Representations of analytic functions and Weihrauch degrees

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$\delta_{\mathcal{M}}$ is continuous and admissible.

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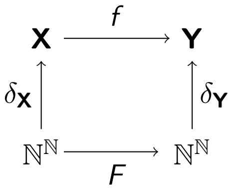

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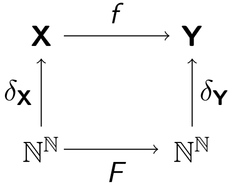

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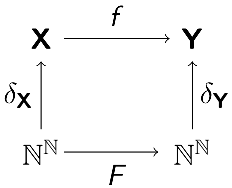

Theorem (computable Weierstraß)

$f : [0, 1] \to \mathbb{R}$ *is computable iff* $f \in C([0, 1])$ *is computable.*

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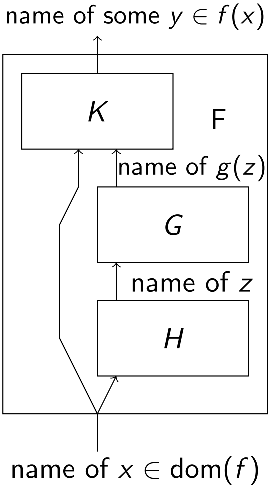

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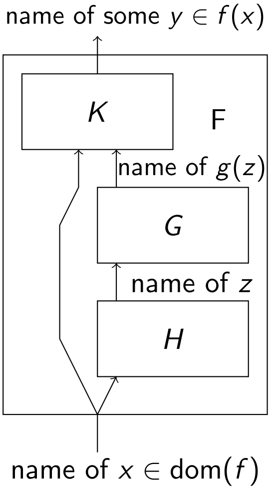

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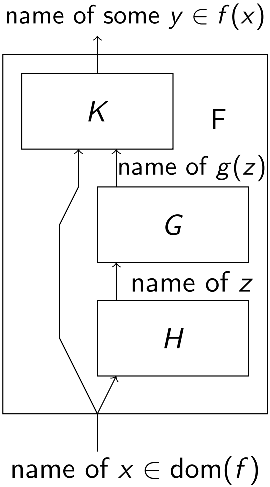

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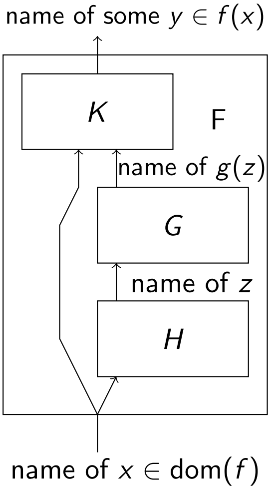

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$$Adv_O((a_k)_{k\in\mathbb{N}}) := \{n \in \mathbb{N} \mid \forall k \in \mathbb{N} : |a_k| \leq n 2^{-\frac{k}{n+1}}\}$$

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$$Adv_O((a_k)_{k\in\mathbb{N}}) := \{n \in \mathbb{N} \mid \forall k \in \mathbb{N} : |a_k| \leq n2^{-\frac{k}{n+1}}\}$$

Analytic functions

Definition

$f : D \to \mathbb{C}$ is analytic iff for all $x_0 \in D : f(x) = \sum_{k\in\mathbb{N}} a_k(x - x_0)^k$

$$a_k = \frac{f^{(k)}(x_0)}{k!} = \frac{1}{2\pi i}\int_0^{2\pi} f(e^{it})e^{-it(k+1)}dt$$

Germ: pair $(x_0, (a_k)_{k\in\mathbb{N}})$. $C^\omega(D) \subseteq C(D)$. ($D$ unit disk.)

Theorem

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Representations

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Name of $f \in C^\omega(D)$:

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The following are computable:

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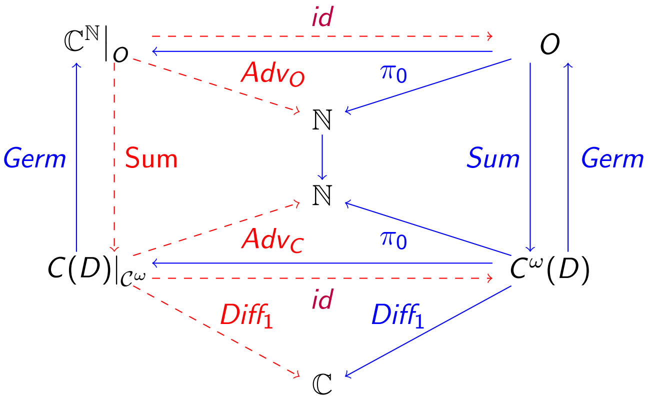
- *Differentiation.*

The results



line: $\equiv_W id_{\mathbb{N}}$

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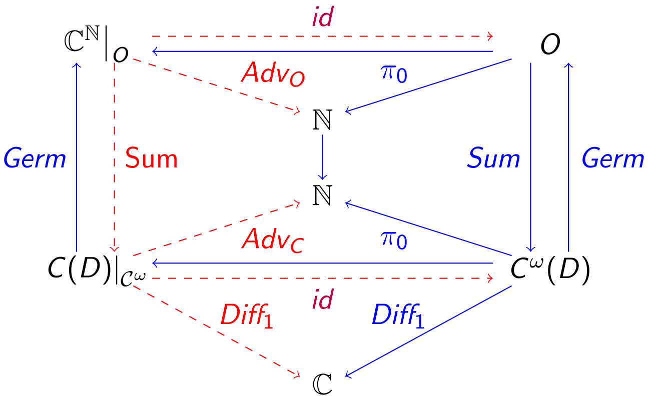

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$$a_k := \begin{cases} 0 & \text{if } p(k) = 0 \\ 1 & \text{if } p(k) > 0 \end{cases}$$

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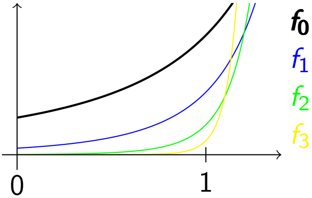
$$f_n(x) := (x - x_n)^{-2^{n+1}},$$

where $x_n := 1 + 2^{\frac{n+1}{2^{n+1}+1}}$



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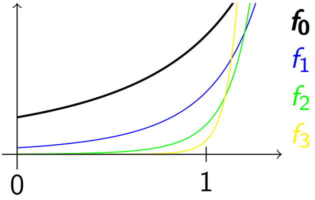

$$f(x) := \sum_{n \in supp(p)} f_n(x).$$

Sum proof idea

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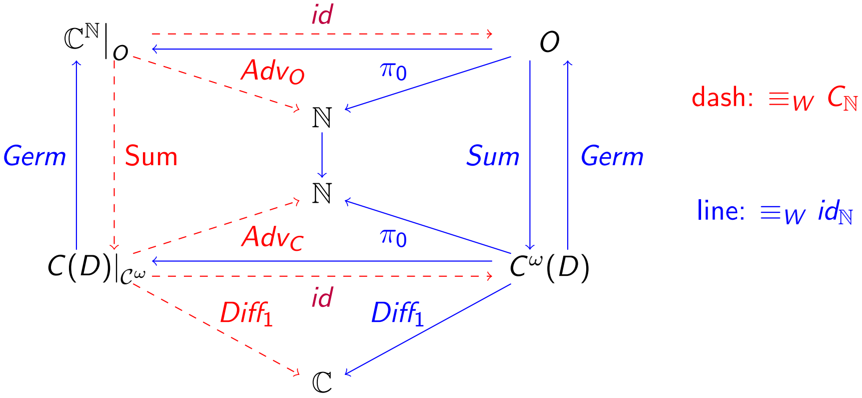

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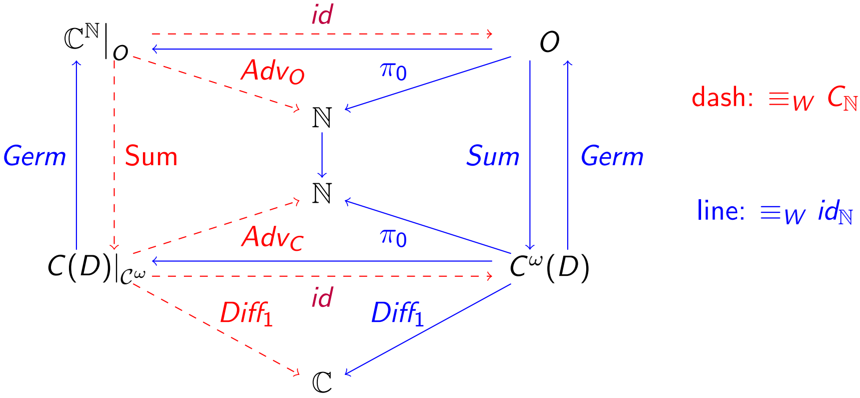

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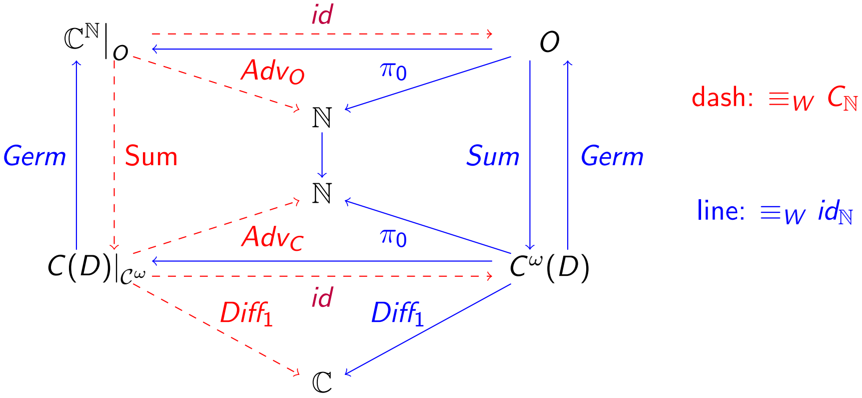

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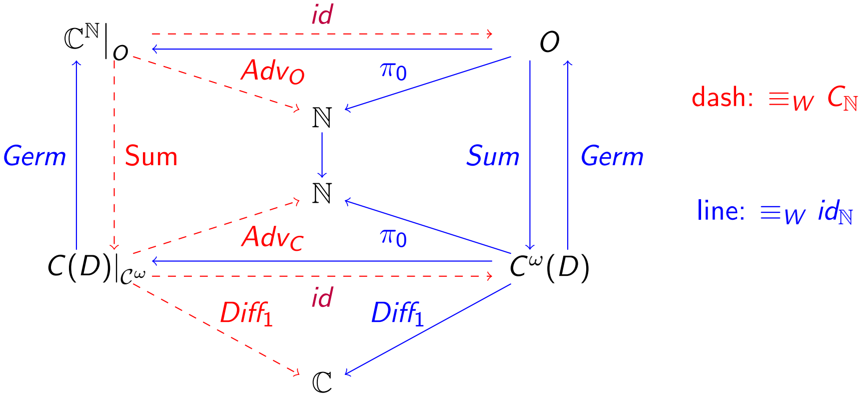
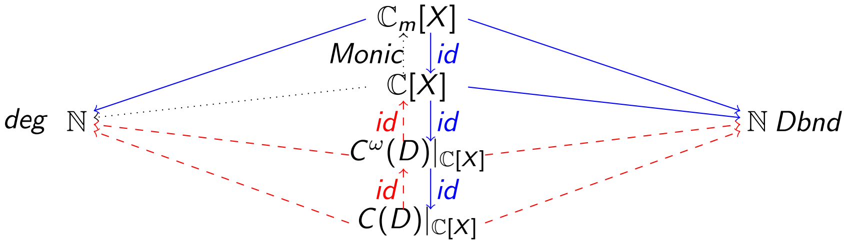


Polynomials/Outlook



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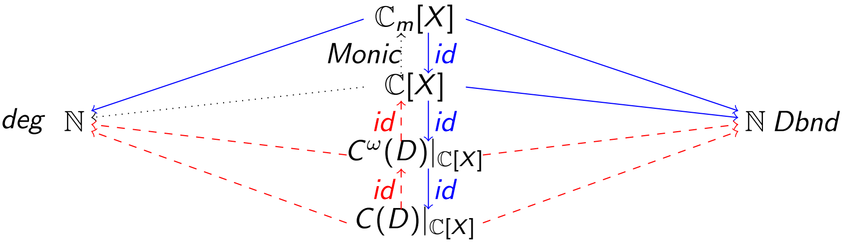
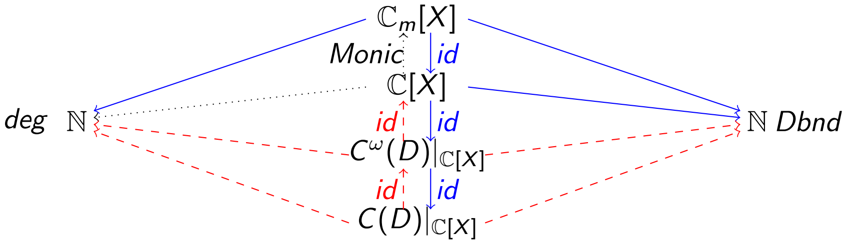


- Inclusions of test function spaces.

Polynomials/Outlook



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Future Work:

- More general domains.

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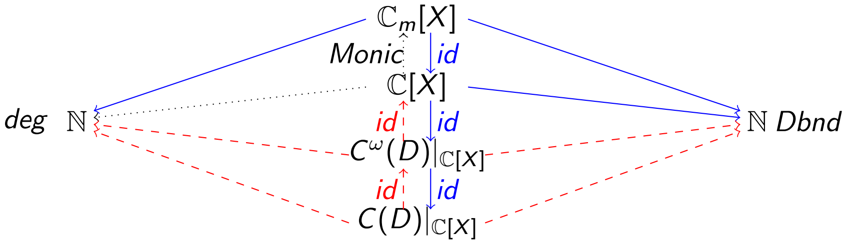

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- More general domains.
- Spaces of distributions.

Thanks!