x_2^2 e^{-4\pi^2 \sigma_{12}^2 f^2} \right),$$

and

$$C = -\sqrt{A_1 A_2} \left(\frac{x_1}{\sqrt{r}} e^{-2\pi^2(\sigma_{21}^2 + \sigma_{22}^2) f^2} + x_2 \sqrt{r} e^{-2\pi^2(\sigma_{12}^2 + \sigma_{11}^2) f^2} \right)$$

The necessary and sufficient condition that the two eigenvalues are less than 1 for all f is therefore $\lambda_{\max} < 1$ or

$$c(f) \stackrel{def}{=} 2 - A - B - \sqrt{(A-B)^2 + 4C^2} > 0 \quad \forall f \tag{7.1}$$

The function $c(f)$ depends on the spatial frequency f and the nine parameters A_1, A_2, x_1, x_2, r, and $\boldsymbol{\sigma}$, the 2×2 matrix σ_{ij}, $i, j = 1, 2$.

We have solved equation (2.6) on $\Omega = [0, 1]$. We have sampled the interval with 100 points corresponding to 100 neural masses. The input $\mathbf{I}_{\text{ext}}$ is equal to $[W_1(t), W_2(t)]^T$, where where the $W_i(t)$s, $i = 1, 2$ are realizations of independent standard Brownian/Wiener processes shown in figure 7.1. We know that the solution

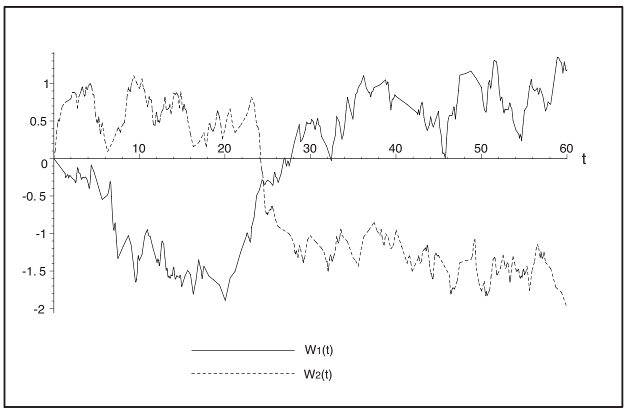


Figure 7.1. *The two coordinates of the input* $\mathbf{I}_{\text{ext}}(t)$ *are realizations of independent standard Brownian/Wiener processes. Time runs along the horizontal axis.*

is not homogeneous because $\mathbf{W}$ is translation invariant. This is illustrated in figure 7.2. The initial conditions are homogeneous and equal to $(0, 0)$ for all neural masses

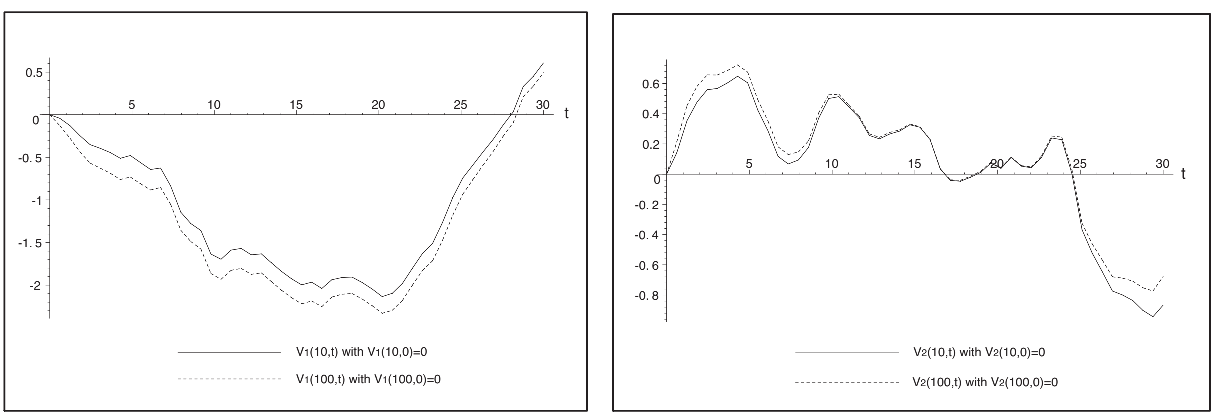


Figure 7.2. *An illustration of the fact that when the connectivity matrix is translation invariant there does not exist in general a homogeneous solution: the state vectors of different neural masses follow different trajectories even when the input and the initial condition are homogeneous (independent of the location x). Left side graph: the time variation of the first coordinate of the solution at points of coordinates* 0.1 *(continuous line) and 1 (dotted line) of the interval* $[0, 1]$. *Right side graph: same for the second coordinate. The initial condition is 0 in both cases.*

state vectors $\mathbf{V}$.

7.1.1. Absolute stability of the solution. Let us now study the absolute stability of the solutions. According to theorem 4.9 and the previous analysis, a sufficient condition for absolute stability is that $c(f) > 0$ for all frequencies f. As shown in figure 7.3, the following choice of the parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\sigma}$ produces a curve $c(f)$ that is positive for all frequencies.

$$\boldsymbol{\alpha} = \begin{bmatrix} 2 & 1.414 \\ 1.414 & 2 \end{bmatrix} \quad \boldsymbol{\sigma} = \begin{bmatrix} 1 & 0.1 \\ 0.1 & 1 \end{bmatrix}$$

We can check that this is indeed the case in figure 7.4 which shows the absolute

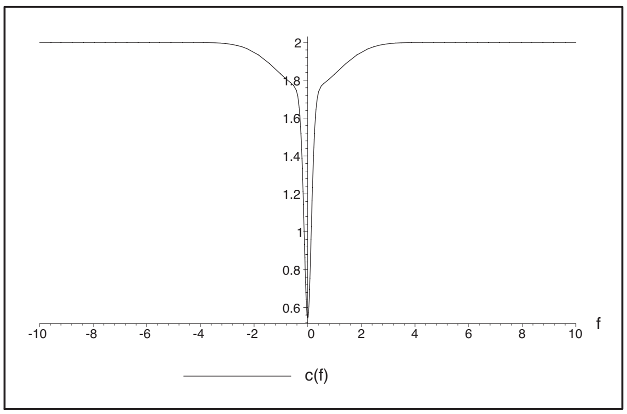
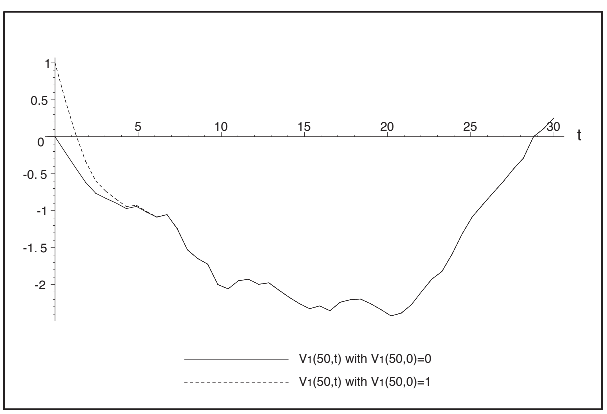


Figure 7.3. *The function $c(f)$ defined in (7.1) is positive for all spatial frequencies f: the system is absolutely stable.*

stability of the solution at the point of coordinate 0.5 of the interval $[0, 1]$.



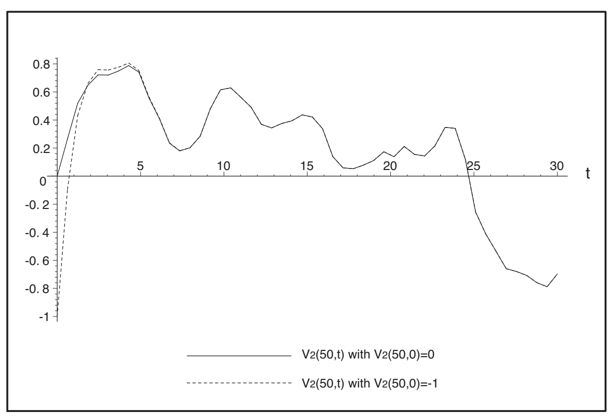


Figure 7.4. *An illustration of the absolute stability of the solution: independently of the choice of the initial condition, the trajectories of the state vector converge to a single trajectory. Results are shown for the neural mass of spatial coordinate* 0.5. *Left: the first coordinate of the state vector. Right: the second coordinate. Initial condition* $(0, 0)$, *continuous curves. Initial condition* $(1, -1)$, *dotted line.*

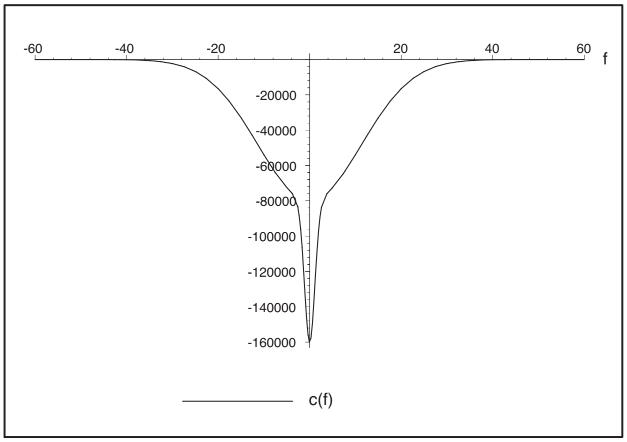


FIGURE 7.5. *The function $c(f)$ defined in (7.1) is not positive for all spatial frequencies f: the system may lose its absolute stability.*

7.1.2. Loss of absolute stability. The following choice of the parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\sigma}$ produces a curve $c(f)$ that is not positive for all frequencies, see figure 7.5.

$$\boldsymbol{\alpha} = \begin{bmatrix} 565.7 & 565.7 \\ 565.7 & 565.7 \end{bmatrix} \quad \boldsymbol{\sigma} = \begin{bmatrix} 0.01 & 0.01 \\ 0.1 & 0.1 \end{bmatrix}$$

Therefore absolute stability is not guaranteed. We show in figure 7.6 that this is indeed the case.

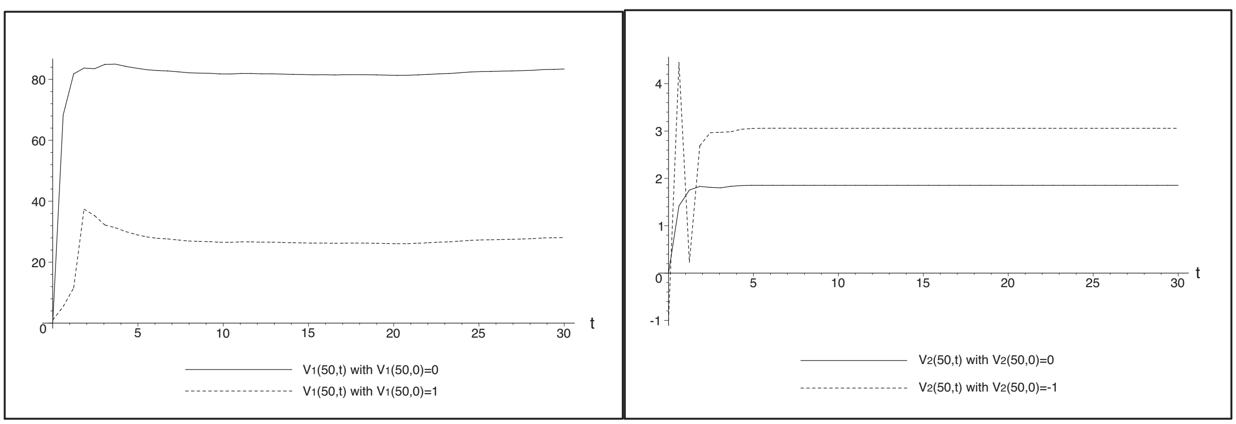


FIGURE 7.6. *An illustration of the lack of absolute stability of the solution: different initial conditions result in different trajectories of the state vectors. Results are shown for the neural mass of spatial coordinate* 0.5. *Left: the first coordinate of the state vector. Right: the second coordinate. Initial condition* $(0,0)$, *continuous curves. Initial condition* $(1,-1)$, *dotted curves.*

7.2. Homogeneous solutions. In the previous case the translation invariance of the connectivity matrix forbids the existence of homogeneous solutions. We can obtain a connectivity matrix satisfying condition (3.5) by defining

$$W_{ij}(x, x') = \pm \alpha \alpha_{ij} \frac{G_{ij}(x - x')}{\int_0^1 G_{ij}(x - y)\, dy} \quad i, j = 1, 2,$$

where α and the α_{ij}s are connectivity weights. These functions are well defined since the denominator is never equal to 0 and the resulting connectivity matrix is in $\mathbf{L}^2_{2\times 2}([0,1]\times[0,1])$. It is shown in figure 7.7. The values of the parameters are given in (7.2). Proposition 6.3 guarantees the existence and uniqueness of a homogeneous

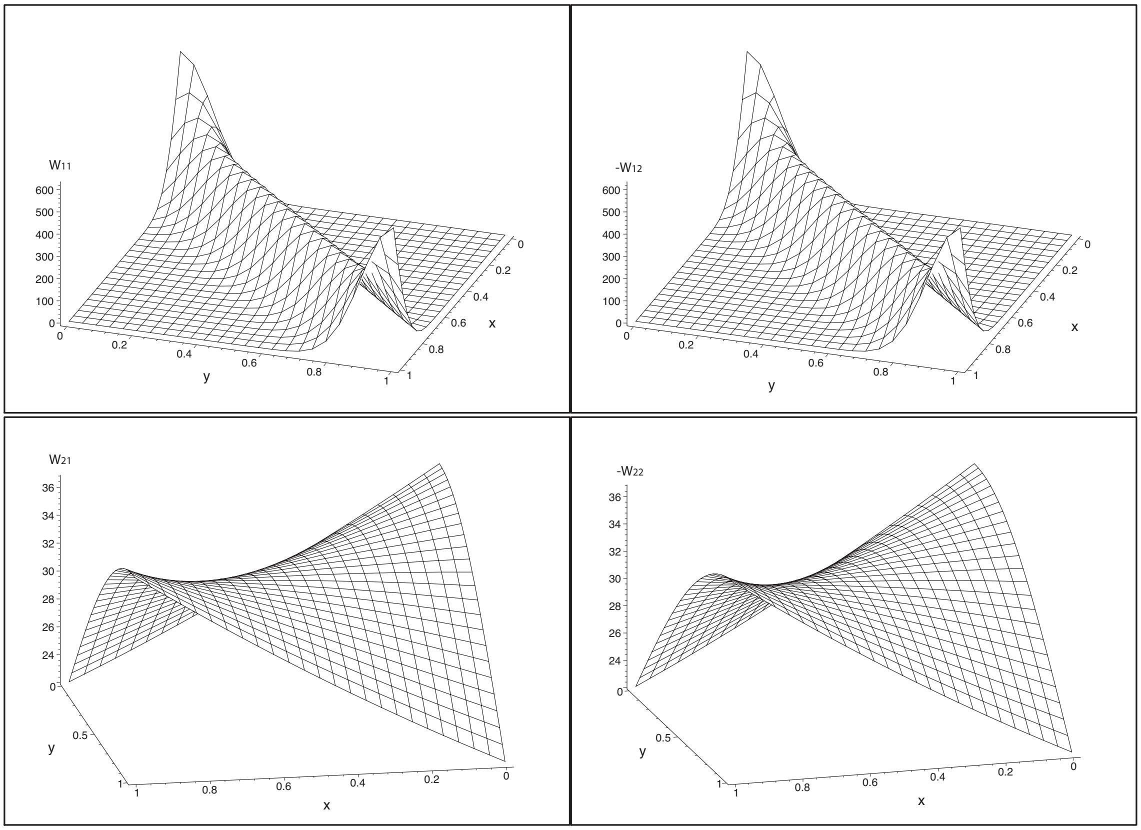


FIGURE 7.7. *The four elements of the matrix* $\mathbf{W}(x,y)$ *in the homogeneous case. Upper left:* $W_{11}(x,y)$*. Upper right:* $-W_{12}(x,y)$*. Lower left:* $W_{21}(x,y)$*. Lower right:* $-W_{22}(x,y)$*.*

solution for an initial condition in $\mathbf{L}^2_2(\Omega)$. According to theorem 5.2 and our choice for the values of $\tau_{\max}$ and DS_m, a sufficient condition for this solution to be absolutely stable is that $\|g*\|_{\mathcal{G}_c^\perp} < 1$.

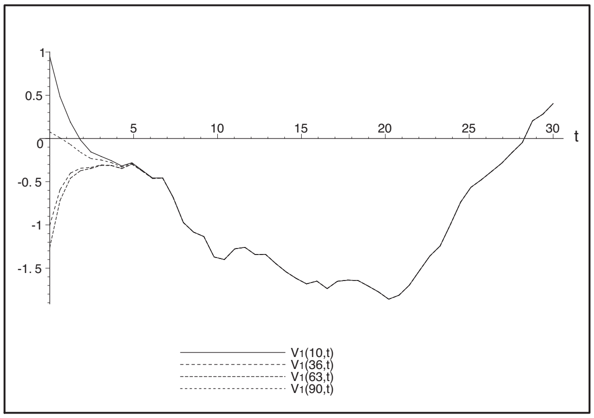
7.2.1. Absolute stability. The following values of the parameters

$$\boldsymbol{\alpha} = \begin{bmatrix} 5.20 & 5.20 \\ 2.09 & 2.09 \end{bmatrix} \quad \boldsymbol{\sigma} = \begin{bmatrix} 0.1 & 0.1 \\ 1 & 1 \end{bmatrix} \quad \tau_1 = \tau_2 = 1 \quad \alpha = 1/20 \qquad (7.2)$$

yield $\|g^*\|_{\mathcal{G}_c^\perp} \simeq 0.01$, hence the homogeneous solutions are absolutely stable. All operator norms have been computed using the method described in appendix A.3.

The initial conditions are drawn randomly and independently from the uniform distribution on $[-2,2]$. The input $\mathbf{I}_{\text{ext}}(t)$ is equal to $[W_1(t), W_2(t)]^T$, where the $W_i(t)$s, $i = 1,2$ are realizations of independent standard Brownian/Wiener processes shown in figure 7.1.

We show in figure 7.8 the complete synchronization of four (numbers 10, 36, 63 and 90) of the hundred neural masses that results from the absolute stability of the homogeneous solution.



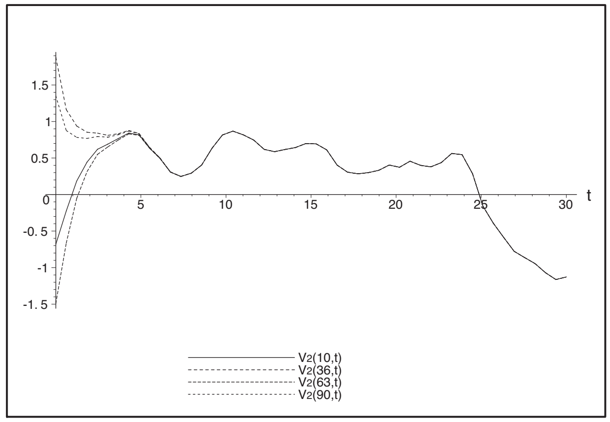


FIGURE 7.8. *The absolute stability of the homogeneous solution results in the complete synchronization of the neural masses. This is shown for four out of the hundred (coordinates 0.1, 0.36, 0.63 and 0.9). The input is shown in figure 7.1. The initial conditions are drawn independently from the uniform distribution on* $[-2, 2]$. *Left: The first components of the four state vectors. Right: the second components.*

7.2.2. Loss of absolute stability. If we increase the value of α it has the effect of increasing $\|g^*\|_{\mathcal{G}_c^\perp}$. The sufficient condition will eventually not be satisfied and we may lose the absolute stability of the homogeneous solution and hence the complete synchronization of the solution. Such a case is shown in figure 7.9 for $\alpha = 15$ corresponding to an operator norm $\|g^*\|_{\mathcal{G}_c^\perp} \simeq 2.62$.

7.3. Locally homogeneous solutions. We partition $\Omega = [0, 1]$ into $\Omega_1 = [0, 1/2[$ and $\Omega_2 = [1/2, 1]$, hence with the notations of section 6.2, $P = 2$. We can obtain a connectivity matrix satisfying condition (6.2) by defining

$$W_{ij}(x, x') = \begin{cases} \pm\alpha\alpha_{ij}(x, x')\dfrac{G_{ij}(x - x')}{\displaystyle\int_0^{1/2} G_{ij}(x - y)\, dy}, \; x' \in \Omega_1 \\ \pm\alpha\alpha_{ij}(x, x')\dfrac{G_{ij}(x - x')}{\displaystyle\int_{1/2}^{1} G_{ij}(x - y)\, dy}, \; x' \in \Omega_2 \end{cases},$$

with $\alpha_{ij}(x, x') = \alpha_{ij}^{kl}, \; x \in \Omega_k, \; x' \in \Omega_l, \; k, l = 1, 2.$
The resulting connectivity matrix is in $\mathbf{L}^2_{2\times 2}([0, 1] \times [0, 1])$. It is shown in figure 7.10.

The input $\mathbf{I}_{\text{ext}}(t)$ is equal to $[W_1(t), W_2(t)]^T$ in Ω_1 and to $[W_3(t), W_4(t)]^T$ in Ω_2, where the $W_i(t)$s, $i = 1, \cdots, 4$ are realizations of independent standard Brownian/Wiener processes shown in figure 7.11. Hence it is homogeneous in Ω_1 (respectively in Ω_2) but not in $\Omega = \Omega_2 \cup \Omega_2$. According to proposition 6.3 there exists a unique solution to (2.6) for a given initial condition in $\mathbf{L}^2_2(\Omega)$. This solution is locally homogeneous if the initial condition is locally homogeneous (theorem 6.5) given the fact that the input is locally homogeneous.

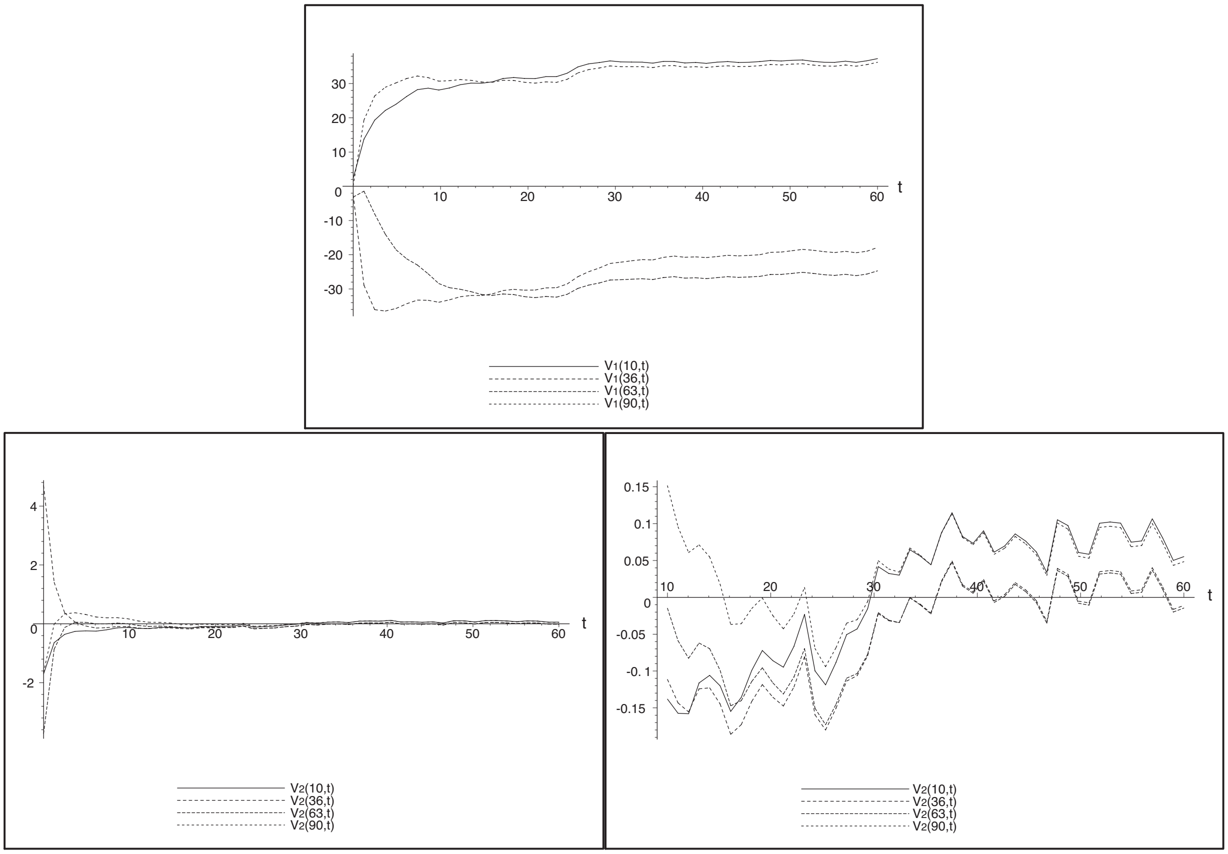


FIGURE 7.9. *The loss of the absolute stability of the homogeneous solution results in the loss of the complete synchronization of neural masses when the sufficient condition of theorem 5.2 is not satisfied. This is shown for four out of the hundred (coordinates 0.1, 0.36, 0.63 and 0.9). The input is the same as in the previous example. Top: The first components of the four state vectors. Bottom left: The second components of the four state vectors for* $0 \leq t \leq 60s$*. Bottom right: Zoom on the second components of the four state vectors for* $10 \leq t \leq 60s$*.*

7.3.1. Absolute stability. The parameters

$$\begin{array}{ll} \boldsymbol{\alpha}^{11} = \begin{bmatrix} 5.21 & 0.23 \\ 0.23 & 5.21 \end{bmatrix} & \boldsymbol{\alpha}^{12} = \begin{bmatrix} 4.98 & 0.34 \\ 0.34 & 4.98 \end{bmatrix} \\ \boldsymbol{\alpha}^{21} = \begin{bmatrix} 4.75 & 0.45 \\ 0.45 & 4.75 \end{bmatrix} & \boldsymbol{\alpha}^{22} = \begin{bmatrix} 5.39 & 0.13 \\ 0.13 & 5.39 \end{bmatrix} \end{array} \quad \boldsymbol{\sigma} = \begin{bmatrix} 0.05 & 0.075 \\ 0.1 & 0.03 \end{bmatrix}$$

result in an operator norm $\|g^*\|_{\mathcal{G}_c^2\perp} \simeq 0.23$. Therefore, according to theorem 6.7, the locally homogeneous solutions are absolutely stable, resulting in the complete local synchronization of the neural masses (within Ω_1 and Ω_2).

We show in figure 7.12 (respectively figure 7.13) the complete synchronization of two neural masses (numbers 10 and 36) in Ω_1 (respectively two neural masses (numbers 63 and 90) in Ω_2). The initial conditions are drawn randomly and independently from the uniform distribution on $[-10, 10]$ and $[-2, 2]$ for Ω_1 and on $[-20, 20]$ and $[-2, 2]$ for Ω_2.

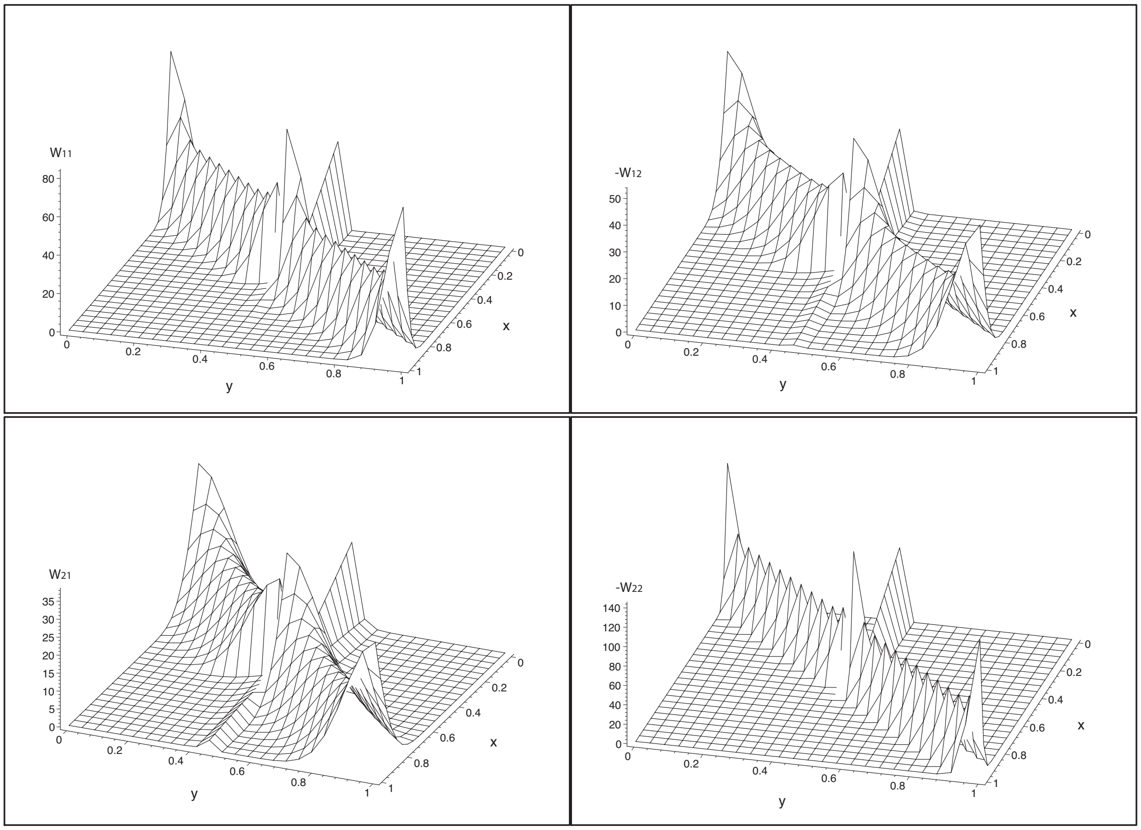


FIGURE 7.10. *The four elements of the matrix* $\mathbf{W}(x, y)$ *in the locally homogeneous case. Upper left:* $W_{11}(x, y)$*. Upper right:* $-W_{12}(x, y)$*. Lower left:* $W_{21}(x, y)$*. Lower right:* $-W_{22}(x, y)$*.*

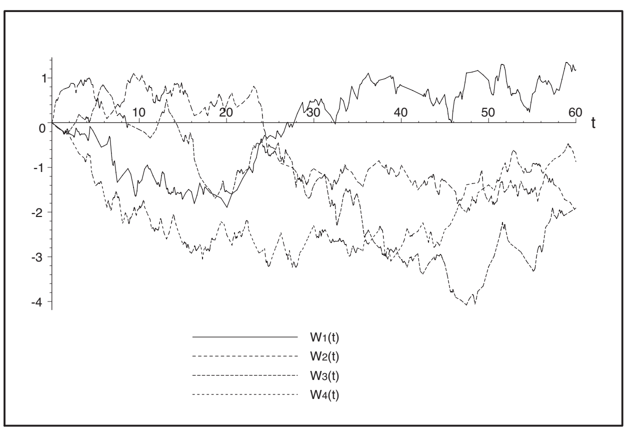


FIGURE 7.11. *The two coordinates of the input* $\mathbf{I}_{\text{ext}}(t)$ *in* Ω_1 *and* Ω_2 *are realizations of four independent Wiener processes (*W_1 *and* W_2 *are identical to those shown in figure 7.1).*

7.3.2. Loss of absolute stability. If we increase the value of α, it has the effect of increasing $\|g^*\|_{\mathcal{G}_c^{2,\perp}}$. The sufficient condition for absolute stability will eventually not be satisfied and we may lose the absolute stability of the locally homogeneous

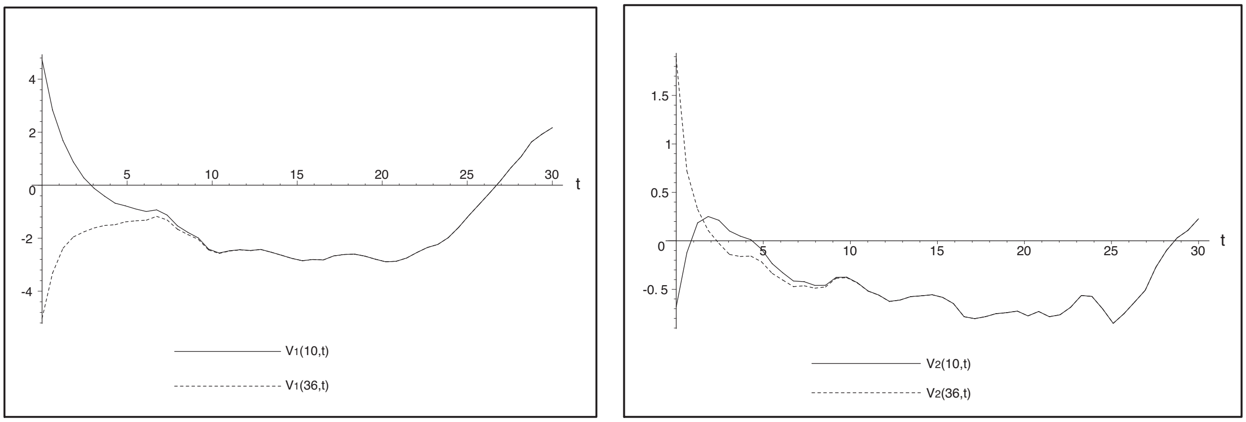


Figure 7.12. *The complete synchronization of two neural masses in Ω_1 of coordinates 0.1 and 0.36. The input is shown in figure 7.11. Left: the first components of the two state vectors. Right: the second components of the two state vectors.*

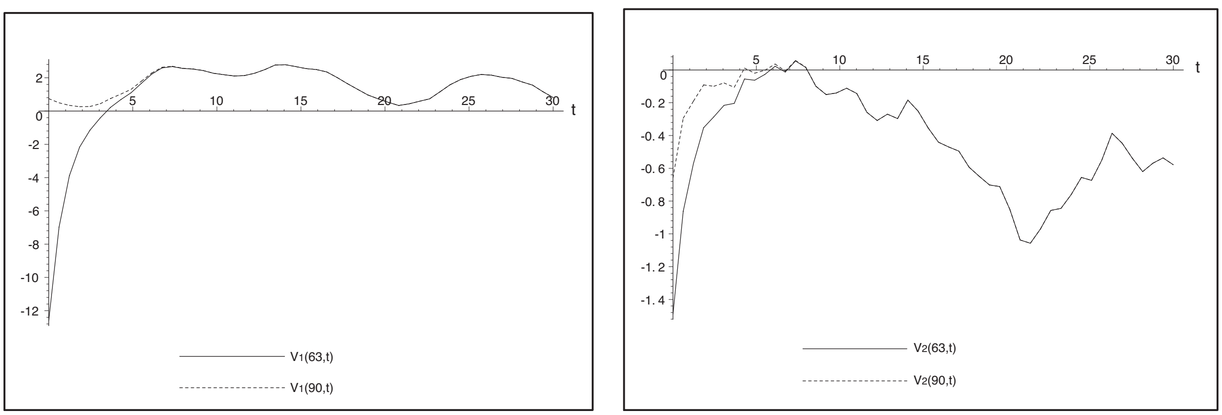


Figure 7.13. *The complete synchronization of two neural masses in Ω_2 of coordinates 0.63 and 0.9. The input is shown in figure 7.11. Left: the first components of the two state vectors. Right: the second components of the two state vectors.*

solution and hence the complete local synchronization of the solution. This is shown in figures 7.14 and 7.15 for $\alpha = 10$ corresponding to an operator norm $\|g_m^{L\,*}\|_{\mathcal{G}_c^{2\,\perp}} \simeq 2.3$.

7.4. Pseudo locally homogeneous solutions and their absolute stability. As mentioned at the end of section 6.2.2, even if the connectivity function does not satisfy condition (6.2) and the operator g^* satifies only the condition of theorem 4.7 but not that of theorem 6.7 the existence of locally homogeneous solutions is not guaranteed but the absolute stability of the solution is, because of proposition 6.8. As shown in figures 7.16 and 7.17 these solutions can be very close to being locally homogeneous and thus enjoy the property of complete local synchronization. This is potentially very interesting from the application viewpoint since one may say that if the system admits homogeneous solutions and if they are absolutely stable it can have locally homogeneous solutions without "knowing" the partition, and they are absolutely stable. These results are illustrated by two animations (files pseudo-

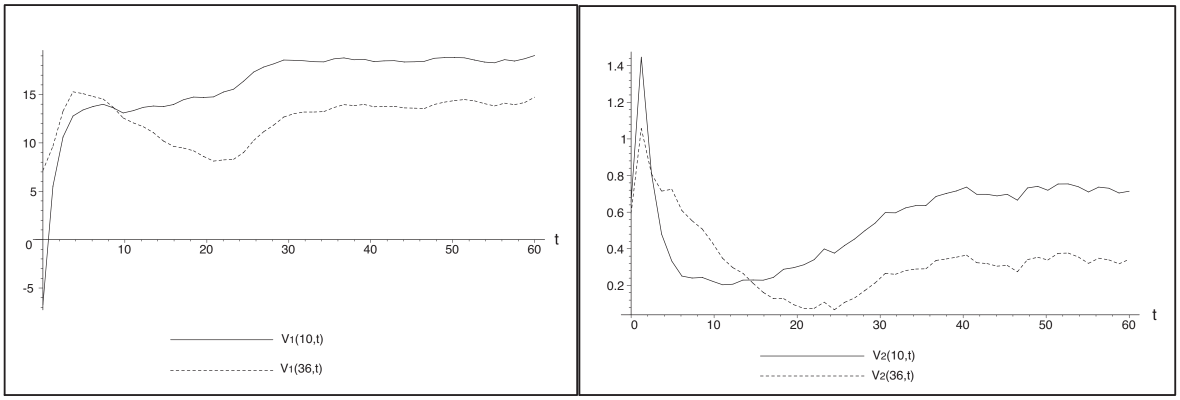


Figure 7.14. *The loss of the absolute stability of the locally homogeneous solution results in the loss of the complete local synchronization of neural masses when the sufficient condition of theorem 6.7 is not satisfied. This is shown for two out of the fifty (coordinates 0.1, 0.36) neural masses in* Ω_1. *The input is the same as in the previous example. Left: The first components of the two state vectors. Right: The second components of the two state vectors.*

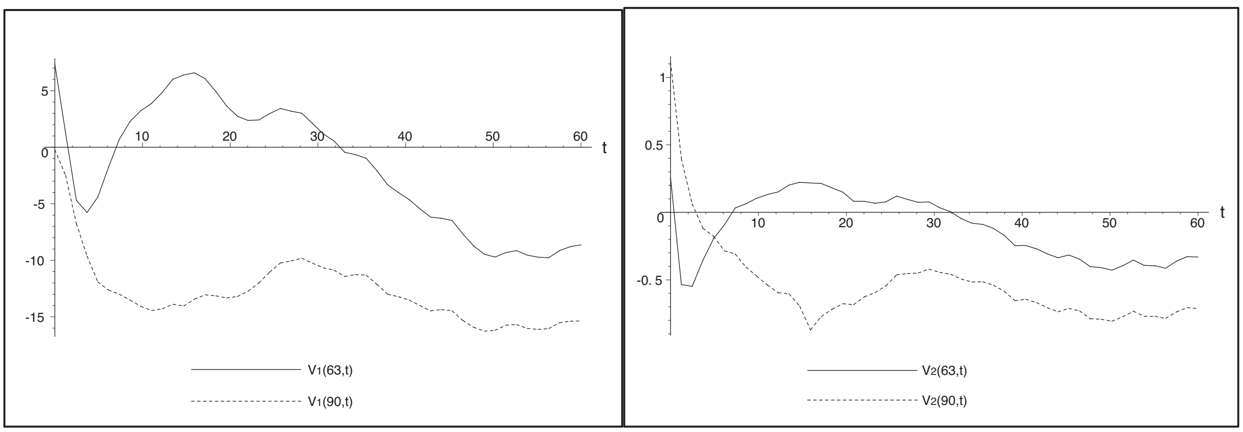


Figure 7.15. *The loss of the absolute stability of the locally homogeneous solution results in the loss of the complete local synchronization of neural masses when the sufficient condition of theorem 6.7 is not satisfied. This is shown for two out of the fifty (coordinates 0.63, 0.9) neural masses in* Ω_2. *The input is the same as in the previous example. Left: The first components of the two state vectors. Right: The second components of the two state vectors.*

local-homogeneous-synchro-1.gif and pseudo-local-homogeneous-synchro-2.gif of the supplemental material). The axes are the same as previously.

8. Conclusion. We have studied the existence, uniqueness, and absolute stability of a solution of two examples of nonlinear integro-differential equations that describe the spatio-temporal activity of sets of neural masses. These equations involve space and time varying, possibly non-symmetric, intra-cortical connectivity kernels. The time dependency of the connectivity kernels opens the door to the study, in this framework, of plasticity and learning. Contributions from white matter afferents are represented by external inputs. Sigmoidal nonlinearities arise from the relation between average membrane potentials and instantaneous firing rates.

The intra-cortical connectivity functions have been shown to naturally define

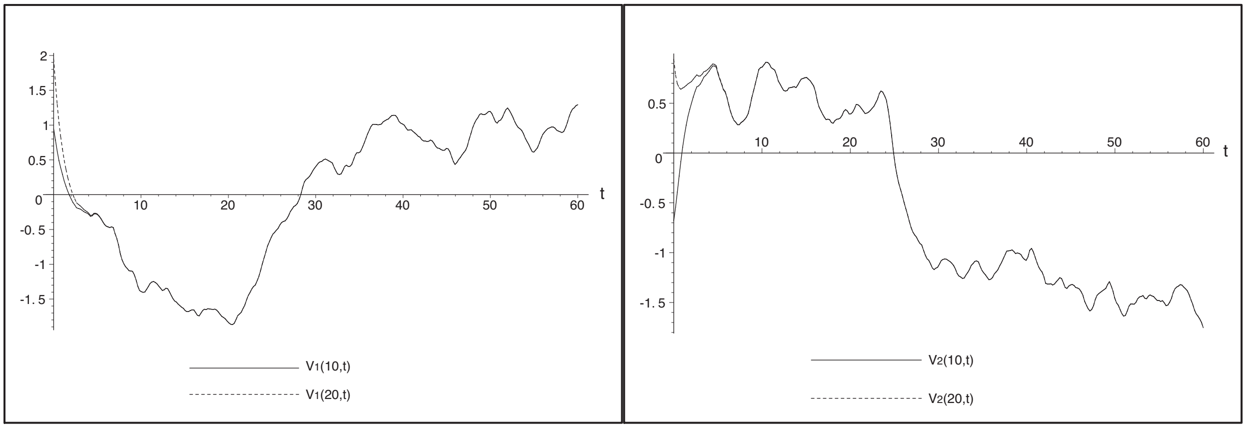


Figure 7.16. *The connectivity function satisfies condition (3.5) but not condition (6.2) and the operator g^* satifies the condition of theorem 5.2, not that of theorem 6.7. The input is locally homogeneous, as in figures 7.12 and 7.13. The solution is absolutely stable, because of theorem 5.2 and almost locally homogeneous. Something very close to complete local synchronization is observed. This is shown for two out of the fifty (coordinates 0.1, 0.2) neural masses in Ω_1. Left: The first components of the two state vectors. Right: The second components of the two state vectors.*

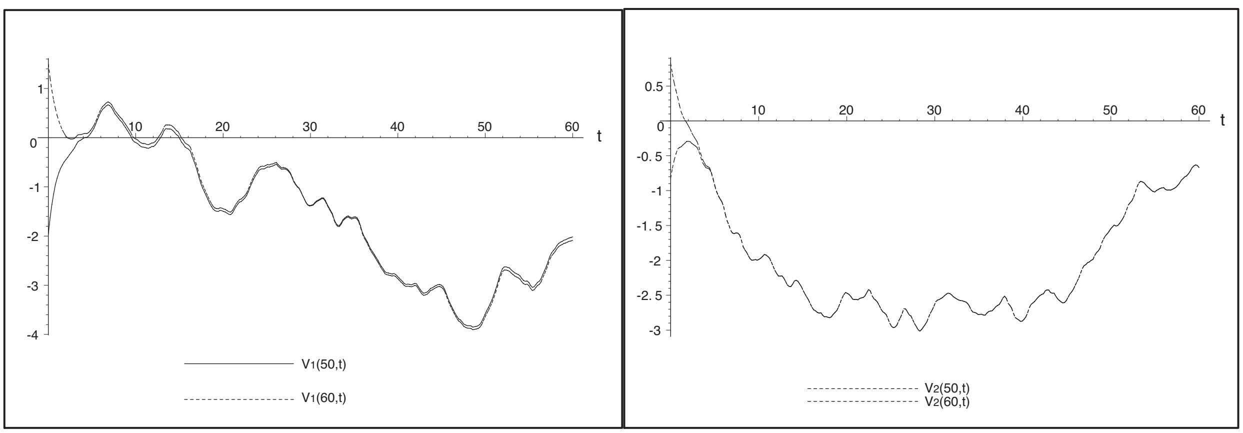


Figure 7.17. *Same as in figure 7.16. The complete local synchronization is shown for two out of the fifty (coordinates 0.5, 0.6) neural masses in Ω_2.*

compact operators of the functional space of interest. Using methods of functional analysis, we have given sufficient conditions for the existence and uniqueness of a solution of these equations for general, homogeneous (i.e. independent of the spatial variable), and locally homogeneous inputs. In all cases we have provided sufficient conditions for the solutions to be absolutely stable, that is to say independent of the initial state of the neural field. These conditions involve the connectivity functions, the maximum slopes of the sigmoids, as well as the time constants used to described the time variation of the postsynaptic potentials. They are very relevant to neuroscience where dynamical neuronal systems that "recognize" a given input regardless of their initial state are quite common.

To our knowledge this is the first time that such a complete analysis of the problem of the existence and uniqueness of a solution of these equations has been obtained. An important contribution also is the analysis of the absolute stability of these solutions which had been considered as much more difficult to perform than the linear stability analysis which it implies.

The reason why we have been able to complete this work programme is our use of the functional analysis framework and the theory of compact operators in a Hilbert space with the effect of providing simple mathematical answers to some of the questions raised by modellers in neuroscience.

Future work includes adding delays to account for the distance travelled by the spikes down the axons and taking into account specific forms of the time variation of the connectivity matrixes in the context of neural plasticity.

Acknowledgments. This work was partially supported by Elekta Instrument AB.

Appendix A. Notations and background material.

A.1. Matrix norms and spaces of functions. We note $\mathcal{M}_{n\times n}$ the set of $n\times n$ real matrices. We consider the matrix norm,

$$\|\mathbf{M}\|_\infty = \max_i \sum_j |M_{ij}|$$

We note $\mathbf{C}_{n\times n}(\Omega)$ the set of continuous functions from Ω to $\mathcal{M}_{n\times n}$ with the infinity norm. This is a Banach space for the norm induced by the infinity norm on $\mathcal{M}_{n\times n}$. Let $\mathbf{M}$ be an element of $\mathbf{C}_{n\times n}(\Omega)$, we note and define $\|\mathbf{M}\|_{n\times n,\infty}$ as

$$\|\mathbf{M}\|_{n\times n,\infty} = \sup_{\mathbf{r}\in\Omega} \max_i \sum_j |M_{ij}(\mathbf{r})| = \max_i \sup_{\mathbf{r}\in\Omega} \sum_j |M_{ij}(\mathbf{r})|$$

We also note $\mathbf{C}_n(\Omega)$ the set of continuous functions from Ω to $\mathbb{R}^n$ with the infinity norm. This is also a Banach space for the norm induced by the infinity norm of $\mathbb{R}^n$. Let $\mathbf{x}$ be an element of $\mathbf{C}_n(\Omega)$, we note and define $\|\mathbf{x}\|_{n,\infty}$ as

$$\|\mathbf{x}\|_{n,\infty} = \sup_{\mathbf{r}\in\Omega} \|\mathbf{x}(\mathbf{r})\|_\infty = \sup_{\mathbf{r}\in\Omega} \max_i |x_i(\mathbf{r})| = \max_i \sup_{\mathbf{r}\in\Omega} |x_i(\mathbf{r})|$$

We can similarly define the norm $\|.\|_{n\times n,\infty}$ (resp. $\|.\|_{n,\infty}$) for the space $\mathbf{C}_{n\times n}(\Omega\times\Omega)$ (resp. $\mathbf{C}_n(\Omega\times\Omega)$).
We have the following

LEMMA A.1. *Given $\mathbf{x}\in\mathbf{C}_n(\Omega)$ and $\mathbf{M}\in\mathbf{C}_{n\times n}(\Omega)$ we have*

$$\|\mathbf{M}\,\mathbf{x}\|_{n,\infty} \le \|\mathbf{M}\|_{n\times n,\infty}\,\|\mathbf{x}\|_{n,\infty}$$

More precisely, we have for all $\mathbf{r}\in\Omega$

$$\|\mathbf{M}(\mathbf{r})\,\mathbf{x}(\mathbf{r})\|_\infty \le \|\mathbf{M}(\mathbf{r})\|_\infty \|\mathbf{x}(\mathbf{r})\|_\infty$$

The same results hold for $\Omega\times\Omega$ instead of Ω.

Proof. Let $\mathbf{y} = \mathbf{M}\,\mathbf{x}$, we have

$$y_i(\mathbf{r}) = \sum_j M_{ij}(\mathbf{r}) x_j(\mathbf{r})$$

and therefore

$$|y_i(\mathbf{r})| \le \sum_j |M_{ij}(\mathbf{r})|\,|x_j(\mathbf{r})| \le \sum_j |M_{ij}(\mathbf{r})|\,\|\mathbf{x}(\mathbf{r})\|_\infty,$$

so, taking the $\max_i$

$$\|\mathbf{y}(\mathbf{r})\|_\infty \le \|\mathbf{M}(\mathbf{r})\|_\infty\,\|\mathbf{x}(\mathbf{r})\|_\infty$$

from which the first statement easily comes. ∎
We also consider the Frobenius norm on $\mathcal{M}_{n\times n}$

$$\|\mathbf{M}\|_F = \sqrt{\sum_{i,j=1}^{n} M_{ij}^2},$$

and consider the space $\mathbf{L}^2_{n\times n}(\Omega\times\Omega)$ of the functions from $\Omega\times\Omega$ to $\mathcal{M}_{n\times n}$ whose Frobenius norm is in $L^2(\Omega\times\Omega)$. If $\mathbf{W}\in\mathbf{L}^2_{n\times n}(\Omega\times\Omega)$ we note $\|\mathbf{W}\|_F^2=\int_{\Omega\times\Omega}\|\mathbf{W}(\mathbf{r},\mathbf{r}')\|_F^2\,d\mathbf{r}\,d\mathbf{r}'$. Note that this implies that each element w_{ij}, $i,\ j=1,\cdots,n$ is in $L^2(\Omega\times\Omega)$. We note $\mathbf{L}^2_n(\Omega)$ the set of square-integrable mappings from Ω to $\mathbb{R}^n$ and $\|\mathbf{x}\|_{n,2}=(\sum_j\|x_j\|_2^2)^{1/2}$ the corresponding norm. We have the following

Lemma A.2. *Given* $\mathbf{x}\in\mathbf{L}^2_n(\Omega)$ *and* $\mathbf{W}\in\mathbf{L}^2_{n\times n}(\Omega\times\Omega)$, *we define* $\mathbf{y}(\mathbf{r})=\int_\Omega\mathbf{W}(\mathbf{r},\mathbf{r}')\mathbf{x}(\mathbf{r}')\,d\mathbf{r}'$. *This integral is well defined for almost all* $\mathbf{r}$, $\mathbf{y}$ *is in* $\mathbf{L}^2_n(\Omega)$ *and we have*

$$\|\mathbf{y}\|_{n,2}\le\|\mathbf{W}\|_F\,\|\mathbf{x}\|_{n,2}.$$

Proof. Since each w_{ij} is in $L^2(\Omega\times\Omega)$, $w_{ij}(\mathbf{r},.)$ is in $L^2(\Omega)$ for almost all $\mathbf{r}$, thanks to Fubini's theorem. So $w_{ij}(\mathbf{r},.)x_j(.)$ is integrable for almost all $\mathbf{r}$ from what we deduce that $\mathbf{y}$ is well-defined for almost all $\mathbf{r}$. Next we have

$$|y_i(\mathbf{r})|\le\sum_j\left|\int_\Omega w_{ij}(\mathbf{r},\mathbf{r}')\,x_j(\mathbf{r}')\,d\mathbf{r}'\right|$$

and (Cauchy-Schwarz):

$$|y_i(\mathbf{r})|\le\sum_j\left(\int_\Omega w_{ij}^2(\mathbf{r},\mathbf{r}')\,d\mathbf{r}'\right)^{1/2}\ \|x_j\|_2,$$

from where it follows that (Cauchy-Schwarz again, discrete version):

$$|y_i(\mathbf{r})|\le\left(\sum_j\|x_j\|_2^2\right)^{1/2}\left(\sum_j\int_\Omega w_{ij}^2(\mathbf{r},\mathbf{r}')\,d\mathbf{r}'\right)^{1/2}=\|\mathbf{x}\|_{n,2}\left(\sum_j\int_\Omega w_{ij}^2(\mathbf{r},\mathbf{r}')\,d\mathbf{r}'\right)^{1/2},$$

from what it follows that $\mathbf{y}$ is in $\mathbf{L}^2_n(\Omega)$ (thanks again to Fubini's theorem) and

$$\|\mathbf{y}\|_{n,2}^2\le\|\mathbf{x}\|_{n,2}^2\sum_{i,j}\int_{\Omega\times\Omega}w_{ij}^2(\mathbf{r},\mathbf{r}')\,d\mathbf{r}'\,d\mathbf{r}=\|\mathbf{x}\|_{n,2}^2\,\|\mathbf{W}\|_F^2.$$

□

A.2. Banach space-valued functions. A useful viewpoint that is used in this article is to consider the state vector of the neural field as a mapping from a closed time interval J containing the origin 0 into one of the spaces discussed in the previous section. We note $C(\mathrm{J};\mathbf{C}_\mathrm{n}(\Omega))$ the set of continuous mappings from J to the Banach space $\mathbf{C}_n(\Omega)$ and $C(\mathrm{J};\mathbf{L}^2_\mathrm{n}(\Omega))$ the set of continuous mappings from J to the Hilbert (hence Banach) space $\mathbf{L}^2_n(\Omega)$, see, e.g., [11].

A.3. Computation of operator norms. We give a method to compute the norms $\|g\|_\mathcal{G}$ and $\|g^*\|_{\mathcal{G}_c^\perp}$ for an operator g of the form

$$g(\mathbf{x})(\mathbf{r})=\int_\Omega\mathbf{W}(\mathbf{r},\mathbf{r}')\,\mathbf{x}(\mathbf{r}')\,d\mathbf{r}'.$$

Since $\mathcal{G}$ (respectively $\mathcal{G}_c^\perp$) is dense in the Hilbert space $\mathbf{L}^2(\Omega)$ (respectively $\mathbf{L}^2_0(\Omega)$, the subspace of $\mathbf{L}^2(\Omega)$ of functions with zero mean), we have $\|g\|_\mathcal{G}$=$\|g\|_{\mathbf{L}^2}$ and $\|g^*\|_{\mathcal{G}_c^\perp}=\|g^*\|_{\mathbf{L}^2_0}$. We consider the compact self-adjoint operators

$$G=g^*g:\mathbf{L}^2\to\mathbf{L}^2$$

and

$$G_c^{\perp} = g^* \mathcal{P} g : \mathbf{L}_0^2 \to \mathbf{L}_0^2,$$

where $\mathcal{P}$ is the orthogonal projection on $\mathbf{L}_0^2$. We compute the norms of the two self-adjoint positive operators G and $G_c^{\perp}$, and use the relations

$$\|G\|_{\mathbf{L}^2} = \|g\|_{\mathbf{L}^2}^2,$$

and

$$\|G_c^{\perp}\|_{\mathbf{L}_0^2} = \|g^* \mathcal{P}^* \mathcal{P} g\|_{\mathbf{L}_0^2} = \|g^* \mathcal{P}^*\|_{\mathbf{L}_0^2}^2 = \|g^*\|_{\mathbf{L}_0^2}^2.$$

Let T be a compact self-adjoint positive operator on a Hilbert space $\mathcal{H}$. Its largest eigenvalue is $\lambda = \|T\|_{\mathcal{H}}$. Let $x \in \mathcal{H}$. If $x \notin \mathrm{Ker}(\lambda \mathrm{Id} - T)^{\perp}$, then, according to, e.g., [9],

$$\lim_{n \to \infty} \|T^n x\|_{\mathcal{H}} / \|T^{n-1} x\|_{\mathcal{H}} = \lambda.$$

This method can be applied to g_m and h_m, and generalized to the computation of the $\|.\|_{\mathcal{G}_c^P{}^{\perp}}$ norm.

Appendix B. Global existence of solutions. In this appendix, we complete the proof of proposition (3.4) by computing the constant $\tau > 0$ such that for any initial condition $(t_0, \mathbf{V}_0) \in \mathbb{R} \times \mathcal{F}$, the existence and uniqueness of the solution $\mathbf{V}$ is guaranteed on the closed interval $[t_0 - \tau, t_0 + \tau]$.
We refer to [2] and exploit the

THEOREM B.1. *Let $\mathcal{F}$ be a Banach space and $c > 0$. We consider the initial value problem:*

$$\begin{cases} \mathbf{V}'(t) & = & f(t, \mathbf{V}(t)) \\ \mathbf{V}(t_0) & = & \mathbf{V}_0 \end{cases}$$

for $|t - t_0| < c$ where $\mathbf{V}_0$ is an element of $\mathcal{F}$ and $f : [t_0 - c, t_0 + c] \times \mathcal{F} \to \mathcal{F}$ is continuous. Let $b > 0$. We define the set $Q_{b,c} \equiv \{(t, \mathbf{X}) \in \mathbb{R} \times \mathcal{F},\ |t - t_0| \le c \quad \text{and} \quad \|\mathbf{X} - \mathbf{V}_0\| \le b\}$. Assume the function $f : Q_{b,c} \to \mathcal{F}$ is continuous and uniformly Lipschitz continuous with respect to its second argument, ie

$$\|f(t, \mathbf{X}) - f(t, \mathbf{Y})\| \le K_{b,c} \|\mathbf{X} - \mathbf{Y}\|,$$

where $K_{b,c}$ is a constant independent of t.
Let $M_{b,c} = \sup_{Q_{b,c}} \|f(t, \mathbf{X})\|$ and $\tau_{b,c} = \min\{b/M_{b,c}, c\}$.
Then the initial value problem has a unique continuously differentiable solution $\mathbf{V}(.)$ defined on the interval $[t_0 - \tau_{b,c}, t_0 + \tau_{b,c}]$.

In our case, $f = f_v$ and all the hypotheses of the theorem hold, thanks to proposition 3.2 and the hypotheses of proposition 3.4, with

$$K_{b,c} = \|\mathbf{L}\|_{\infty} + |\Omega|\, D S_m \sup_{|t - t_0| \le c} \|\mathbf{W}(\cdot, \cdot, t)\|_{n \times n, \infty},$$

where the sup is well defined (continuous function on a compact domain).
We have

$$M_{b,c} \le \|\mathbf{L}\|_{\infty} \left(\|\mathbf{V}_0\|_{n,\infty} + b\right) + |\Omega|\, S_m\, W + I,$$

where $W = \sup_{|t-t_0|\le c} \|\mathbf{W}(\cdot,\cdot,t)\|_{n\times n,\infty}$ and $I = \sup_{|t-t_0|\le c} \|\mathbf{I}_{\text{ext}}(\cdot,t)\|_{n,\infty}$.
So

$$b/M_{b,c} \ge \frac{1}{\|\mathbf{L}\|_\infty + \frac{\|\mathbf{L}\|_\infty \|\mathbf{V}_0\|_{n,\infty} + |\Omega|\, S_m\, W + I}{b}}.$$

Hence, for $c \ge \frac{1}{2\|\mathbf{L}\|_\infty}$ and b big enough, we have $\tau_{b,c} \ge \frac{1}{2\|\mathbf{L}\|_\infty}$ and we can set $\tau = \frac{1}{2\|\mathbf{L}\|_\infty}$.

A similar proof applies to the case $f = f_a$ and the one of proposition 6.3.

Appendix C. Complements on $\boldsymbol{\mathcal{M}}$ and $\lambda_{\max}$. Expressing the exponential as a power series in the definition of $\boldsymbol{\mathcal{M}}$ and computing the powers of the block matrix $\boldsymbol{\mathcal{L}}$, we easily find a block expression of $\boldsymbol{\mathcal{M}}$ depending on $\mathbf{L}$

$$\boldsymbol{\mathcal{M}} = \begin{pmatrix} \mathbf{L}/4 + 5\mathbf{L}^{-1}/4 & \mathbf{L}^{-2}/2 \\ \mathbf{L}^{-2}/2 & \mathbf{L}^{-1}/4 + \mathbf{L}^{-3}/4 \end{pmatrix}.$$

$\boldsymbol{\mathcal{M}}$ is diagonalizable, as a symmetric positive definite matrix, and has at most $2n$ distinct eigenvalues. More precisely, these eigenvalues are the roots of the second order polynomials

$$\lambda^2 - \left(\frac{1}{4\,\tau_i} + \frac{3\,\tau_i}{2} + \frac{\tau_i^3}{4}\right)\lambda + \frac{1}{16} + \frac{3\,\tau_i^2}{8} + \frac{\tau_i^4}{16}, \quad 1 \le i \le n.$$

The largest eigenvalue of each of these polynomials is

$$\lambda(\tau_i) = \frac{1}{8\,\tau_i}\left(1 + 6\,\tau_i^2 + \tau_i^4 + \sqrt{1 + 8\,\tau_i^2 + 14\,\tau_i^4 + 8\,\tau_i^6 + \tau_i^8}\right),$$

so that $\lambda_{\max}$ is simply $\max_i \lambda(\tau_i)$. Note that since the function $\lambda(\tau)$ is not monotonous, $\lambda_{\max}$ is not necessarily equal to $\lambda(\tau_{\max})$.

Appendix D. Summary of important notations. Table D.1 summarizes some notations which are introduced in the article and are used in several places.

Matrix functions	Definition (if applicable)	Where defined	Operators (if applicable)
$\mathbf{L}$	diagonal matrix of the inverse synaptic time constants	equation 2.4	
$\tau_{\max}$	largest time constant	definition 2.2	
$D\mathbf{S}_m$		definition 2.1	
$\mathbf{W}$		equations (3.2), (3.3), (3.4)	f_v, f_a, g_v
$\mathbf{W}_{cm}$	$\mathbf{W}D\mathbf{S}_m$	definition 4.1	g_m
$\mathbf{W}_{mc}$	$D\mathbf{S}_m\mathbf{W}$	definition 4.1	h_m

TABLE D.1
Summary of some important definitions.

REFERENCES

[1] Fatihcan M. Atay and Axel Hutt, *Stability and bifurcations in neural fields with finite propagation speed and general connectivity*, SIAM Journal on Applied Mathematics, 65 (2005), pp. 644–666.
[2] Kendall Atkinson and Weimin Han, *Theoretical Numerical Analysis*, Springer-Verlag, 2001.
[3] D.P. Buxhoeveden and M.F. Casanova, *The minicolumn hypothesis in neuroscience*, Brain, 125 (2002), pp. 935–951.
[4] L. M. Chalupa and J.S. Werner, eds., *The visual neurosciences*, MIT Press, 2004.
[5] T.P. Chen and S.I. Amari, *Exponential convergence of delayed dynamical systems*, Neural Computation, 13 (2001), pp. 621–635.
[6] Stephen Coombes, *Waves, bumps, and patterns in neural fields theories*, Biological Cybernetics, 93 (2005), pp. 91–108.
[7] S. Coombes, N.A Venkov, L. Shiau, I. Bojak, D.T.J. Liley, and C.R. Laing, *Modeling electrocortical activity through local approximations of integral neural field equations*, Physical Review E, 76 (2007).
[8] P. Dayan and L. F. Abbott, *Theoretical Neuroscience : Computational and Mathematical Modeling of Neural Systems*, MIT Press, 2001.
[9] Jean Dieudonné, *Foundations of Modern Analysis*, Academic Press, 1960.
[10] Bard Ermentrout, *Neural networks as spatio-temporal pattern-forming systems*, Reports on Progress in Physics, 61 (1998), pp. 353–430.
[11] L.C. Evans, *Partial Differential Equations*, vol. 19 of Graduate Studies in Mathematics, AMS, 1998.
[12] Y. Fang and T.G. Kincaid, *Stability analysis of dynamical neural networks*, IEEE Transactions on Neural Networks, 7 (1996), pp. 996–1006.
[13] Stefanos E. Folias and Paul C. Bressloff, *Breathing pulses in an excitatory neural network*, SIAM Journal on Applied Dynamical Systems, 3 (2004), pp. 378–407.
[14] W.J. Freeman, *Mass action in the nervous system*, Academic Press, New York, (1975).
[15] W. Gerstner and W. M. Kistler, *Mathematical formulations of hebbian learning.*, Biological Cybernetics, 87 (2002), pp. 404–415.
[16] F. Grimbert and O. Faugeras, *Bifurcation analysis of Jansen's neural mass model*, Neural Computation, 18 (2006), pp. 3052–3068.
[17] F. Grimbert, O. Faugeras, and F. Chavane, *Neural field model of VSD optical imaging signals*, Research Report 6398, INRIA, 12 2007.
[18] Stefan Haeusler and Wolfgang Maass, *A statistical analysis of information-processing properties of lamina-specific cortical microcircuits models*, Cerebral Cortex, 17 (2007), pp. 149–162.
[19] J. J. Hopfield, *Neurons with graded response have collective computational properties like those of two-state neurons*, Proceedings of the National Academy of Sciences, USA, 81 (1984), pp. 3088–3092.
[20] Ben H. Jansen and Vincent G. Rit, *Electroencephalogram and visual evoked potential generation in a mathematical model of coupled cortical columns*, Biological Cybernetics, 73 (1995), pp. 357–366.
[21] E.R. Kandel, J.H. Schwartz, and T.M. Jessel, *Principles of Neural Science*, McGraw-Hill, 4th ed., 2000.
[22] S. Kubota and K. Aihara, *Analyzing global dynamics of a neural field model*, Neural Processing Letters, 21 (2005), pp. 133–141.
[23] C.R. Laing, W.C. Troy, B. Gutkin, and G.B. Ermentrout, *Multiple Bumps in a Neuronal Model of Working Memory*, SIAM Journal on Applied Mathematics, 63 (2002), pp. 62–97.
[24] F.H. Lopes da Silva, A. Hoeks, and L.H. Zetterberg, *Model of brain rhythmic activity*, Kybernetik, 15 (1974), pp. 27–37.
[25] F.H. Lopes da Silva, A. van Rotterdam, P. Barts, E. van Heusden, and W. Burr, *Model of neuronal populations. the basic mechanism of rhythmicity.*, M.A. Corner, D.F. Swaab (eds) Progress in brain research, Elsevier, Amsterdam, 45 (1976), pp. 281–308.
[26] A.M. Lyapunov, *Stability of Motion*, Academic Press, 1966.
[27] Kiyotoshi Matsuoka, *Stability conditions for nonlinear continuous neural networks with asymmetric connection weights*, Neural Networks, 5 (1992), pp. 495–500.
[28] V.B. Mountcastle, *Modality and topographic properties of single neurons of cat's somatosensory cortex*, Journal of Neurophysiology, 20 (1957), pp. 408–434.
[29] ———, *The columnar organization of the neocortex*, Brain, 120 (1997), pp. 701–722.
[30] Q.C. Pham and J.J.E. Slotine, *Stable concurrent synchronization in dynamic system net-*

works, Neural Networks, 20 (2007).

[31] Arkady Pikovsky, Jurgen Kurths, and Michael Rosenblum, *Synchronization: A Universal Concept in Nonlinear Sciences*, Cambridge University Press, 2001.

[32] D.J. Pinto and G.B. Ermentrout, *Spatially structured activity in synaptically coupled neuronal networks: 1. traveling fronts and pulses.*, SIAM J. of Appl. Math., 62 (2001), pp. 206–225.

[33] ———, *Spatially structured activity in synaptically coupled neuronal networks: 2. standing pulses.*, SIAM J. of Appl. Math., 62 (2001), pp. 226–243.

[34] D. Pinto, R. Jackson, and C.E. Wayne, *Existence and stability of traveling pulses in a continuous neuronal network*, SIAM J. of Appl. Dyn. Syst., 4 (2005), pp. 954–984.

[35] J.J.E. Slotine and W. Li, *Applied nonlinear control*, Prentice Hall Englewood Cliffs, NJ, 1991.

[36] Alex M. Thomson and A. Peter Bannister, *Interlaminar connections in the neocortex*, Cerebral Cortex, 13 (2003), pp. 5–14.

[37] A. van Rotterdam, F.H. Lopes da Silva, J. van den Ende, M.A. Viergever, and A.J. Hermans, *A model of the spatial-temporal characteristics of the alpha rhythm*, Bulletin of Mathematical Biology, 44 (1982), pp. 283–305.