Towards Computational Complexity Theory on Advanced Function Spaces in Analysis

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representations

Representation: Surjective, partial $\delta :\subseteq {\Sigma^*}^{\Sigma^*} \to X$.

Elements of $\delta^{-1}(x)$: **names** of x.

Representation of $X \rightsquigarrow$ complexity classes in X.

Complexity of $f : X \to Y$?

Definition

$F :\subseteq {\Sigma^*}^{\Sigma^*} \to {\Sigma^*}^{\Sigma^*}$ is **realizer** of f iff $F(\psi) \in \delta_Y^{-1}(f(\delta_X(\psi)))$.

ψ name of x

$\rightsquigarrow$

$F(\psi)$ name of $f(x)$

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ \delta_X \uparrow & & \uparrow \delta_Y \\ {\Sigma^*}^{\Sigma^*} & \xrightarrow[\quad F \quad]{} & {\Sigma^*}^{\Sigma^*} \end{array}$$

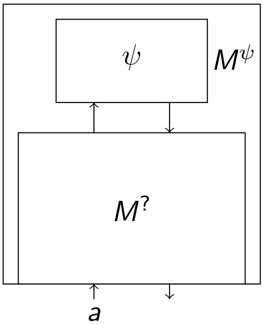
Mehlhorn (70s): feasible functionals on Baire space.

Kapron and Cook (90s):

Feasible functionals $\rightsquigarrow$ resource bounded oracle Turing machines.

Kawamura and Cook (2010s): $\rightsquigarrow$ Complexity in analysis.

Second order complexity



Definition

length $|\psi| : \omega \to \omega$ of ψ:

$$|\psi|(n) = \min\{|\psi(a)| \mid |a| < n\}$$

Often: ψ length-monotone $\rightsquigarrow$

$$|\psi|(n) = |\psi(a)| \quad \text{if} \quad |a| = n$$

M runs in time $T : \omega^\omega \times \omega \to \omega$, if $M^\psi(a)$ terminates after $T(|\psi|, |a|)$ steps.

A second order polynomial:

$$(l, n) \mapsto l(n^5 + 4l(l(n)^2)) + 3.$$

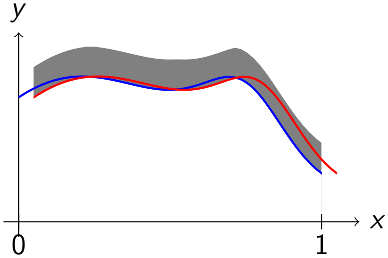
Example: continuous functions

$\mu : \omega \to \omega$ is **modulus of continuity** of $f : [0,1] \to \mathbb{R}$ iff

$$|x - y| \leq 2^{-\mu(n)} \quad \Rightarrow \quad |f(x) - f(y)| < 2^{-n}.$$

$$|h| \leq 2^{-\mu(n)} \quad \Rightarrow \quad |f(x) - f(x-h)| < 2^{-n}.$$

$$x \in [h, 1]$$



Identify dyadic rationals with encodings.

Definition

$\psi \in {\Sigma^*}^{\Sigma^*}$ is δ_C-name of f iff:

- $|f(r) - \psi(\langle r, 1^n \rangle)| < 2^{-n}$
- $|\psi|$ is modulus of continuity of f.

- Introduced and investigated by Kawamura and Cook in 2012.
- Comp. equiv. to the standard representation as metric space.
- Right complexity classes.
- Weakest representation such that evaluation is polytime.

L^p-spaces

$$L^p := L^p([0,1]):$$

Measurable functions $f : [0,1] \to \mathbb{R}$ such that

$$\|f\|_p := \left(\int_0^1 |f(t)|^p \, dt \right)^{\frac{1}{p}} < \infty$$

$\|f\|_p$ is a norm and L^p a Banach space.
(if we identify functions that are coincide almost everywhere).

Also L^∞ can be defined.

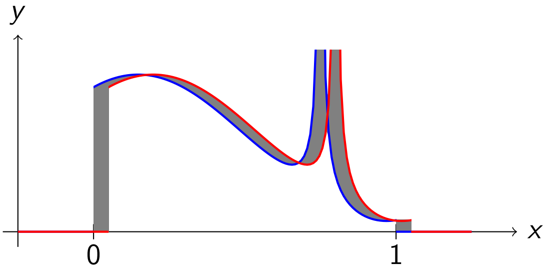
$$L^1 \supsetneq L^p \supsetneq L^\infty \supsetneq C([0,1])$$

L^p-moduli

Definition

A function $\mu : \omega \to \omega$ is an L^p-modulus of f if

$$|h| \leq 2^{-\mu(n)} \Rightarrow \left\|\tilde{f} - \tau_h \tilde{f}\right\|_p := \left(\int_{-\infty}^{\infty} \left|\tilde{f}(t) - \tilde{f}(t-h)\right|^p dt\right)^{\frac{1}{p}} < 2^{-n}$$



representing L^p

Definition

$\psi \in {\Sigma^*}^{\Sigma^*}$ is δ_p-name of f iff

- $\left|\int_q^r f(t)dt - \psi(\langle q, r, 1^n\rangle)\right| < 2^{-n}$
- and $|\psi|$ is an L^p-modulus of f.

Properties:

- Comp. equiv. to the standard representation as metric space.
- Integration operator is polynomial time computable.
- However, it is not the minimal such representation.

Definition

$g \in L^1$ is the weak derivative of $f \in L^1$ if

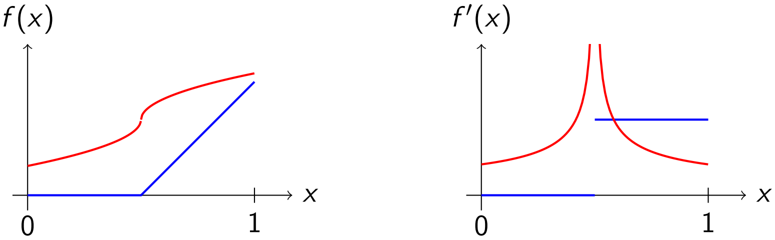
$$\int_0^1 g(t)\varphi(t)dt = -\int_0^1 f(t)\varphi'(t)dt,$$

φ differentiable, $\varphi(0) = 0 = \varphi(1)$. $\rightsquigarrow$ $f' := g$ unique.

f differentiable $\rightsquigarrow$ f' is the usual derivative. (Integration by parts)

sobolev spaces

$$W^{1,p} := \{f \in L^p \mid f' \in L^p\}$$



$$f(x) - f(y) = \int_x^y f'(t)dt,$$

$$C^1([0,1]) \subsetneq W^{1,p} \subsetneq C([0,1]),$$

$$\|f\|_{1,p} := \|f\|_p + \|f'\|_p \quad \rightsquigarrow \quad W^{1,p} \text{ Banach space.}$$

representing $W^{1,p}$

Definition

ψ is $\delta_{1,p}$-name of f iff

- $\left|\int_q^r f(t)dt - \psi(\langle q, r, 1^n\rangle)\right| < 2^{-n}$
- $|\psi|$ is an L^p-modulus of f'.

Theorem

$\frac{d}{dx} : W^{1,p} \to L^p, \quad f \mapsto f'$

is polytime.

Lemma

L^p-modulus of f' $\rightsquigarrow$ modulus of continuity of f.

sketch of the proof of the theorem.

Use the modulus of continuity from the lemma to obtain approximations $f(x)$ from integrals. Recall

$$\int_x^y f'(t)dt = f(x) - f(y).$$

□

Higher derivatives

$W^{m,p}$ is inductively defined by

$$W^{m,p} := \{f \in W^{m-1,p} \mid f' \in W^{m-1,p}\}.$$

Definition

$\psi \in \Sigma^{*\Sigma^*}$ is $\delta_{m,p}$-name of f if

- $\left|\int_q^r f(t)dt - \psi(\langle q, r, 1^n\rangle)\right| < 2^{-n}$
- $|\psi|$ is an L^p-modulus of $f^{(m)}$.
- $|\psi|(0) \geq \log_2(\left\|f^{(k)}\right\|_\infty)$ for any $0 < k < m$.

Theorem

The k-fold differentiation operator

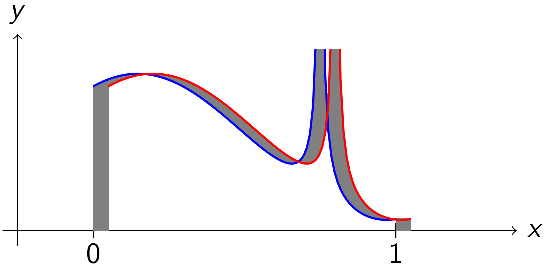
$$\frac{d^k}{dx^k} : W^{m,p} \to W^{m-k,p}, \quad f \mapsto f^{(k)}$$

is polynomial time computable.

proof sketch

Lemma

From a bound on $\|f\|_\infty$ and a modulus of continuity we can obtain an L^p-modulus.



Corollary

$W^{1,p} \hookrightarrow L^p$ *and* $W^{1,p} \hookrightarrow C([0,1])$ *are polytime.*

Conclusion/Outlook

- We specified representations of the spaces L^p and $W^{1,p}$.
- These representations seem to perform reasonably well.
- δ_p can be further justified: Both the modulus of continuity and the L^p-modulus can be connected to best approximation properties.
- For $W^{1,p}$ something like this should also be possible.
- From the one dimensional case it is not clear how to treat higher dimension.
- This is in contrast to the case L^p where higher dimensions are no problem.
- There is a computability theory for Sobolev spaces on the whole space. A comparision would be nice.
- It would be nice to have applications.

last frame

Thanks for the attention