Spy Game on Graphs

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2-Player Combinatorial Games

- Mobile agents in a graph.
- Turn-by-turn with 2 players.
- Coordination for common goal, e.g.,

Cops and Robbers (capture) (Quilliot, 1978; Nowakowski, Winkler, 1983; Bonato, Nowakowski, 2011)

Eternal Domination (protection) (Goddard et al, 2005; Klostermeyer, MacGillivray, 2009)
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Spy Game

Spy \((1^{st})\) vs guards \((2^{nd})\) in a graph \(G\).

**Start**: Spy placed at a vertex. Then, guards placed.

**Turn-by-turn**: Spy traverses up to \(s \geq 2\) edges. Guards traverse up to 1 edge.

**Goal**: Spy wants to be at least distance \(d + 1\) from all guards.

Ex : \(s = 2\) and \(d = 1\).
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Spy (1st) vs guards (2nd) in a graph $G$.

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Ex: $s = 2$ and $d = 1$. 
Guard Number: $gn_{s,d}(G)$

**Definition**

For all $s \geq 2$, $d \geq 0$ and a graph $G$, $gn_{s,d}(G)$ is the minimum number of guards guaranteed to win vs the spy.
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$gn_{2,1}(G) = 2$

$gn_{s,1}(G) \leq \gamma(G)$
Our Results: Computing $gn$

**Complexity**
Calculating $gn_{s,d}$ is NP-hard in general.

**Tight bounds for paths**
$$gn_{s,d}(P_n) = \left\lceil \frac{n}{2d + 2 + q} \right\rceil \text{ where } q = \left\lfloor \frac{2d}{s-1} \right\rfloor.$$  

**Almost tight bounds for cycles**
$$\left\lfloor \frac{n+2q}{2(d+q)+3} \right\rfloor \leq gn_{s,d}(C_n) \leq \left\lceil \frac{n+2q}{2(d+q)+1} \right\rceil \text{ where } q = \left\lfloor \frac{2d}{s-1} \right\rfloor.$$  

**Polynomial time Linear Program for trees**
Can calculate $gn_{s,d}(T)$ and a corresponding strategy in polynomial time.

**Grids**
$$\exists \beta > 0, \text{ s.t. } \Omega(n^{1+\beta}) \leq gn_{s,d}(G_{n \times n}).$$
Related Work

- Cops vs robber (capture at a distance) (Bonato et al, 2010).

  \[ \gamma_m(G) = \gamma_{\text{ns}}(G) = \gamma_{\text{ds}}(G) \] when \( s = \infty \) and \( d = 0 \).
Related Work

- Cops vs robber (capture at a distance) (Bonato et al, 2010).
- Cops vs fast robber (Fomin et al, 2010).

\[ \gamma_m(G) = g_{n^s, d}(G) \text{ when } s = \infty \text{ and } d = 0. \]
Related Work

- Cops vs robber (capture at a distance) (Bonato et al, 2010).
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  - How many cops needed in an $n \times n$ grid?

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  - $\gamma^m(m \times n \text{ grid}) \leq \left\lceil \frac{mn}{5} \right\rceil + O(m + n)$ (Lamprou et al, 2016).
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  - $\gamma^m(m \times n \text{ grid}) \leq \left\lceil \frac{mn}{5} \right\rceil + O(m + n)$ (Lamprou et al, 2016).
  - $\gamma^m(G) = gn_{s,d}(G)$ when $s = \infty$ and $d = 0$. 
**Theorem**

For all $s \geq 2$, $d \geq 0$, and a path $P_n$ on $n$ vertices,

$$gn_{s,d}(P_n) = \left\lceil \frac{n}{2d+2 + \lfloor \frac{2d}{s-1} \rfloor} \right\rceil$$

Ex: $s = 3$ and $d = 1$. 

\[ gn_{3,1}(P_{10}) = 2 \]
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![Diagram of a path with marked vertices]
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\[ \text{Diagram of a path with侦察员和间谍} \]
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Each guard protects a subpath of $2d + 2 + \lfloor \frac{2d}{s-1} \rfloor$ vertices.
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Each guard protects a subpath of \( 2d + 2 + \left\lfloor \frac{2d}{s-1} \right\rfloor \) vertices.
Cycles: Upper Bound Case $2d < s - 1$

**Theorem**

For all $s \geq 2$, $d \geq 0$ s.t. $2d < s - 1$, and a cycle $C_n$ on $n$ vertices,

$$gn_{s,d}(C_n) = \left\lceil \frac{n}{2d+3} \right\rceil.$$

Ex: $s = 6$ and $d = 0$.

$$gn_{6,0}(C_{12}) = 4$$
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For all $s \geq 2$, $d \geq 0$ s.t. $2d < s - 1$, and a cycle $C_n$ on $n$ vertices,
\[ gn_{s,d}(C_n) = \left\lceil \frac{n}{2d+3} \right\rceil. \]

Ex : $s = 6$ and $d = 0$.

\[ gn_{6,0}(C_{12}) = 4 \]
Cycles : Upper Bound Case $2d < s - 1$

Theorem

For all $s \geq 2$, $d \geq 0$ s.t. $2d < s - 1$, and a cycle $C_n$ on $n$ vertices,

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Cycles

Theorem
For all $s \geq 2$, $d \geq 0$ s.t. $q = 0$, and a cycle $C_n$ on $n$ vertices,
\[ gn_{s,d}(C_n) = \left\lceil \frac{n}{2d+3} \right\rceil. \]

Theorem
For all $s \geq 2$, $d \geq 0$ s.t. $q \neq 0$, and a cycle $C_n$ on $n$ vertices,
\[ \left\lceil \frac{n+2q}{2(d+q)+3} \right\rceil \leq gn_{s,d}(C_n) \leq \left\lceil \frac{n+2q}{2(d+q)+1} \right\rceil. \]

Reminder: $q = \left\lfloor \frac{2d}{s-1} \right\rfloor$. 
Trees are Harder

Paths: 1 guard per subpath of $2d + 2 + \left\lfloor \frac{2d}{s-1} \right\rfloor$ vertices.
Trees are Harder

Can’t always divide tree into subtrees protected by a certain number of guards.

Example of a tree $T$ where $s = 2$, $d = 1$ and $gn_{2,1}(T) = 4$. 
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Fractional Version of the Game

- Guards may be fractional entities; movements rep. by flows.

- Unchanged for spy. Total fraction of guards distance $\leq d$ from spy must be $\geq 1$. 

Linear program to compute optimal fractional strategy. Optimal fractional strategy $\Rightarrow$ optimal integral strategy in trees. $s = 2, d = 1$. 

$C^2_g(\mathcal{G}) = 2$ but $1.5$ guards suffice.
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$s = 2, \ d = 1$.

$ gn_{2,1}(C_6) = 2$

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Linear program to compute optimal fractional strategy.

$g_{n_2,1}(C_6) = 2$ but 1.5 guards suffice.
**Fractional Version of the Game**

- Guards may be *fractional* entities; movements rep. by flows.

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- **Linear program** to compute optimal fractional strategy.

- Optimal fractional strategy $\Rightarrow$ optimal integral strategy in trees.

$s = 2, d = 1$. 

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1.5 guards suffice.
Theorem: Can transform optimal fractional strategy into optimal integral strategy in polynomial time.

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Tree’s protection and guards’ movements preserved.

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Restricted Strategies

\[ f : V^k \times V \Rightarrow V^k \text{ (Unrestricted strategy)} \]

\[ \omega : V \Rightarrow V^k \text{ (Restricted strategy)} \]
Restricted Strategies

\[ f : V^k \times V \Rightarrow V^k \] (Unrestricted strategy)

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- Guards’ positions depend only on position of spy.
- 1 Unique configuration for guards for each position of spy.

Theorem
Optimal fractional strategy \( \Rightarrow \) optimal fractional restricted strategy in trees.

Can calculate optimal restricted fractional strategies with Linear Program in polynomial time.

Cohen, Martins, Mc Inerney, Nisse, Pérennes, Sampaio
Spy Game on Graphs
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Optimal fractional strategy \( \Rightarrow \) optimal fractional restricted strategy in trees.

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Restricted strategy: $\omega: V \Rightarrow V^k$

$\omega_{x,u}$: quantity of guards on $u$ when spy is on $x$.

$f_{x,x',u,u'}$: quantity of guards that go from $u$ to $u'$ when spy goes from $x$ to $x'$. 
Linear Program to Compute Restricted Strategy

Restricted strategy : \( \omega : V \Rightarrow V^k \)

\( \omega_{x,u} \) : quantity of guards on \( u \) when spy is on \( x \).

\( f_{x,x',u,u'} \) : quantity of guards that go from \( u \) to \( u' \) when spy goes from \( x \) to \( x' \).

(1) Minimize \( \sum_{v \in V} \omega_{x_0,v} \)

Minimize number of guards.
Restricted strategy: $\omega : V \Rightarrow V^k$

$\omega_{x,u} :$ quantity of guards on $u$ when spy is on $x$.

$f_{x,x',u,u'} :$ quantity of guards that go from $u$ to $u'$ when spy goes from $x$ to $x'$.

\[ (2) \sum_{v \in N_d[x]} \omega_{x,v} \geq 1 \quad \forall x \in V \]

Guarantees always at least 1 guard within distance $d$ of spy.
Restricted strategy: $\omega : V \Rightarrow V^k$

$\omega_{x,u}$: quantity of guards on $u$ when spy is on $x$.

$f_{x,x',u,u'}$: quantity of guards that go from $u$ to $u'$ when spy goes from $x$ to $x'$.

\begin{align*}
(3) \quad \sum_{u' \in N[u]} f_{x,x',u,u'} &= \omega_{x,u} \quad \forall u \in V, x' \in N_s[x] \\
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\end{align*}

Guarantees validity of moves of guards when spy moves.
Linear Program

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Guarantees validity of moves of guards when spy moves.

$O(n^4)$ real variables and constraints.
Main Result: $gn$ in Trees

**Theorem**

$\forall s > 1, \ d \geq 0$ and all trees $T$, $gn_{s,d}(T)$ and a corresponding strategy can be calculated in polynomial time.

**Idea of proof**: Linear Program can compute opt. frac. restr. strategy in polynomial time.

Run LP. From previous theorem, strategy is opt. frac.

Can transform opt. frac. into opt. int. in polynomial time.
Grids

Theorem

\[ \exists \beta > 0, \text{ s.t. } \forall s > 1, d \geq 0, \Omega(n^{1+\beta}) \leq gn_{s,d}(G_{n \times n}). \]

Idea of proof: Lower bound holds for fractional version.
Grids

Theorem

\[ \exists \beta > 0, \text{ s.t. } \forall s > 1, d \geq 0, \Omega(n^{1+\beta}) \leq g_{n,s,d}(G_{n \times n}). \]

**Idea of proof**: Lower bound holds for fractional version.

Torus and grid have *same order* of number of guards.

Theorem

\[ \exists \alpha \geq \log(3/2) \approx 0.58, \text{ s.t. } \forall s > 1, d \geq 0, \]
\[ fgn_{s,d}(G_{n \times n}) \leq O(n^{2-\alpha}). \]

**Idea of proof**: Density function \( \omega^*(v) = \frac{c}{(dist(v,v_0)+1)^{\log 3/2}} \) for a constant \( c > 0 \) satisfies LP.
Distribution of Guards in the Torus for an optimal symmetrical spy-positional strategy when $n = 100$, $m = 100$, $s = 2$ and $d = 1$
Further Work

- Determine $gn_{s,d}(G_{n \times n})$.

- Approximate $gn_{s,d}(G)$ in polynomial time in certain classes of graphs?

- Fractional approach applied to other combinatorial games.
Thanks!