Interval observer with near optimal adaptation dynamics. Application to the estimation of lipid quota in microalgae

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SUMMARY

In this paper, the design of an interval observer with an adaptive dynamical gain is presented. The observer is formulated in the framework of robust state estimation of uncertain dynamical systems, where an interval that encloses the unknown state variables is provided. For a specific type of interval observer design, an optimal gain could provide the narrowest interval. Since this gain depends on the (unknown) state, it can however not be determined, and only intervals for this optimal gain can be estimated. Here we propose a strategy to track, with a high gain observer, the optimal gain together with the computation of its derivative. The observer performance is first illustrated with the simple case of an uncertain bioreactor model. Then, we propose the real case of the estimation of microalgal oil in the framework of biofuel production. The proposed observer design, when applied to experimental data of Isochrysis affinis galbana, appears to be a suitable robust estimation technique. Copyright © 0000 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Robust state estimation of uncertain bioprocesses is a trending topic that has received increasing attention in the last decades. Observer techniques have been studied e.g. in [1, 2], proposing methods for the estimation of state variables, kinetics (reaction rates) and fundamental parameters. The high uncertain nature of this kind of processes and moreover the lack of specific instrumentation and sensors have made crucial the development of robust estimation techniques. Interval observers [3, 4] offer a way to deal with uncertainty (on the inputs–outputs and the dynamics) in a guaranteed state estimation framework.

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Indeed, interval observers have become a leading state estimation technique. They are inserted in the class of guaranteed state estimation methods, providing a region of the state space where the unknown state variables are sure to be. They are constructed on the basis of positive differential systems [5] and have been successfully applied to the estimation of variables of biological systems [6, 7, 8], chaotic dynamics [9, 10], vehicle positioning [11], linear systems with additive disturbances [12], etc. Assuming that a guaranteed bound of the initial unknown state and bounds on the uncertainties (inputs, disturbances, parameters, ...) are provided, a framer basically consists in an auxiliary dynamical system whose trajectories always stay above or below (component by component) those of the original system. This definition is general and does not impose any further constraint on the qualitative behavior of the framer. Therefore, an interval observer is a stable framer (with bounded error dynamics).

One of the main advantages of interval observers is that the estimation performance can be evaluated online. Thanks to this property, several framers can be run in parallel [6] (bundle of framers) and then the best estimates can be selected. In [13] a strategy is proposed in order to improve the observer performance through the coupling of framers. The main advantage is that the stability of the framer envelope is guaranteed by a stable framer within the bundle [6].

In [7], an interval observer is designed based on the work of [14]. It is shown that there exists an optimal observer gain providing the narrowest interval. However, this optimal gain is directly dependent on the (unknown) state variable. Moreover, its derivative is also needed to implement the observer. In [7] a strategy is proposed by bounding the gain interval and running several interval observers with constant gains in parallel. Here, we propose an alternative strategy, where the optimal gain is tracked with a high gain observer (so we call it adaptive interval observer). This dynamics ensures high gain convergence of the observer gain towards the optimal value for which interval is the narrowest. Moreover, the proposed adaptation dynamics is constrained to guarantee that some appropriate sign conditions (for the framer design) stay valid. Finally, when dealing with a time varying gain, the observer equation depends on the time derivative of the gain, which is provided by the high-gain dynamics.

The proposed approach is illustrated first with a simple bioreactor model, in order to improve the performance of the classical asymptotic observer [15]. Then, we propose an interval observer on a real case, for the estimation of the microalgal oil. Such process may become more and more used since microalgae are potentially considered as one of the main biofuel producers in the future [16, 17]. Nevertheless, the lack of on-line sensor for monitoring the nutrient status of the cell and its subsequent lipid accumulation makes it difficult to control and optimize lipid production. Some software sensors have been proposed to address this issue and estimate the internal nitrogen quota in microalgae [1, 8]. In the present paper we use a recently developed model [18] predicting both the nitrogen status of the microalgal cells and their lipid content to support an adaptive interval observer. The observer performance is illustrated with experimental data of *Isochrysis affinis galbana* (clone T-iso) cultures [19].

The paper is organized in four main sections. Section 2 is devoted to recalls on interval observer design, which are then illustrated in Section 3. In Section 4, an improved interval observer design
is described. Application of the method to an uncertain bioreactor model is developed in Section 5. Finally, after a presentation of the microalgal lipid growth model, Section 6 is devoted to design the observer for the microalgal model, and its validation with experimental data.

2. RECALL ON INTERVAL OBSERVER DESIGN

2.1. Notations and definitions

We consider a differential system given by:

\[
\begin{align*}
\left\{ \begin{array}{l}
\dot{x}(t) = f(x(t), u(t), \lambda) \\
y(t) = h(x(t))
\end{array} \right.
\end{align*}
\]

(1)

where \(x(t) \in \Omega \subset \mathbb{R}^n_x\) is the vector of state variables with \(\Omega\) a convex compact set of \(\mathbb{R}^n_x\), \(u(t) \in \mathcal{U} \subset \mathbb{R}^n_u\) is the vector of inputs with \(\mathcal{U}\) the set of admissible controls and \(\lambda \in \Lambda \subset \mathbb{R}^n_\lambda\) is a parameter vector. \(f\) is a \(C^1\) mapping. The initial state vector \(x(0)\) is in an admissible domain \(\Omega_0 \subset \Omega\). The output \(y\) is related to the state via the \(C^1\) mapping \(h\).

2.2. Framers

In the framework of parameter uncertainties, we assume that the real inputs and parameters are unknown, but that they can be enclosed by known quantities:\footnote{All the inequalities in \(\mathbb{R}^n\) must be understood component by component.}

\[
\forall t \geq 0, \ u(t) \leq \underline{u}(t) \leq \overline{u}(t), \ \underline{\lambda} \leq \lambda \leq \overline{\lambda}
\]

(2)

A framer associated to the differential system \((\Sigma)\) is a differential system whose solutions generate guaranteed bounds for the state variables of \((\Sigma)\) [3].

Definition 1

A framer for system (1) where inputs and parameters satisfy equation (2) is a pair of coupled dynamical systems:

\[
\begin{align*}
\dot{\underline{x}}(t) &= f(\underline{\tau}, \underline{z}, \underline{u}(t), \underline{\lambda}, \theta(t), y), \\
\dot{\overline{x}}(t) &= f(\overline{\tau}, \overline{z}, \overline{u}(t), \overline{\lambda}, \theta(t), y), \\
\underline{\tau}(t) &= \underline{g}(\underline{x}(t), \underline{z}(t), \underline{u}(t), \underline{\lambda}, \theta(t), y), \\
\overline{\tau}(t) &= \overline{g}(\overline{x}(t), \overline{z}(t), \overline{u}(t), \overline{\lambda}, \theta(t), y), \\
\underline{z}(0) &= \underline{x}_0; \ \underline{\tau}(0) = \tau_0, \\
\overline{z}(0) &= \overline{h}(\overline{x}_0, \overline{z}_0, \overline{u}(0), \overline{\lambda}, \theta(0)), \\
\underline{z}(0) &= \underline{h}(\underline{x}_0, \underline{z}_0, \underline{u}(0), \underline{\lambda}, \theta(0)).
\end{align*}
\]

(3)
with \( \overline{f} \) and \( f \) of class \( C^1 \), such that, for \( x_0 \leq x_\theta \leq \overline{x}_0 \), and for any \( \theta(t) \in \Theta \) (\( \Theta \) being a subset of \( C^1 : \mathbb{R} \rightarrow \mathbb{R}^n \)), we have \( \forall t \geq 0 \):

\[
\underline{x}(t) \leq x(t) \leq \overline{x}(t)
\]

The \( C^1 \) functions \( \overline{g}, \overline{h}, \) and \( h \) define a coordinate change that guarantee the following interval conditions (with shorten notations):

\[
\overline{g} \left( \overline{h}(\underline{x}_0, \overline{x}_0), \overline{h}(\underline{x}_0, \overline{x}_0) \right) \leq \underline{x}_0 \text{ and } \overline{x}_0 \leq \overline{g} \left( \overline{h}(\underline{x}_0, \overline{x}_0), \overline{h}(\underline{x}_0, \overline{x}_0) \right)
\]

The time–varying parameter vector \( \theta \in \Theta \) can be used in order to tune the framer performance. Note that this definition is rather general, highlighting the fact that a framer is simply designed to give an upper and a lower bound of the unknown state. The following definition also includes stability properties.

**Definition 2**

A framer (3) with bounded predictions \( \overline{x} \) and \( \underline{x} \) is called an interval observer.

### 3. APPLICATION TO A SIMPLE BIOPROCESS MODEL

**3.1. Model presentation**

As an example, we apply the presented approach to the framers proposed by Moisan et al. [7] to estimate biomass concentration considering substrate measurement. These framers have been designed using a classical bioreactor model [20, 15], which describes the growth of a biomass \( x \) that uptakes a substrate \( s \) in a perfectly mixed bioreactor:

\[
\begin{cases}
\dot{x} = \mu(s)x - Dx \\
\dot{s} = D(s_{in} - s) - k\mu(s)x \\
x(0) = \underline{x}_0, \quad s(0) = s_0
\end{cases}
\]

where \( D(t) \) is the dilution rate, \( s_{in} \) is the input substrate concentration and \( k \) is the pseudo yield coefficient. \( \mu(s) \) is the specific growth rate (which is generally poorly known).

**Hypothesis 1**

We consider the following assumptions:

- The specific growth rate is unknown but bounded by known functions \( \mu(s) \) and \( \overline{\mu}(s) \):
  \[
  \underline{\mu}(s) \leq \mu(s) \leq \overline{\mu}(s)
  \]

- The input substrate concentration is also unknown, with known bounds\( \dagger \):
  \[
  \underline{s}_{in}(t) \leq s_{in}(t) \leq \overline{s}_{in}(t)
  \]

\( \dagger \) In the following, the time dependence of these values will be omitted for sake of brevity.
3.2. Observer design

A bounded error observer can be written on the basis of the new variable $z$ (see [14] for more details):

$$z = kx + \theta(t)s$$

whose dynamics are given by:

$$\dot{z} = (1 - \theta)\mu(s)(z - \theta s) + D(\theta s_{in} - z) + \dot{\theta} s.$$ 

On this basis, Moisan et al. [7] propose a framer for system (4).

**Proposition 1**

Given $x_0', x_0)$ such that $x \in [x_0, x_0')$, the following system with $\theta_i(t) > 1$, $i = 1, 2$ is a framer for system (4):

$$\dot{z}_1 = (1 - \theta_1)(\mu(s)\bar{x} - \theta_1\bar{\mu}(s)s) + D(\theta_1 s_{in} - \bar{z}) + \dot{\theta}_1 s$$

$$\dot{z}_2 = (1 - \theta_2)(\bar{\mu}(s)\bar{x} - \theta_2\mu(s)s) + D(\theta_2 s_{in} - \bar{z}) + \dot{\theta}_2 s$$

(5)

with $\bar{x} = (\bar{x} - \theta_1s)/k$ and $\bar{z} = (\bar{z} - \theta_2s)/k$.

**Proof.** See [7]. ☐

In [7], the optimal values for gain $\theta_i > 1$, denoted $\tilde{\theta}_1$ and $\tilde{\theta}_2$, which respectively minimizes and maximizes $\bar{x}$ and $\bar{z}$ have been computed and read as follows:

$$\tilde{\theta}_1(t) = \max\{1, \frac{1}{2} - \frac{D(s_{in} - s_{in}) + k(\mu(s)x - \mu(s)\bar{x})}{2(\bar{\mu}(s) - \mu(s))}\}$$

$$\tilde{\theta}_2(t) = \max\{1, \frac{1}{2} - \frac{D(s_{in} - s_{in}) - k(\mu(s)x - \bar{\mu}(s)\bar{x})}{2(\bar{\mu}(s) - \mu(s))}\}$$

(6)

**Remark 1**

As it is shown in [7], the gain value $\theta(t) = 0$ is also a solution of the optimization problem for a framer similar to (5). The framer associated with this gain value is not considered in this paper for the sake of clarity, however it can be easily incorporated within the bundle (see [13] for more details on the coupling of framers).

4. TRACKING OF THE OPTIMAL INTERVAL OBSERVER GAIN

4.1. Motivation

In the following, we assume that there exists some optimal interval observer gain, noted $\tilde{\theta}(t) \in \Theta$, which optimizes some observer performance (e.g. the interval decreasing rate, see section 3.2 or [7]). The constraint when using a time varying parameter in the design of the framer is that its derivative $\dot{\theta}(t)$ must be used to compute $\bar{x}$ and $\bar{z}$. When dealing with an optimal value of this parameter, the computation of this derivative may reveal delicate since it may depend on the (unknown) state.
variable of system ($\Sigma$) (see section 3.2 or [7] for an example of the computation of this optimal parameter). Here we propose to use bounds on this optimal gain, and to introduce a dynamics of adaptation so that the gain $\theta$ converges towards the optimal gain $\tilde{\theta}(t)$:

$$\dot{\theta} = \phi(\tilde{\theta}, \theta, \bar{\theta}, \bar{\theta}) \quad (7)$$

where adaptation dynamics of $\theta$ are driven by the mapping $\phi$. Functions $\bar{\theta}$ and $\bar{\theta}$ are bounds to ensure that $\theta$ stays in the parametric domain $\Theta$ for which system (3) is guaranteed to be a framer. Moreover, we assume that there exists a positive constant $\epsilon$ such that, for any time $t$, $\theta(t) + \epsilon < \tilde{\theta} < \bar{\theta}(t) - \epsilon$.

The mapping $\phi$ should be defined to guaranty that $\theta$ converges to $\tilde{\theta}$. In practice, this convergence must be fast compared to the original system (3), a high gain strategy in the adaptation dynamics has therefore been chosen.

Since $\theta$ must be bounded between two known bounds $\underline{\theta}$ and $\bar{\theta}$ to satisfy sign conditions required for the framer, a possible choice of $\phi$ can be (for each component $\theta_i$)

$$\dot{\theta}_i = K_i^a (\tilde{\theta}_i - \theta_i) \left(1 + \frac{\epsilon}{\tilde{\theta}_i - \theta_i} + \frac{\epsilon}{\theta_i - \bar{\theta}_i}\right) \quad (8)$$

where $K_i^a$ are positive adaptation gains and $\epsilon$ a small constant.

For sake of simplicity, since the dynamics of the observer gains $\theta_i$ are uncoupled, we will focus on one of the components and omit the subscript $i$ in the following. Thus, we will write generically $\theta$ instead of $\theta_i$.

Property 1
Assuming that the derivatives of $\theta$ and $\bar{\theta}$ are bounded in norm by a constant $N$, and that there exists a positive constant $\epsilon$ such that $\theta(t) + \epsilon < \tilde{\theta} < \bar{\theta}(t) - \epsilon$, then Equation (8) guarantees that for any $\theta(t_0)$ such that $\theta(t_0) < \theta(t_0) < \bar{\theta}(t_0)$, we have $\theta(t) < \theta(t) < \bar{\theta}(t)$, $\forall t > t_0$.

Proof. Let us consider $\bar{\tau}_\theta = \bar{\theta} - \theta$ (remember that these equations indeed refer to one of the gain components, and that indexes are omitted). Its dynamics is:

$$\dot{\bar{\tau}}_\theta = \dot{\bar{\theta}} - K^a (\bar{\theta} - \bar{\theta} + \bar{\tau}_\theta) \left(1 + \frac{\epsilon}{\bar{\tau}_\theta} + \frac{\epsilon}{\bar{\theta} - \bar{\theta} - \bar{\tau}_\theta}\right)$$

We focus on the dynamics of $\bar{\tau}_\theta$ for $\bar{\tau}_\theta < \epsilon$ to evaluate the sign of $\bar{\tau}_\theta$ when $\bar{\tau}_\theta$ cancels. Note that $\dot{\bar{\theta}} = \bar{\theta} - \bar{\theta} + \bar{\tau}_\theta < -\epsilon + \bar{\tau}_\theta$. Since $\bar{\theta} \geq -N$, we get

$$-N + K^a (\epsilon - \bar{\tau}_\theta) \left(1 + \frac{\epsilon}{\bar{\tau}_\theta} + \frac{\epsilon}{\bar{\theta} - \bar{\theta} - \bar{\tau}_\theta}\right) < \dot{\bar{\tau}}_\theta \quad (9)$$

If $\bar{\tau}_\theta$ tends to 0, then the left hand side of (9) tends to infinity. Therefore, there exists a positive $\epsilon_t$ small enough such that, when $\bar{\tau}_\theta < \epsilon_t$, $\bar{\tau}_\theta > 0$. As a consequence, $\bar{\tau}_\theta$ cannot cancel when it has been positively initiated. $\square$
In order to study the convergence of $\theta$ towards the optimal value $\bar{\theta}$, let us define $\delta(t) = \theta(t) - \bar{\theta}(t)$ whose dynamics is:

$$
\dot{\delta} = -K^\alpha \delta \left( 1 + \frac{\epsilon}{\bar{\theta} - \theta - \delta} + \frac{\epsilon}{\delta + \bar{\theta} - \theta} \right) - \dot{\bar{\theta}}
$$

(10)

Given Property 1, we can restrict our analysis to the positively invariant (time-varying) set $\Delta = \{ \delta \in \mathbb{R} | \bar{\theta} - \bar{\theta} < \delta < \bar{\theta} - \bar{\theta} \}$.

**Property 2**

In the case where, for all times, $\dot{\bar{\theta}} = 0$, equation (10) admits a unique GAS equilibrium $\delta^* = 0$ in the positively invariant set $\Delta$.

**Proof.** The roots of Equation (10) with $\dot{\bar{\theta}} = 0$ are $\delta_a = 0$ and the solutions of the polynomial equation:

$$-\delta^2 + \delta(\bar{\theta} + \theta - 2\bar{\theta} - (\bar{\theta} - \bar{\theta})(\theta - \bar{\theta}) + \epsilon(\bar{\theta} - \bar{\theta}) = 0
$$

(11)

The discriminant of (11) reads:

$$\Delta = (\bar{\theta} - \theta)^2 + 4\epsilon(\bar{\theta} - \theta) > 0,
$$

so Equation (10) admits 2 other solutions:

$$\delta_b = \frac{\bar{\theta} + \theta - 2\bar{\theta} + \sqrt{(\bar{\theta} - \theta)^2 + 4\epsilon(\bar{\theta} - \theta)}}{2} = \bar{\theta} - \bar{\theta}
$$

and:

$$\delta_c = \frac{\bar{\theta} + \theta - 2\bar{\theta} - \sqrt{(\bar{\theta} - \theta)^2 + 4\epsilon(\bar{\theta} - \theta)}}{2} = \bar{\theta} - \bar{\theta}
$$

Therefore, $\delta = 0$ is the unique equilibrium in $\Delta$.

Since it is locally stable in dimension 1, it is globally stable. □

We can now extend this property to the perturbation framework, i.e. where $\dot{\bar{\theta}}$ is time-varying.

**Property 3**

Considering $\dot{\bar{\theta}}$ as a (bounded) perturbation input, the system (10) in the invariant set $\Delta$ is input-to-state stable (ISS), i.e. there exists a class $\mathcal{K}$ function $\beta$ and a class $\mathcal{K}$ function $\gamma$ such that for any initial state $\delta(t_0) \in \Delta$ and any bounded input $\dot{\bar{\theta}}(t)$, the solution $\delta(t)$ satisfies:

$$\|\delta(t)\| \leq \beta(\|\delta(t_0)\|, t - t_0) + \gamma \left( \sup_{t_0 \leq \tau \leq t} \|\dot{\bar{\theta}}(\tau)\| \right)
$$

Moreover, the mapping $\gamma(r) = \frac{r}{(1 - \alpha)K^\alpha}$, (with $0 < \alpha < 1$) can be as small as desired on $\Delta$ by an appropriate choice of $K^\alpha$.

**Proof.** Taking $V = \delta^2/2$ as an ISS-Lyapunov function candidate, the derivative of $V$ with respect to Equation (10) is given by:

$$
\dot{V} = -K^\alpha \delta^2 \left( 1 + \frac{\epsilon}{\bar{\theta} - \theta - \delta} + \frac{\epsilon}{\delta + \bar{\theta} - \theta} \right) - \delta \dot{\bar{\theta}}
$$

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Since $\delta \in \Delta$, the term in brackets is greater than one, so we get:

$$\dot{V} < -(1 - \alpha)K^a\delta^2 - \alpha K^a\delta^2 - \delta \dot{\theta}$$

where $0 < \alpha < 1$.

Let us denote $M$ the upper bound of the perturbation: $|\dot{\theta}| \leq M$

$$\dot{V} < -(1 - \alpha)K^a\delta^2 - \alpha K^a\delta^2 + M|\delta|$$

Then, $\forall|\delta| \geq \frac{M}{(1 - \alpha)K^a}$, we have

$$\dot{V} < -\alpha K^a\delta^2.$$  

Thus, applying Theorem 4.19 from [21], the system is ISS with

$$\gamma(r) = \frac{r}{(1 - \alpha)K^a}.$$  

Properties 1, 2 and 3 ensure that $\theta$ will stay in the parametric domain $\Theta$ and will converge towards the optimal time-varying value $\tilde{\theta}$, the attraction domain can be as small as desired by the choice of a large gain $K^a$. In particular, from Property 3, $\delta(t)$ remains bounded for bounded input $\dot{\theta}$, with an ultimate bound which is a function of the input magnitude. Moreover, if the optimal gain $\tilde{\theta}$ is constant, then system (10) is globally asymptotically stable. In practice, $\epsilon$ can be chosen small enough such that the two rational functions of Equations (8) affect the dynamic only if $\theta$ is very close to a bound.

5. APPLICATION TO A SIMPLE BIOPROCESS MODEL (CONTINUED)

The implementation of the optimal framer defined by equations (6) and (5) requires computation of $\tilde{\theta}_i$ and their derivatives, $\tilde{\theta}_i$, depend on $s_{in}$ and $\mu(s)x$ which are unknown, hence a direct computation is not possible. In [7], it has been proposed to bound a region that contains the optimal values, and then run a bundle of framers with gain values belonging to this region.

In the following, we propose to estimate $\tilde{\theta}_i$ with an high gain observer–based estimator. In this purpose, Equations (6) can be rewritten as follows:

$$\tilde{\theta}_1 = \max\left\{1, \frac{1}{2} + \frac{s + D(s - \pi) + k\mu(s)\pi}{2s(\pi(s) - \mu(s))}\right\}$$

$$\tilde{\theta}_2 = \max\left\{1, \frac{1}{2} - \frac{s - D(s - s_{in}) + k\mu(s)x}{2s(\pi(s) - \mu(s))}\right\}$$  

(12)

In order to overcome the computation of the derivative of the output $s$, an estimation of $\dot{s} = \psi$, noted $\hat{\psi}$, assuming that $\hat{\psi}$ is bounded, is provided by a high gain observer–based estimator:

$$\begin{cases} \dot{s} = \hat{\psi} - (\eta + \eta'2)(\hat{s} - y) \\ \dot{\hat{\psi}} = -\eta'2(\hat{s} - y) \end{cases}$$  

(13)
with exponential convergence rate for positive and large enough values of $\eta$ for a constant $\psi$ (see [15] for more details). When noise is present in the measurements, $\eta$ should be chosen as a trade-off between estimation error and amplification of noise measurement [22]. In practice, considering limited gain values, a varying $\psi$ and measurement noise will lead to a biased estimate of $\dot{s}$, leading to a suboptimal observer. However, the observer is designed so that it keeps its bounding properties.

With the estimate of $\dot{s}$, we are able to provide estimations of $\hat{\hat{\theta}}_i$, called $\hat{\tilde{\theta}}_i$:

$$\hat{\tilde{\theta}}_1 = \max \left\{ 1, \frac{1}{2} + \frac{\dot{\psi} + D(y - s_{\text{in}}) - k\mu(y)\pi}{2\psi(\mu(y) - \mu(y))} \right\}$$

$$\hat{\tilde{\theta}}_2 = \max \left\{ 1, \frac{1}{2} - \frac{\dot{\psi} + D(y - s_{\text{in}}) + k\mu(y)\pi}{2\psi(\mu(y) - \mu(y))} \right\}$$

A direct use of this estimation in framer (5) will require derivative computation of $\hat{\tilde{\theta}}_i$, or at least bounded estimations, which are difficult to provide accurately. Therefore, in application of the main idea presented in Section 4, framer (5) is modified introducing the adaptation dynamics for $\theta$:

**Proposition 2**

Given $x_0, \pi_0$ such that $x \in [x_0, \pi_0]$, the following system is a framer for system (4):

$$\begin{align*}
\hat{z} &= (1 - \theta_1)(\mu(s)\pi - \theta_1\pi(s)s) + D(\theta_1 s_{\text{in}} - \pi) + \theta_1 s \\
\dot{\hat{z}} &= (1 - \theta_2)(\mu(s)\pi - \theta_2\mu(s)s) + D(\theta_2 s_{\text{in}} - \pi) + \theta_2 s \\
\dot{\theta}_1 &= \phi(\hat{\theta}_1 + \epsilon, \theta_1, +\infty, 1) \\
\dot{\theta}_2 &= \phi(\hat{\theta}_2 + \epsilon, \theta_2, +\infty, 1) \\
\theta_i(t_0) &> 1, i = 1, 2.
\end{align*}$$

with $\pi = (\pi - \theta_1 s)/k$ and $\zeta = (\zeta - \theta_2 s)/k$, the mapping $\phi$ is defined by (8) and $\hat{\hat{\theta}}_i$ by Equations (14).

**Proof.** From Property 1, the mapping $\phi$ ensures that $\theta_i > 1$ for any positive time. Therefore, in direct consequence of Proposition 1, system (15) is a framer for system (4). □

Note that we use the near optimal gains $\hat{\hat{\theta}}_i + \epsilon$, for a small $\epsilon$ in order to satisfy the hypotheses of Property 1 (i.e. the optimal gain should never be equal to the lower bound).

### 5.1. Extension to biased output

In this section, we consider the more realistic case where the output is biased by a bounded multiplicative noise.

**Hypothesis 2**

Online measurement $y(t)$ is perturbed by a noise $\delta_s(t)$. We assume that this perturbation is of multiplicative nature:

$$y(t) = s(t)(1 + \delta_s(t))$$

Moreover, this noise signal is bounded such that $|\delta_s(t)| \leq \Delta_s < 1$.

We can define dynamic bounds for the (positive) substrate:

$$\underline{y}(t) \leq s(t) \leq \overline{y}(t)$$
with

\[
\begin{align*}
\dot{y}(t) &= \frac{y(t)}{(1+\Delta_y)} \\
\dot{\theta}(t) &= \frac{\theta(t)}{(1+\Delta_\theta)}
\end{align*}
\]

In [7], the framer equations (5) have been modified in order to take into account the output uncertainty. Although the accuracy of the estimation will suffer from noise measurement, the computation of \(\hat{\theta}_i\) remains valid, and thus framer (15) can be modified as follows:

**Proposition 3**

Given \(x_0\), \(x_0\) such that \(x \in [x_0, x_0]\), the following system is a framer for system (4) under Hypothesis 2:

\[
\begin{cases}
\dot{z} = (1 - \theta_1)(\nu(y, \bar{y}) z - \theta_1 \nu(y, \bar{y}) \bar{y}) + u(\theta_1 \bar{y} - z) + \dot{\theta}_1 (\sigma_1 \bar{y} + (1 - \sigma_1) y) \\
\dot{\bar{z}} = (1 - \theta_2)(\nu(y, \bar{y}) \bar{z} - \theta_2 \nu(y, \bar{y}) y) + u(\theta_2 \bar{y} - \bar{z}) + \dot{\theta}_2 (\sigma_2 \bar{y} + (1 - \sigma_2) \bar{y}) \\
\dot{\theta}_1 = \phi(\hat{\theta}_1 + \epsilon, 1, +\infty, 1) \\
\dot{\theta}_2 = \phi(\hat{\theta}_2 + \epsilon, 1, +\infty, 1) \\
\theta_i(t_0) > 1, i = 1, 2.
\end{cases}
\]

(16)

with \(\bar{x} = (z - \theta_1 \bar{y}(t))/k\) and \(\bar{x} = (z - \theta_2 y(t))/k\), the mapping \(\phi\) is defined by (8) and \(\hat{\theta}_i\) by Equations (14).

The functions \(\nu(.)\) and \(\nu(.)\) are defined such that:

\[
\nu(y, \bar{y}) = \min_{q \in [y, \bar{y}]} \{\mu(q)\}
\]

\[
\nu(y, \bar{y}) = \max_{q \in [y, \bar{y}]} \{\mu(q)\}
\]

and \(\sigma_i = \begin{cases} 1 & \text{if } \dot{\theta}_i \geq 0 \\ 0 & \text{otherwise} \end{cases}\), uses the sign of \(\dot{\theta}_i\) in order to provide the correct bounding.

**Proof.** See [7] (considering that \(\theta_i(t) > 1, \forall t > t_0\)). \(\square\)

### 5.2. Simulation results

An application of the estimation scheme is presented, inspired from a simple microbial culture in a continuous bioreactor. The specific growth rate is taken as a classical Monod function [20]:

\[
\mu(s) = \mu_{\text{max}} \frac{s}{K_s + s}
\]

(17)

where \(\mu_{\text{max}}\) and \(K_s\) are respectively the maximum specific growth rate and the half–saturation constant. We considered a \(\pm 20\%\) uncertainty for the kinetic parameter \(\mu_{\text{max}}\), a \(\pm 10\%\) uncertainty for the input substrate concentration and a multiplicative noise on the measurements up to a \(5\%\). Substrate measurements \(y(t)\) with and without noise are shown in Figure 1-A. The interval estimation of biomass concentration is presented in Figure 1-B. In comparison with the interval version of the classical asymptotic observer (which corresponds to \(\theta = 1\), fast convergence of the
6. STUDY OF A REAL EXAMPLE: MICROALGAE LIPID PRODUCTION

6.1. Mathematical modelling

In [18], a mathematical model which describes microalgal lipid production under nitrogen stresses has been presented. It is based on the Droop model, which represents the effect of a limited nutrient on phytoplankton growth [23, 24, 25]. Droop model considers that the growth of the biomass \( x \) is related to the limited nutrient quota \( q_n \), while nutrient uptake depends on the external concentration of nutrient \( s \) (nitrogen). Mairet et al. [18] have introduced a simplified carbon metabolism: the organic carbon is split into a functional pool (proteins, nucleic acids, membranes) and two storage pools: sugar and neutral lipid. Carbon from \( \text{CO}_2 \) is first incorporated as sugar. These carbohydrates are mobilized to produce functional carbon (mainly proteins) when microalgae uptake nitrogen. In parallel, carbohydrates are used to produce neutral lipid \( l \) which can be stored or mobilized to produce functional carbon (membranes). Considering this simplified carbon metabolism, the Droop model is completed with the dynamic of the neutral lipid quota \( q_l \). In a perfectly mixed reactor, the lipid model reads:
Figure 2. Evolution of the adaptive parameters $\theta_i$. Thick blue line: Optimal values $\tilde{\theta}_i$, dashed green line: estimations of the optimal value $\hat{\theta}_i$, thin red line: adaptive parameters $\theta_i$.

\[
\begin{align*}
\dot{s} &= Ds_{in} - \rho(s)x -Ds \\
\dot{q}_n &= \rho(s) - \mu(q_n)q_n \\
\dot{x} &= \mu(q_n)x - Dx \\
\dot{q}_l &= (\beta q_n - q_l)\mu(q_n) - \gamma \rho(s)
\end{align*}
\]

where $D$ is the dilution rate and $s_{in}$ the influent nitrogen concentration.

In this model, the absorption rate $\rho(s)$ and growth rate $\mu(q_n)$ are respectively taken as Michaelis-Menten and Droop functions:

\[
\begin{align*}
\rho(s) &= \bar{\rho} \frac{s}{s + K_s} \\
\mu(q_n) &= \bar{\mu} (1 - \frac{Q_0}{q_n})
\end{align*}
\]

where $K_s$ is the half saturation constant for substrate uptake and $Q_0$ the minimal cell quota. $\bar{\rho}$ and $\bar{\mu}$ are the maximum inorganic nitrogen uptake rate and the maximum growth rate, respectively.

6.2. Framer design

For the sake of simplicity, the framer is first presented considering that $s$, $q_n$, and $x$ are perfectly measured and that the specific growth ($\mu(q_n)$) and uptake ($\rho(s)$) rates are the only uncertain term in system (18):

Hypothesis 3

The specific growth rate is unknown but bounded by known functions $\underline{\mu}(q_n)$ and $\bar{\mu}(q_n)$:

$$\underline{\mu}(q_n) \leq \mu(q_n) \leq \bar{\mu}(q_n)$$
Let us consider the change of coordinate $z = (\theta - q_l)x + \gamma s$, where $\theta$ is a time-varying gain. The dynamics of $z$ is given by:

$$
\dot{z} = D(\gamma s_{in} - z) + (\theta - \beta q_n)x + \theta x
$$

which let us introduce the following property.

**Proposition 4**

Given $\theta_i(t) \geq \beta q_n$ for $i = 1, 2$ and $z_0$ and $z_0$ such that $z(t_0) = z_0 \in [\underline{z}, \bar{z}]$, the system

$$
\begin{align*}
\dot{\underline{z}} &= D(\gamma s_{in} - \underline{z}) + (\theta_2 - \beta q_n)\underline{\mu}(q_n)x + \theta_2 x \\
\dot{\bar{z}} &= D(\gamma s_{in} - \bar{z}) + (\theta_1 - \beta q_n)\bar{\mu}(q_n)x + \theta_1 x \\
\dot{\underline{\theta}} &= \theta_1 - (\bar{z} - \gamma s)/x \\
\dot{\bar{\theta}} &= \theta_2 - (\underline{z} - \gamma s)/x
\end{align*}
$$

is a framer of system (18).

**Proof.** Let us introduce the comparisons $\underline{\tau}_q = \underline{\theta} - q_l$ and $\bar{\tau}_q = q_l - \bar{\theta}$. Their dynamics read:

$$
\begin{align*}
\dot{\underline{\tau}}_q &= -\mu(q_n)\tau_q + (\theta_1 - \beta q_n)\left[\mu(q_n) - \underline{\mu}(q_n)\right]x \\
\dot{\bar{\tau}}_q &= -\mu(q_n)\tau_q + (\theta_2 - \beta q_n)\left[\bar{\mu}(q_n) - \mu(q_n)\right]x
\end{align*}
$$

At time $t_0$ it is possible to check that $\tau_q \geq 0$. Now consider the time instant $t^*$ such that $\tau_q(t^*) = 0$, from Equation (22) we have $\underline{\tau}_q(t^*) > 0$ and therefore the error will always stay positive, i.e. $\underline{\tau} \geq q_l$, $\forall t \geq t_0$. Similarly, one can easily check that $\bar{\tau}_q$ will also stay positive, so the system (21) is a framer of system (18). □

Note that it is not necessary to know the absorption rate $\rho(s)$ for implementing framer (21).

### 6.3. Optimal framers

As it has been stated in [7], the optimal pair of gains $\hat{\theta}_1$ and $\hat{\theta}_2$ respectively minimizes $\hat{\tau}_q$, and $\hat{\omega}_q$, at any time instant $t \geq t_0$, that is:

$$
\begin{align*}
\hat{\theta}_1 &= \arg \min_{\theta_1 \geq \beta q_n} \{ J_1(\theta_1, q_n, x, \tau_q) \} \\
\hat{\theta}_2 &= \arg \min_{\theta_2 \geq \beta q_n} \{ J_2(\theta_2, q_n, x, \omega_q) \}
\end{align*}
$$

with $J_1(\theta_1, q_n, x, \tau_q) = \bar{\tau}_q$ and $J_2(\theta_2, q_n, x, \tau_q) = \bar{\tau}_q$.

Recalling Equations (22), it can be easily verified that the optimal gain values in $(\beta q_n, +\infty)$ are $\hat{\theta}_1 = \hat{\theta}_2 = \beta q_n$. Given that the optimal value and the bounds are the same for $\theta_1$ and $\theta_2$, we will use $\theta_1 = \theta_2 = \theta$.

A direct use of the optimal gain in framer (21) will require derivative computation of $q_n$, or at least bounded estimations, which are difficult to provide accurately. Therefore, in application of the
main idea presented in Section 4, framer (21) is modified introducing an adaptive dynamics for $\theta$:

$$\begin{align*}
\dot{z} &= D(\gamma s_{in} - z) + (\theta - \beta q_n)\pi(q_n)x + \dot{\theta}x \\
\dot{\theta} &= \phi(\beta q_n + \epsilon, \theta, +\infty, \beta q_n) \\
\theta(t_0) &> \beta q_n(t_0) \\
\overline{q} &= \theta - (z - \gamma s)/x \\
\lambda &= \theta - (\overline{z} - \gamma s)/x
\end{align*}$$

where the mapping $\phi$ is defined by Equation (8).

Note that in this case, the optimal gain should be chosen as close as possible to the lower bound. Nevertheless, as for the first example, we select the near optimal gain: $\dot{\theta} = \beta q_n + \epsilon$, for a small $\epsilon$ in order to satisfy the hypotheses of Property 1.

### 6.4. Including the uncertainties on parameters and the measurement noise

In the following, we consider a realistic framework where all model parameters suffer from uncertainties and only $s$ and $x$ are measured, with noise perturbation.

**Hypothesis 4**

Online measurements $y_s(t)$ and $y_x(t)$ are perturbed by noises $\delta_s(t)$ and $\delta_x(t)$. We assume that these perturbations are of multiplicative nature:

$$y_s(t) = s(t)(1 + \delta_s(t)) \quad \text{and} \quad y_x(t) = x(t)(1 + \delta_x(t))$$

Moreover, these noise signals are bounded such that $|\delta_s(t)| \leq \Delta_s < 1$ and $|\delta_x(t)| \leq \Delta_x < 1$.

We can define dynamic bounds for the substrate and the biomass:

$$\underline{y}_s(t) \leq s(t) \leq \overline{y}_s(t) \quad \text{and} \quad \underline{y}_x(t) \leq x(t) \leq \overline{y}_x(t)$$

with

$$\begin{align*}
\underline{y}_s(t) &= \frac{y_s(t)}{1 + \Delta_s} \\
\overline{y}_s(t) &= \frac{y_s(t)}{1 - \Delta_s} \\
\underline{y}_x(t) &= \frac{y_x(t)}{1 + \Delta_x} \\
\overline{y}_x(t) &= \frac{y_x(t)}{1 - \Delta_x}
\end{align*}$$

#### 6.4.1. Interval estimation of $q_n$

An asymptotic interval estimator [15, 3] of the nitrogen quota $q_n$ is designed using a change of variable to eliminate the reaction rates $\rho(s)$ and $\mu(q)$

$$\zeta = s + q_n x$$

whose dynamics is $\dot{\zeta} = D(s_{in} - \zeta)$.

**Property 4**

Given $s_{in}$ and $\overline{s}_{in}$ such that $s_{in} \in [\underline{s}_{in}, \overline{s}_{in}]$, and $\overline{\zeta}_0$ and $\underline{\zeta}_0$ such that $\zeta_0 \in [\overline{\zeta}_0, \underline{\zeta}_0]$, the following
framer will provide bounds for the nitrogen quota $q_n$:

$$
\begin{align*}
\dot{\zeta} &= D(s_{in} - \zeta) \\
\dot{\zeta} &= D(s_{in} - \zeta) \\
\bar{q}_n &= (\zeta - y_s) / y_x \\
\underline{q}_n &= (\zeta - y_s) / y_x
\end{align*}
$$  \hspace{1cm} (25)

**Proof.** Computing the dynamics of the errors $\bar{\zeta} = \zeta - \zeta$ and $\underline{\zeta} = \zeta - \zeta$, it is straightforward to show that they stay positive after a positive initialization. Then, we have:

$$
\begin{align*}
\bar{q}_n &= (\zeta - y_s) / y_x \geq (\zeta - y_s) / y_x = q_n \\
\underline{q}_n &= (\zeta - y_s) / y_x \leq (\zeta - y_s) / y_x = q_n
\end{align*}
$$  \hspace{1cm} (26)

which concludes the proof. □

### 6.4.2. Interval estimation of $q_l$

The framer equations (24) have been modified in order to take into account all the uncertainties:

**Property 5**

Given $\zeta_0$ and $\zeta_0$ such as $\zeta_0 \in [\zeta_0; \zeta_0]$ and $z_0$ and $\pi_0$ such as $z_0 \in [\pi_0; \pi_0]$, the following framer will

provide bounds for the lipid quota $q_l$:

$$
\begin{align*}
\dot{z} &= D(\gamma s_{in} - z) + (\theta - \beta q_n) \bar{y}(\bar{q}_n) y_x + \dot{\theta}(\sigma y_x + (1 - \sigma) y_x) \\
\dot{\bar{z}} &= D(\gamma s_{in} - \bar{z}) + (\theta - \beta \bar{q}_n) \mu(q_n) y_x + \dot{\theta}(\sigma y_x + (1 - \sigma) y_x) \\
\dot{\zeta} &= D(s_{in} - \zeta) \\
\dot{\zeta} &= D(s_{in} - \zeta) \\
\dot{\theta} &= \phi(\beta q_n + \epsilon, \theta, +\infty, \beta q_n) \\
\theta(t_0) &> \beta q_n(t_0) \\
\bar{y} &= \theta - (z - \gamma s_{in})/y_x \\
q_l &= \theta - (\bar{z} - \gamma \bar{s}_{in})/y_x \\
q_n &= (\bar{z} - \bar{s}_{in})/y_x \\
q_n &= (\bar{z} - \bar{s}_{in})/y_x \\
\end{align*}
$$

(27)

where the mapping $\phi$ is defined by Equation (8) and $\sigma = \begin{cases} 1 & \text{if } \dot{\theta} \geq 0 \\ 0 & \text{otherwise} \end{cases}$, uses the sign of $\dot{\theta}$ in order to provide the correct bounding.

**Proof.** First, from Property 4, we have $q_n(t) \leq q_l(t) \leq \bar{q}_n(t)$. Using the dynamics of the errors $\bar{e}_z = z - z$ and $\bar{e}_z = z - \bar{z}$, one can show that they stay positive after a positive initialization. Then, considering model (18), we have $q_l < \beta q_n$, and therefore $z - \gamma s = (\theta - q_l) > 0$. We can finally conclude the proof:

$$
\begin{align*}
\bar{q}_l &= \theta - (\bar{z} - \gamma \bar{s}_{in})/y_x \\
q_l &= \theta - (z - \gamma s_{in})/y_x \\
\end{align*}
$$

6.5. Experimental validation

The framer performances are assessed with experimental data of *Isochrysis affinis galbana* (clone T iso) cultures. The experiment consists in imposing various nitrogen limitations through a succession of dilution rates changes. More details about the experiment and model parameter values can be found in [18, 19]. We considered a $\pm 10\%$ uncertainty for the maximum specific growth rate $\bar{\mu}$, a $\pm 2\%$ uncertainty for the other model parameters and a multiplicative noise on the measurements up to a $2\%$. We take for adaptation gain value $K^a = 10$. Figure 3 presents the substrate and biomass measurements $y_s(t)$ and $y_x(t)$ which are used in framer (27) to estimate the nitrogen and lipid quotas. In Figures 4 and 5, the performances of the framer are assessed with independent experimental data (i.e. data not used in the framer). The framer (27) provides an accurate interval estimation of the nitrogen and lipid quotas where almost all the measurements lie (actually, all the measurement error bars intersect with the estimated interval). Finally, we can observe perturbations...
7. CONCLUSIONS

We have proposed a new design for near optimal interval estimation. The idea relies on a high gain adaptive dynamics ensuring tracking of the optimal gain and of its derivative. It is worth noting that,
because of the uncoupling between the gain dynamics, and since the optimal gain is maintained within a bounded interval, no strong peaking phenomenon is observed. A crucial point is that, even if the observer may not be optimal during a first transient of the adaptive gain, the conditions for framer design are always guaranteed and the predicted interval is always encompassing the real state.

The application of the proposed design to the real case of lipid production by microalgae in photobioreactors demonstrates the efficiency of the approach on a non-trivial case. The proposed estimates of lipid content tend to be very accurate and may complement or even replace, the very difficult and expensive analytical measurements. This approach is significantly simpler than the one proposed in [7].

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