Spy Game on Graphs

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Corner Brook, Nfld., August 8, 2017

GRASCan 2017

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- Turn-by-turn with 2 players.
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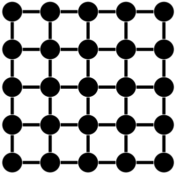

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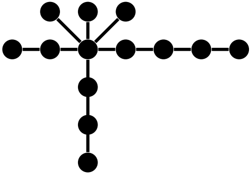

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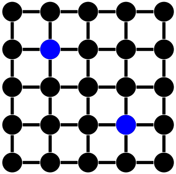

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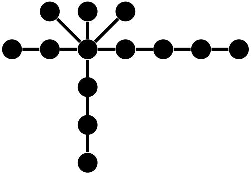

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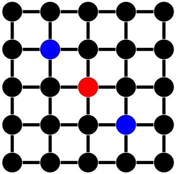

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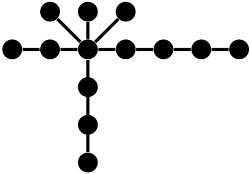

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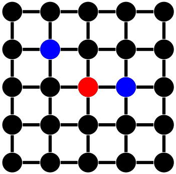

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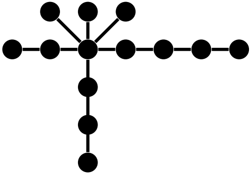

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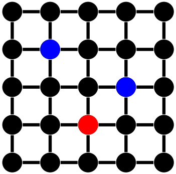

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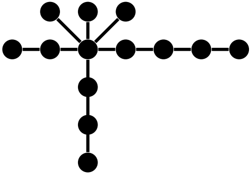

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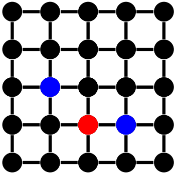

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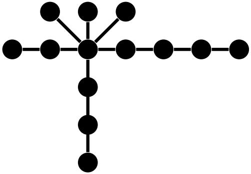

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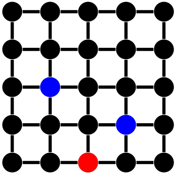

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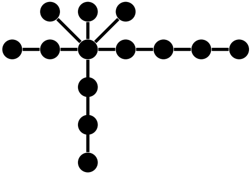

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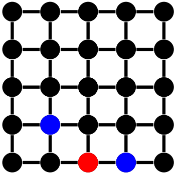

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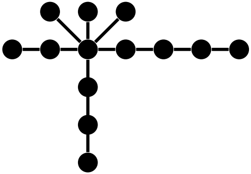

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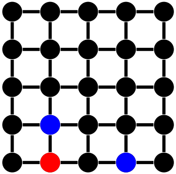

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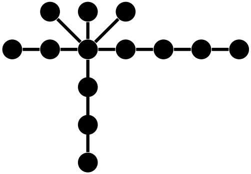

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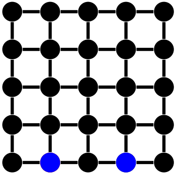

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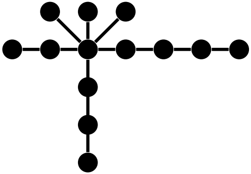

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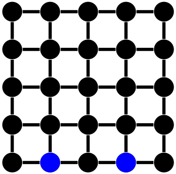

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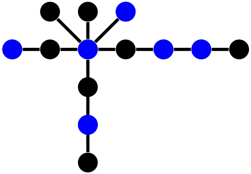

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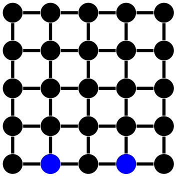

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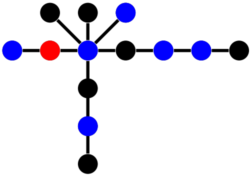

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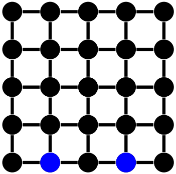

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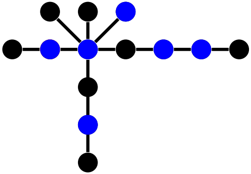

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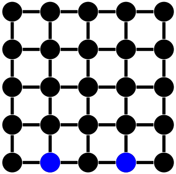

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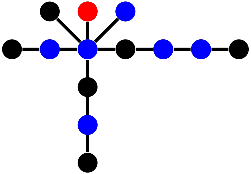

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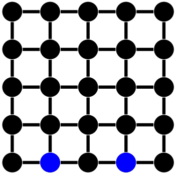

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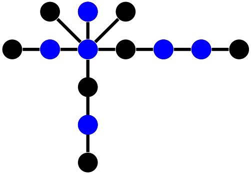

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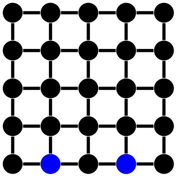

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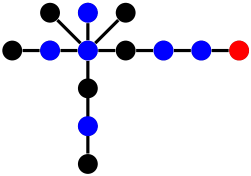

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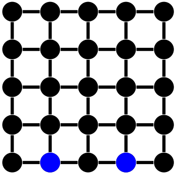

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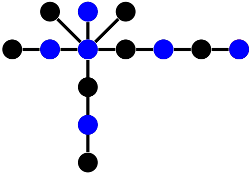

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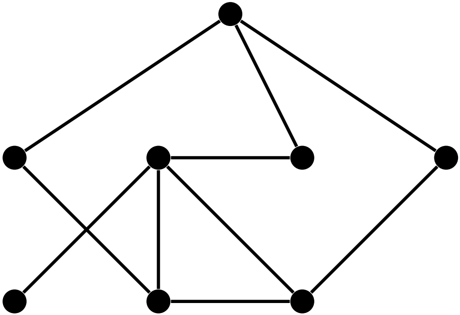
Spy (1^{st}) vs guards (2^{nd}) in a graph G.

Start : Spy placed at a vertex. Then, guards placed.

Turn-by-turn : Spy traverses up to $\mathbf{s} \geq 2$ edges. Guards traverse up to 1 edge.

Goal : Spy wants to be at least distance $\mathbf{d} + 1$ from all guards.

Ex : $\mathbf{s} = 2$ and $\mathbf{d} = 1$.



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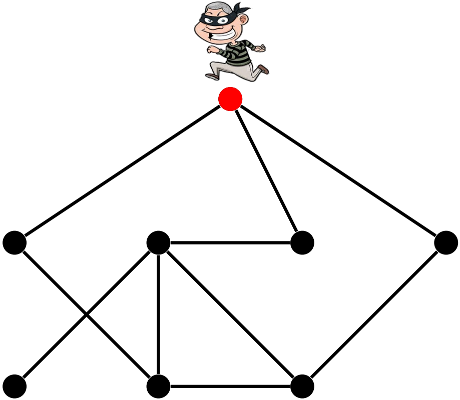

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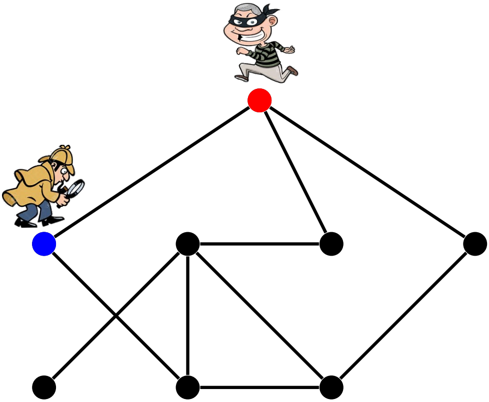

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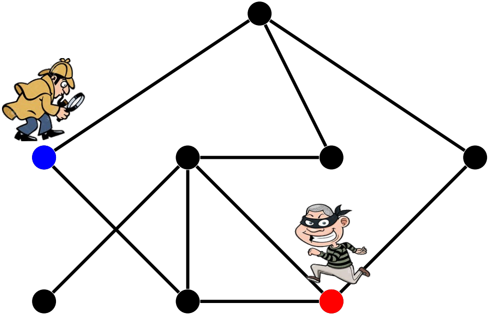

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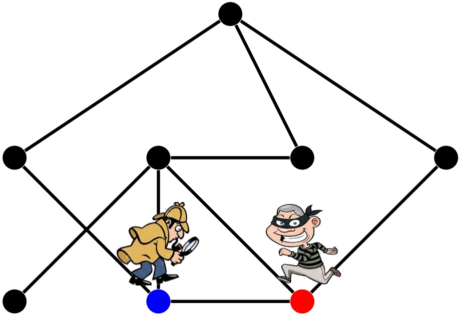

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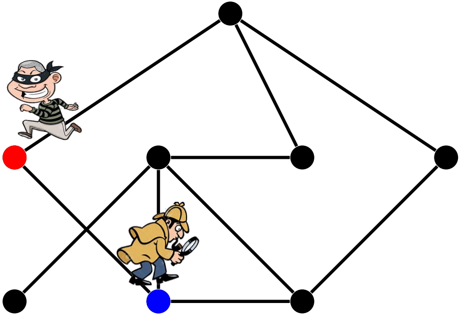

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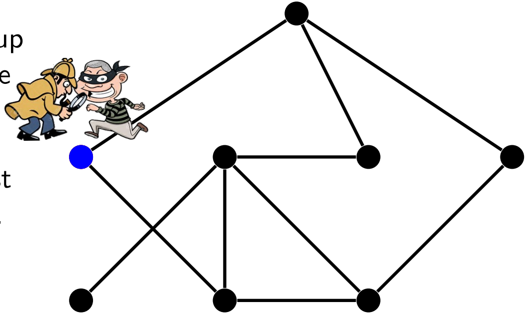

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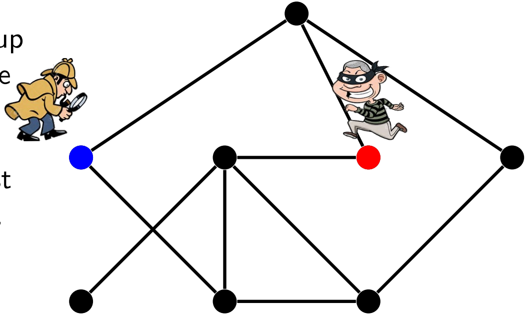

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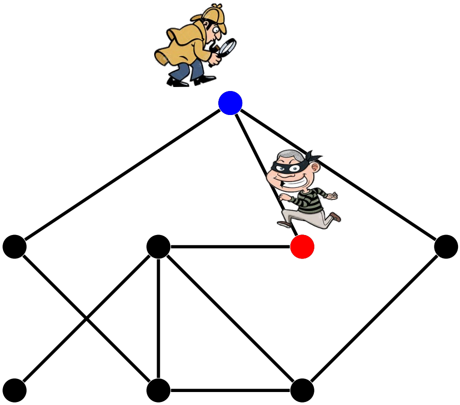

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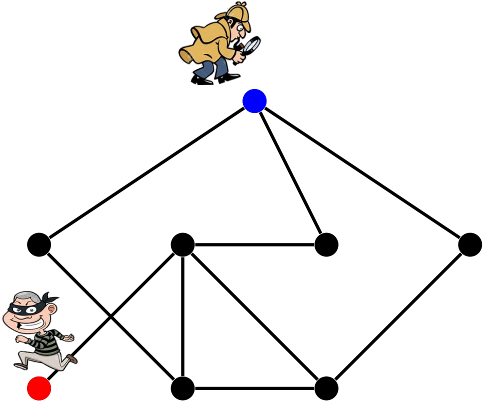

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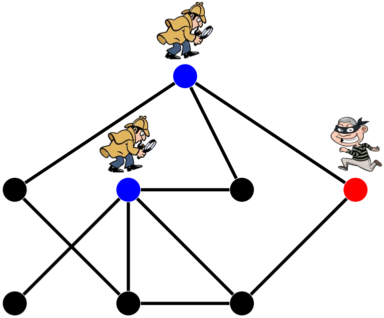

$$gn_{2,1}(G) = 2$$

$$gn_{s,1}(G) \leq \gamma(G)$$

Our Results : Computing *gn*

Complexity

Calculating $gn_{s,d}$ is NP-hard in general.

Tight bounds for paths

$gn_{s,d}(P_n) = \left\lceil \frac{n}{2d+2+q} \right\rceil$ where $q = \lfloor \frac{2d}{s-1} \rfloor$.

Almost tight bounds for cycles

$\left\lceil \frac{n+2q}{2(d+q)+3} \right\rceil \leq gn_{s,d}(C_n) \leq \left\lceil \frac{n+2q}{2(d+q)+1} \right\rceil$ where $q = \lfloor \frac{2d}{s-1} \rfloor$.

Polynomial time Linear Program for trees

Can calculate $gn_{s,d}(T)$ and a corresp. strategy in polynomial time.

Grids

$\exists \beta > 0$, s.t. $\Omega(n^{1+\beta}) \leq gn_{s,d}(G_{n \times n})$.

Related Work

- Cops vs robber (capture at a distance) (Bonato et al, 2010).

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Paths

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Paths : Lower bound

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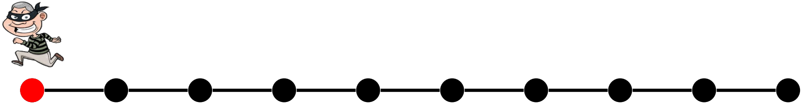

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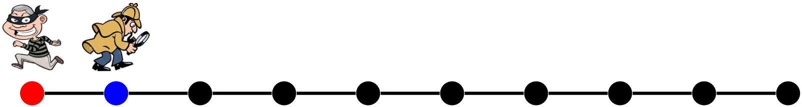

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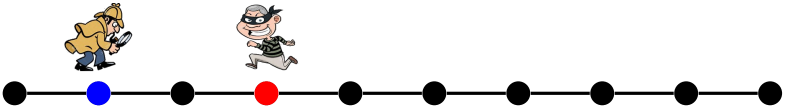

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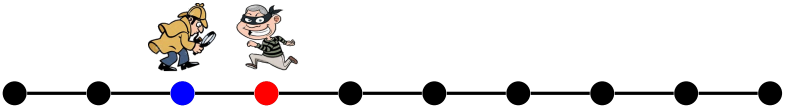

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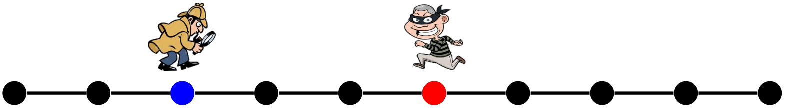

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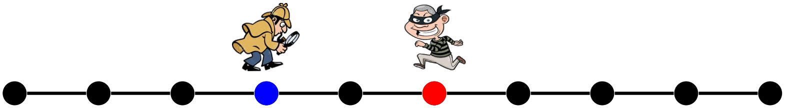

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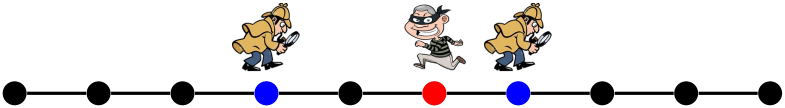

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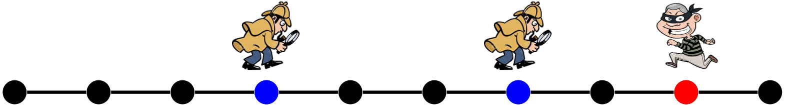

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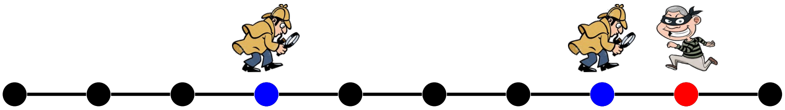

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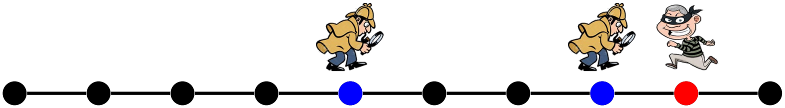

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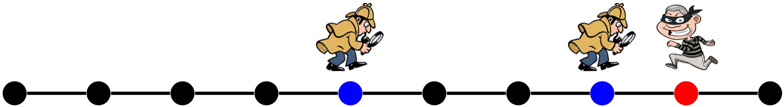

$$gn_{3,1}(P_{10}) = 2$$

Paths : Upper bound

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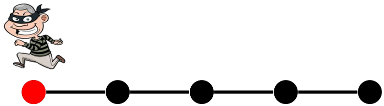

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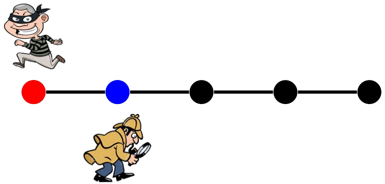

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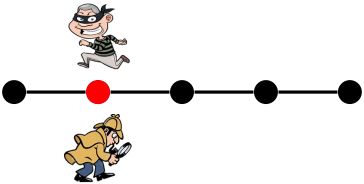

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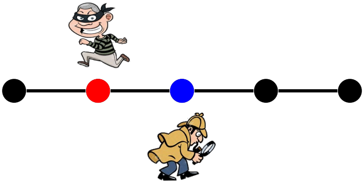

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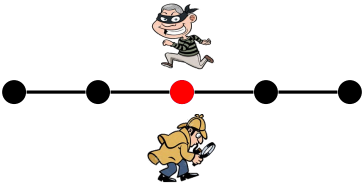

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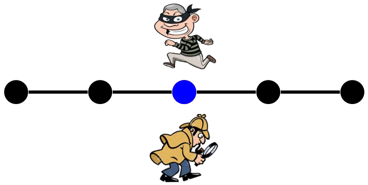

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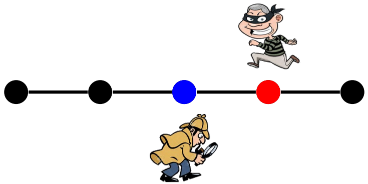

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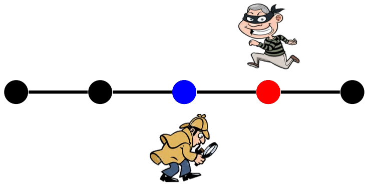

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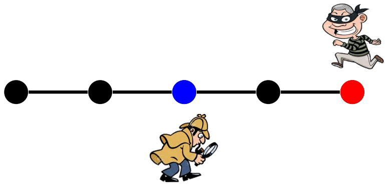

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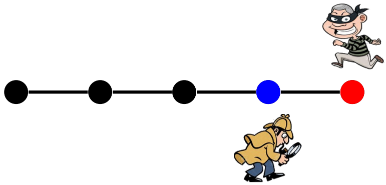

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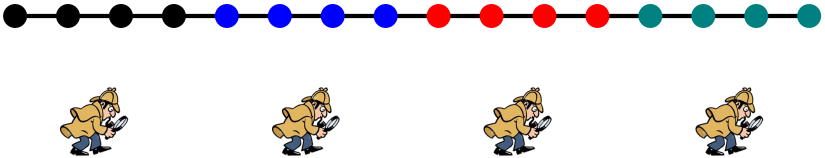


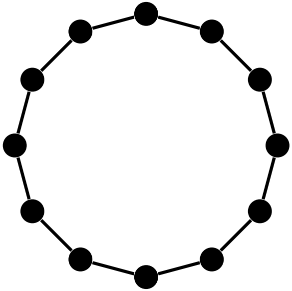
Cycles : Upper Bound Case $2d < s - 1$

Theorem

For all $s \geq 2$, $d \geq 0$ s.t. $2d < s - 1$, and a cycle C_n on n vertices, $gn_{s,d}(C_n) = \left\lceil \frac{n}{2d+3} \right\rceil$.

Ex : $s = 6$ and $d = 0$.

$$gn_{6,0}(C_{12}) = 4$$



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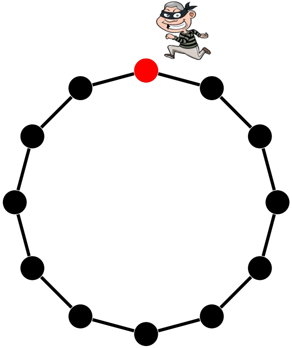

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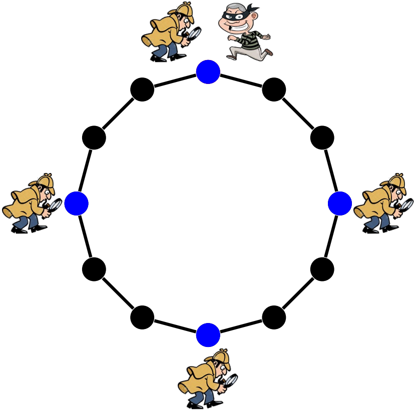

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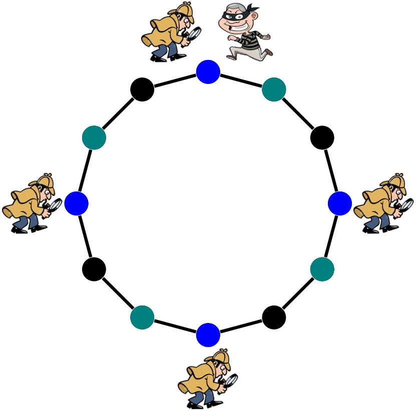

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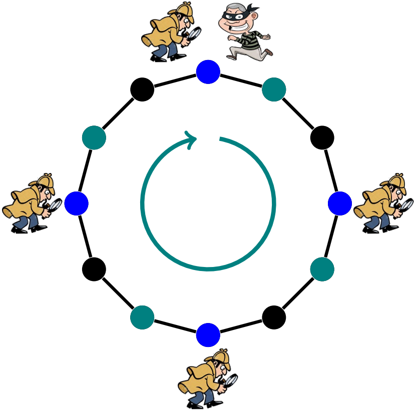

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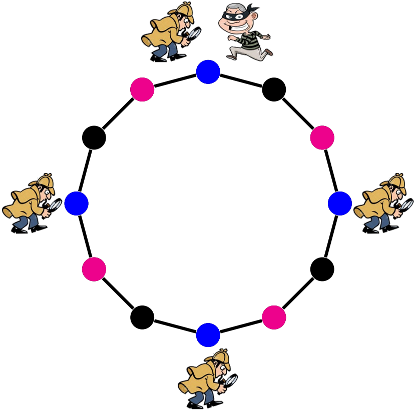

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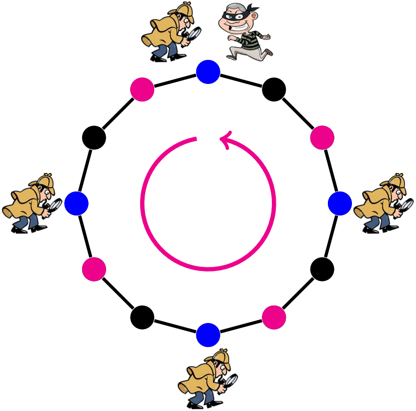


Cycles

Theorem

For all $s \geq 2$, $d \geq 0$ s.t. $q = 0$, and a cycle C_n on n vertices, $gn_{s,d}(C_n) = \left\lceil \frac{n}{2d+3} \right\rceil$.

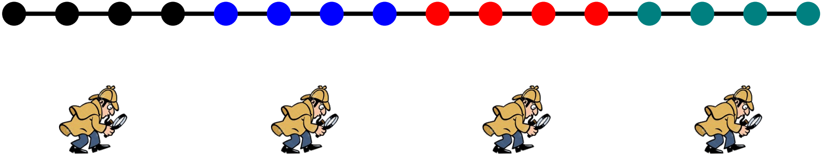
Theorem

For all $s \geq 2$, $d \geq 0$ s.t. $q \neq 0$, and a cycle C_n on n vertices, $\left\lceil \frac{n+2q}{2(d+q)+3} \right\rceil \leq gn_{s,d}(C_n) \leq \left\lceil \frac{n+2q}{2(d+q)+1} \right\rceil$.

Reminder : $q = \left\lfloor \frac{2d}{s-1} \right\rfloor$.

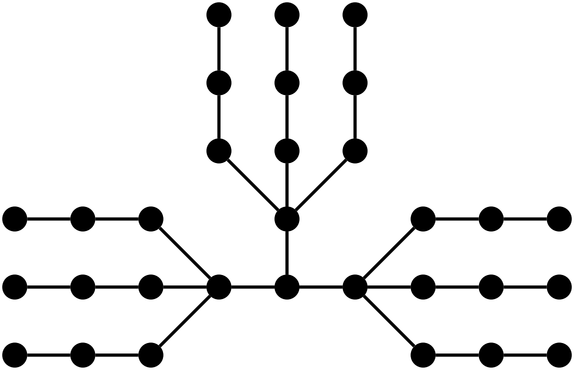
Trees are Harder

Paths : 1 guard per subpath of $2d + 2 + \lfloor \frac{2d}{s-1} \rfloor$ vertices.



Trees are Harder

Can't always divide tree into subtrees protected by a certain number of guards.



Example of a tree T where $s = 2$, $d = 1$ and $gn_{2,1}(T) = 4$.

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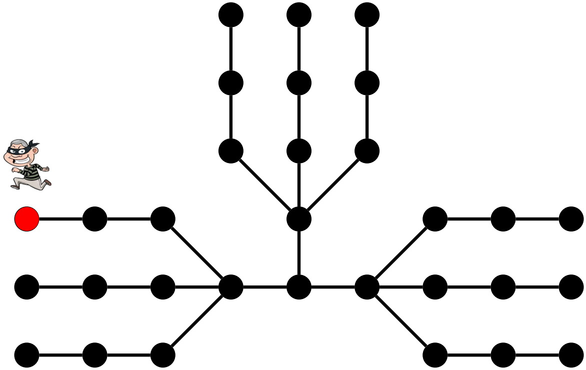

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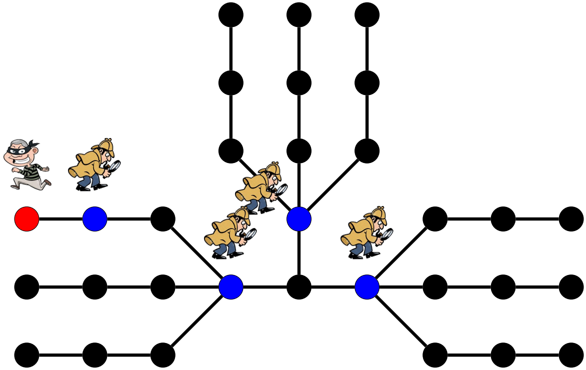

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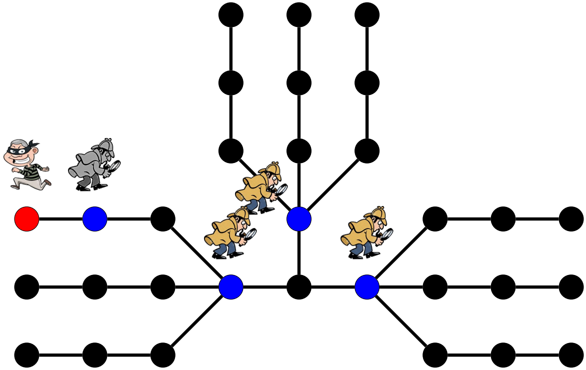

Example of a tree T where $s = 2$, $d = 1$ and $gn_{2,1}(T) = 4$.

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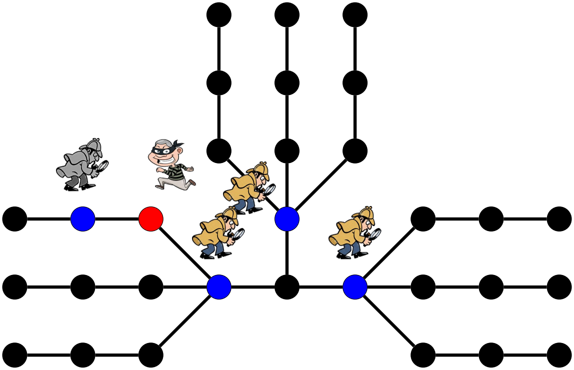

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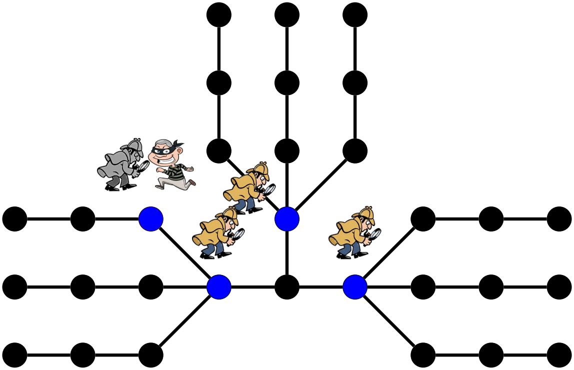

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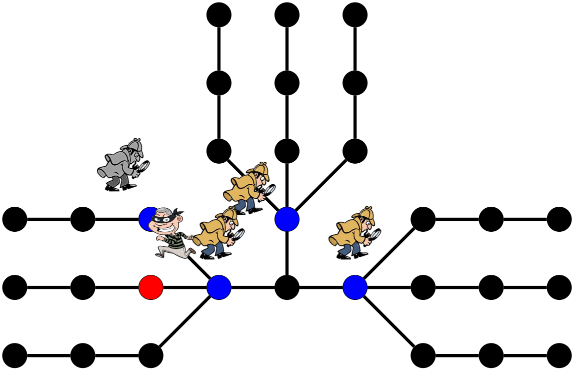

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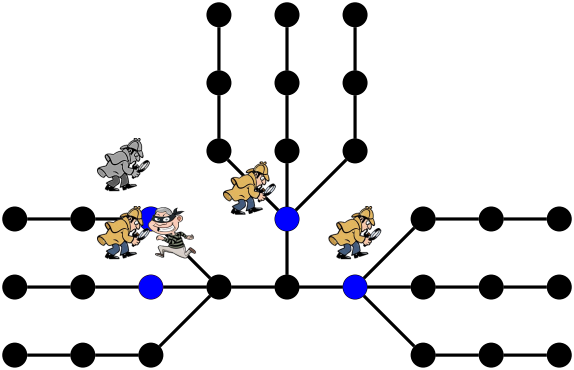

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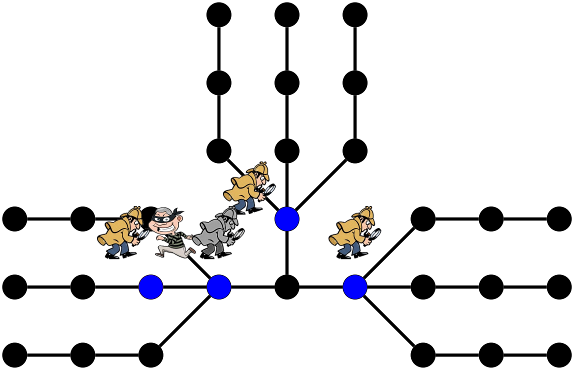

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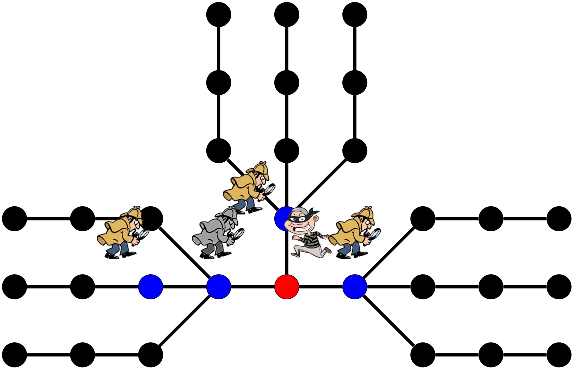

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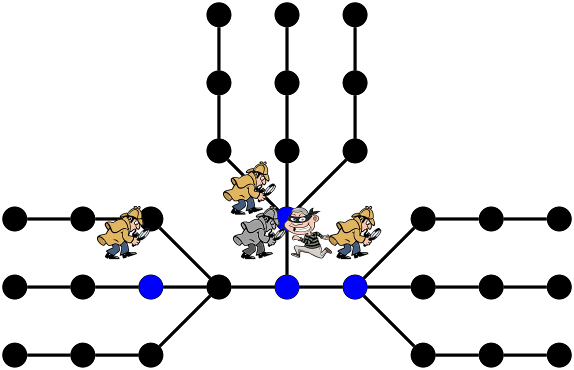

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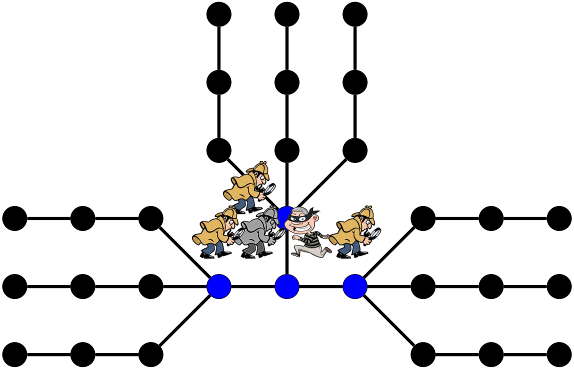

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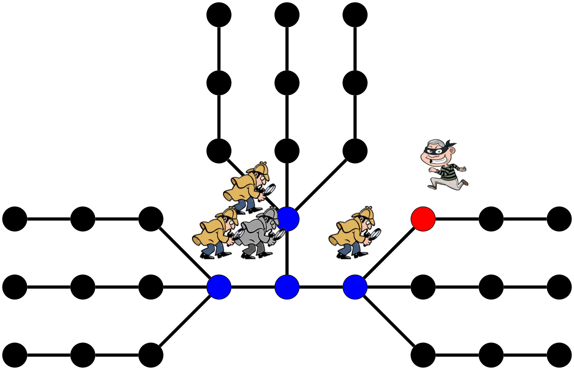

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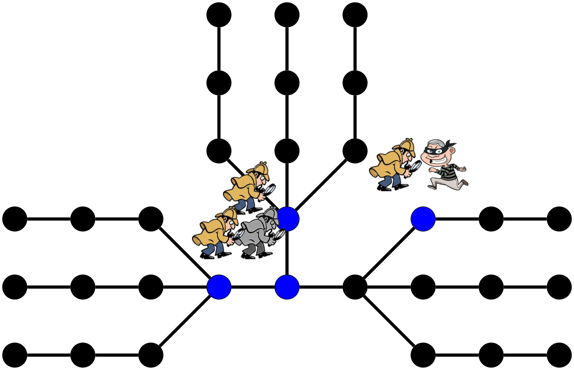

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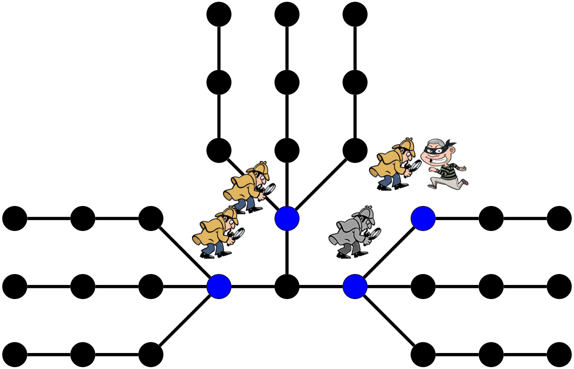

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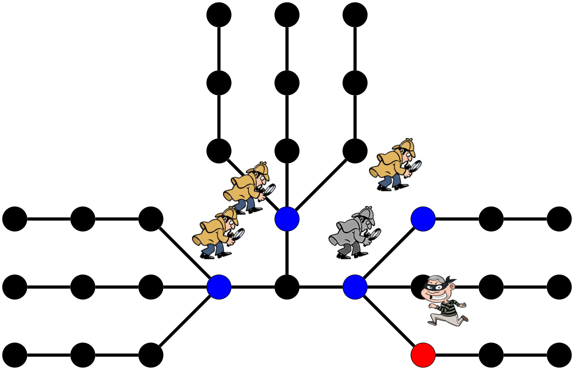

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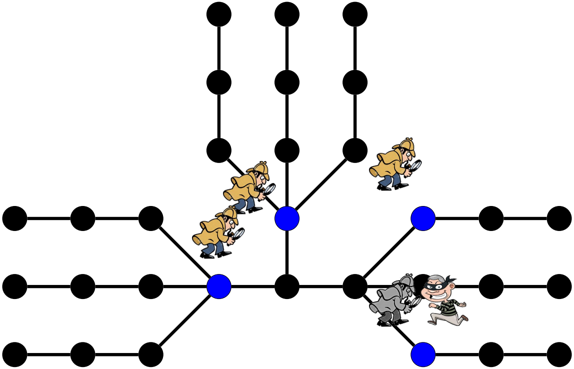

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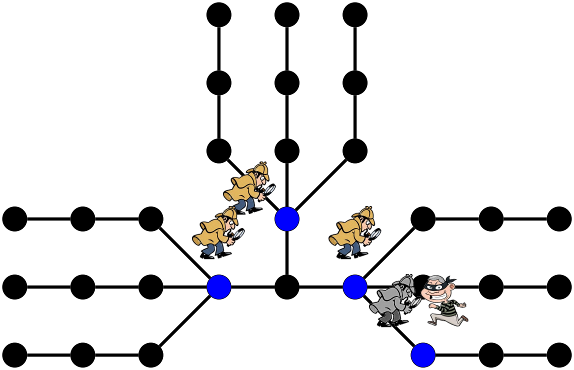

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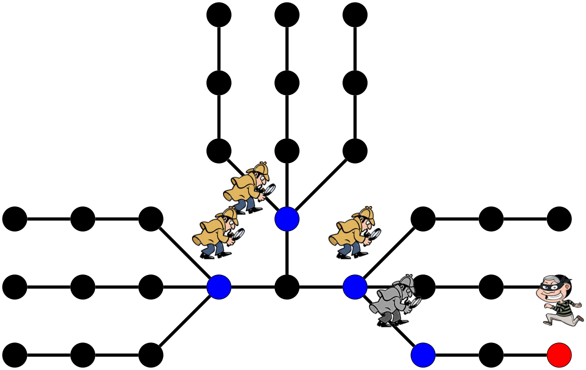

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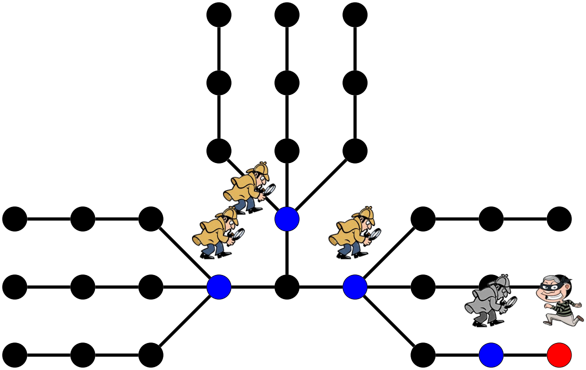

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Fractional Version of the Game

- Guards may be fractional entities ; movements rep. by flows.
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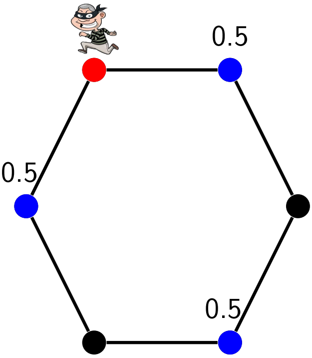

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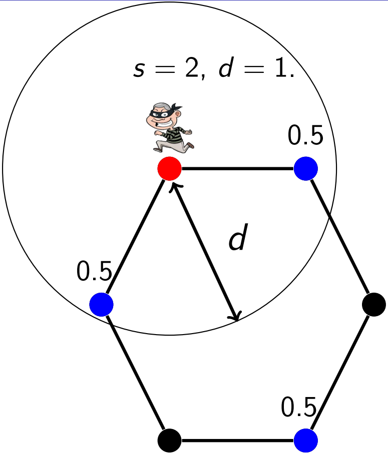

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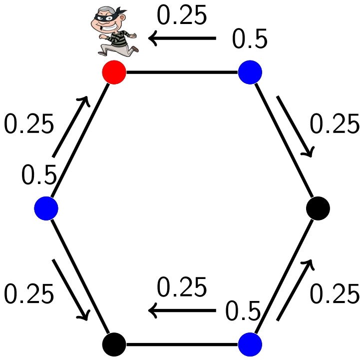

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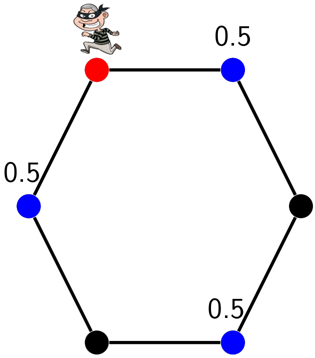

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- Optimal fractional strategy $\Rightarrow$ optimal integral strategy in trees.

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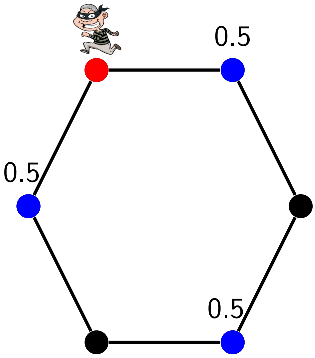

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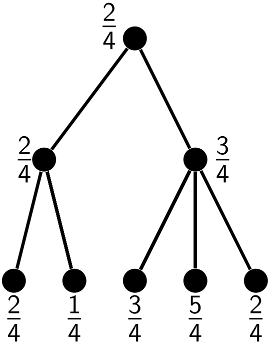
Opt. Fractional Strategy ⇒ Opt. Integral Strategy in Trees

Theorem : Can transform optimal fractional strategy into optimal integral strategy in polynomial time.



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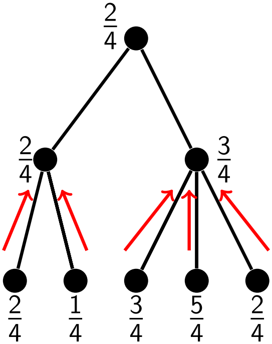

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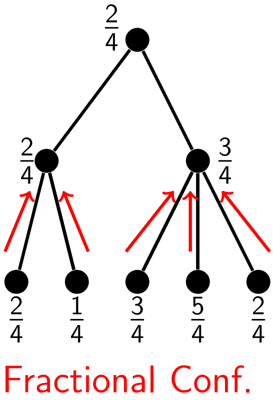

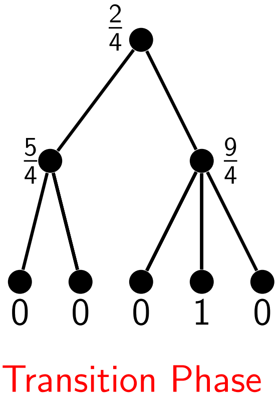

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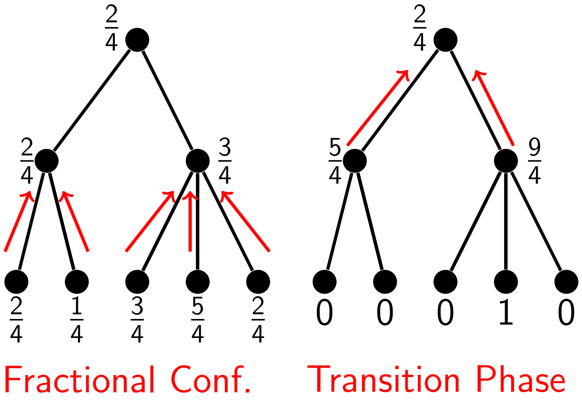

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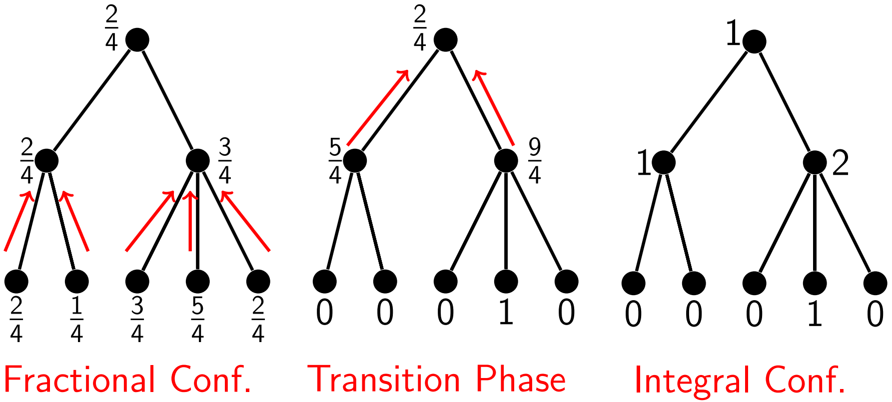

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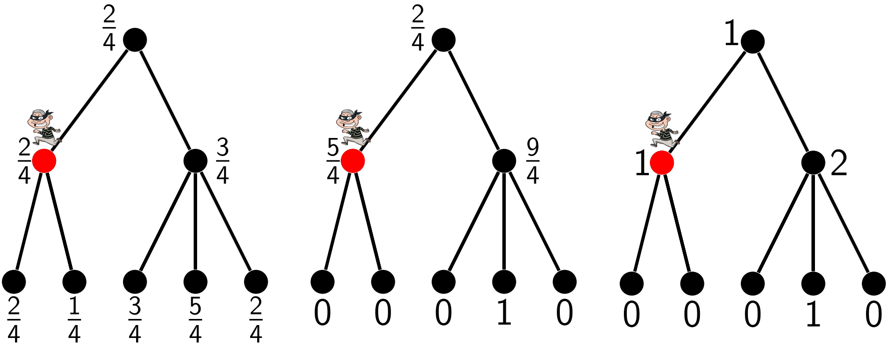
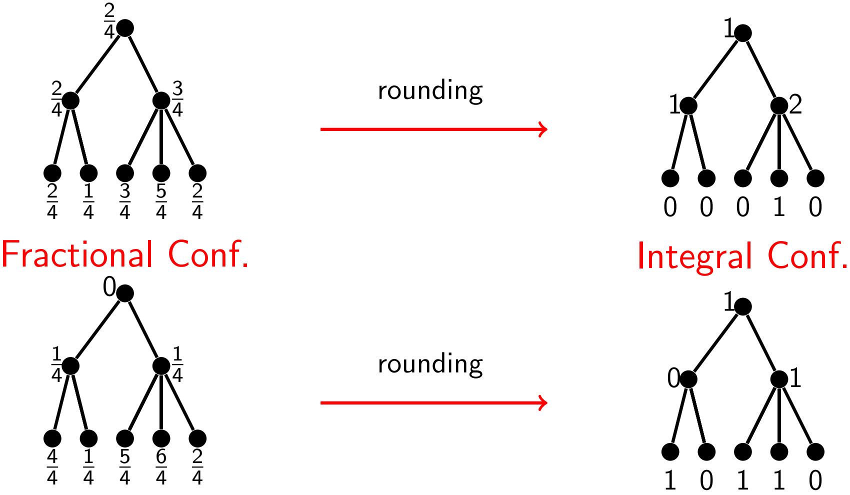


Fractional Conf. Transition Phase Integral Conf.

Tree's protection and guards' movements preserved.

Opt. Fractional Strategy ⇒ Opt. Integral Strategy in Trees



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Restricted Strategies

$f : V^k \times V \Rightarrow V^k$ (Unrestricted strategy)

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Optimal fractional strategy $\Rightarrow$ optimal fractional restricted strategy in trees.

Can calculate optimal restricted fractional strategies with Linear Program in polynomial time.

Linear Program to Compute Restricted Strategy

Restricted strategy : $\omega : V \Rightarrow V^k$

$\omega_{x,u}$: quantity of guards on u when spy is on x.

$f_{x,x',u,u'}$: quantity of guards that go from u to u' when spy goes from x to x'.

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$$(1)\ \text{Minimize} \sum_{v \in V} \omega_{x_0,v}$$

Minimize number of guards.

Linear Program

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$$(2) \sum_{v \in N_d[x]} \omega_{x,v} \geq 1 \qquad \forall x \in V$$

Guarantees always at least 1 guard within distance d of spy.

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$$(3) \sum_{u' \in N[u]} f_{x,x',u,u'} = \omega_{x,u} \qquad \forall u \in V,\ x' \in N_s[x]$$

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$O(n^4)$ real variables and constraints.

Main Result : *gn* in Trees

Theorem

$\forall s > 1$, $d \geq 0$ and all trees T, $gn_{s,d}(T)$ and a corresponding strategy can be calculated in polynomial time.

Idea of proof : Linear Program can compute opt. frac. restr. strategy in polynomial time.

Run LP. From previous theorem, strategy is opt. frac.

Can transform opt. frac. into opt. int. in polynomial time.

Grids

Theorem

$\exists \beta > 0$, s.t. $\forall s > 1$, $d \geq 0$, $\Omega(n^{1+\beta}) \leq gn_{s,d}(G_{n\times n})$.

Idea of proof : Lower bound holds for fractional version.

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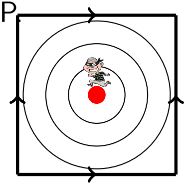
Idea of proof : Lower bound holds for fractional version.

Torus and grid have same order of number of guards.

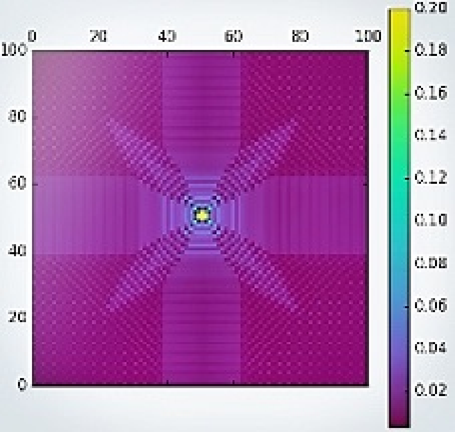
Theorem

$\exists \alpha \geq log(3/2) \approx 0.58$, s.t. $\forall s > 1$, $d \geq 0$,
$fgn_{s,d}(G_{n \times n}) \leq O(n^{2-\alpha})$.

Idea of proof : Density function $\omega^*(v) = \frac{c}{(dist(v,v_0)+1)^{log 3/2}}$ for a constant $c > 0$ satisfies LP.



Distribution of Guards in the Torus for an optimal symmetrical spy-positional strategy when $n = 100$, $m = 100$, $s = 2$ and $d = 1$



Further Work

- Determine $gn_{s,d}(G_{n\times n})$.
- Approximate $gn_{s,d}(G)$ in polynomial time in certain classes of graphs ?
- Fractional approach applied to other combinatorial games.

Thanks !