Graph Algorithms

TD1 : Introduction

1 To begin

1. Show that a graph always has an even number of odd degree vertices.
2. Show that a graph with at least 2 vertices contains 2 vertices of equal degree.

   Hint: If G contains no isolated vertex, how many different values are possible for the degree of a vertex in G?
3. Let G be a graph of minimum degree $\delta(G) \geq 2$. Show that G contains a cycle.
4. Let G be a graph of minimum degree d, and of girth $2t+1$. Given any vertex $v \in V(G)$, show that there are at least $d(d-1)^i$ vertices at distance exactly i from v in G, for every $1 \leq i \leq t$. Deduce a lower bound on the number of vertices of G.

2 Dense subgraphs

1. Show that every graph of average degree d contains a subgraph of minimum degree at least $\frac{d}{2}$.

   Hint: Consider a subgraph of maximum average degree.
2. Can you find a similar relation between the maximum degree and the minimum degree? And between the maximum degree and the average degree?
3. Show that every graph of average degree d contains a bipartite subgraph of average degree at least $\frac{d}{2}$.

   Hint: Consider a maximal cut.

3 Cuts and trees

1. If G is connected, and $e = uv$ is a bridge in G, how many connected components does $G \setminus e$ contain? Show that u and v are cut-vertices.
2. Show that a graph G is a tree if and only if there exists a unique path from u to v in G, for every pair of vertices $u, v \in G$.
3. Let T a BFS tree of a graph G. Show that every edge of G is contained either within a layer of T, or between two consecutive layers of T.
4. Let T be a DFS tree of a graph G. Show that, for every edge $e \in E(G)$, there is a branch of T that contains both extremities of e.