Graph Algorithms

TD2 : Graph colouring

1 Some properties of colouring

1. What is the chromatic number of an even cycle C_{2n} ? Of an odd cycle C_{2n+1}?
2. Show that a graph is bipartite if and only if it contains no odd cycle.
3. Show that for every graph G, there exists an order on the vertices such that the greedy algorithm applied in this order returns a colouring with $\chi(G)$ colours.
4. Prove that $\chi(G) \geq |V(G)|/\alpha(G)$, for every graph G.

2 Interval graphs

Given a set of intervals $\mathcal{I} = \{I_1, \ldots, I_n\}$ where $I_i = [a_i, b_i]$ for every $1 \leq i \leq n$, the interval graph associated with $\mathcal{I}$ is the graph $G = (V, E)$ where $V = \{1, \ldots, n\}$ and $ij \in E$ iff I_i and I_j intersect, i.e. $a_i \leq b_j$ and $a_j \leq b_i$, for every $1 \leq i, j \leq n$.

1. Show that in an interval graph, there exists a simplicial vertex, i.e. a vertex v such that $N[v]$ induces a clique.
2. Write an algorithm that computes an optimal proper colouring of an interval graph G. You may assume that we know the intervals. The goal complexity is $O(n \ln n + m)$.
3. We now want to write an algorithm which computes a proper colouring of any graph G, and uses $\chi(G)$ colours if G is an interval graph (so in particular we don't know the intervals if this is the case). Show that this can be done with the greedy colouring algorithm applied with a reverse degeneracy ordering.

3 Mycielski graphs

In this exercise, we construct a family of triangle-free graphs (so of clique number at most 2) with increasing chromatic number. This shows that $\chi(G)$ can be arbitrarily larger than $\omega(G)$.

Given a graph G, the Mycielskian of G, denoted $M(G)$, is obtained by adding a copy u_i of each vertex $v_i \in V(G)$, and adding a vertex w adjacent to all those copies. In the end, $N_{M(G)}(u_i) = \{w\} \cup N_G(v_i)$, and the vertices $\{u_i\}_i$ form an independent set.

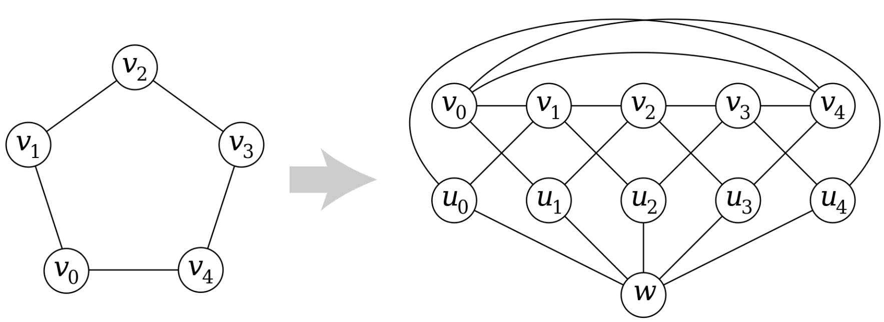

Figure 1: The Mycielskian operation performed on C_5

The Mycielski graphs are a family of graphs $(M_i)_{i\geq 2}$, with $M_2 = K_2$, and $M_{i+1} = M(M_i)$ for every $i \geq 2$.

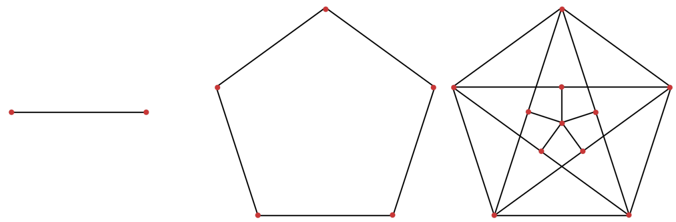

Figure 2: The first Mycielski graphs

1. Let G be a k-chromatic graph, and c a proper k-colouring of G. Show that for every colour i, there exists a vertex $v \in V(G)$ such that $c(v) = i$ and all the other colours appear in its neighbourhood.

   Hint: Show that you can reduce the number of colours otherwise.

2. Show that for all $i \geq 2$, the graph M_i contains no triangle (i.e. a copy of the complete graph K_3).

3. Show by induction that $\chi(M_i) \leq i$, for all $i \geq 2$.

4. Show that $\chi(M_i) \geq i$, for all $i \geq 2$.

   Hint: Use the result of Question 3.1.