Spy Game on Graphs

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COATI Seminar

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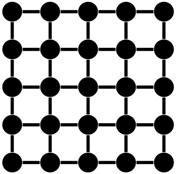
- Turn-by-turn with 2 players.
  - Coordination for common goal, e.g.,



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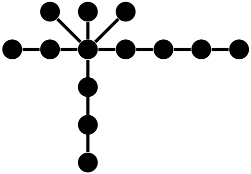

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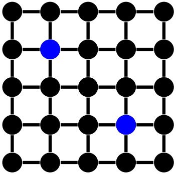

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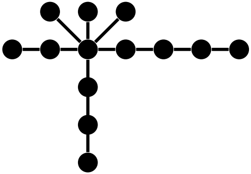

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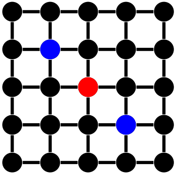

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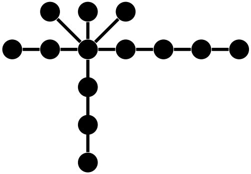

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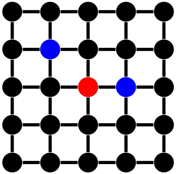

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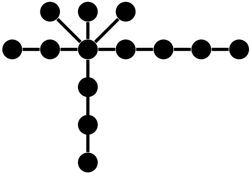

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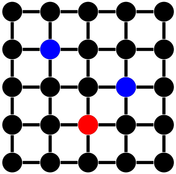

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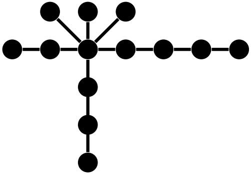

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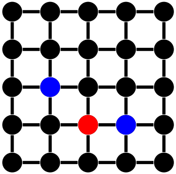

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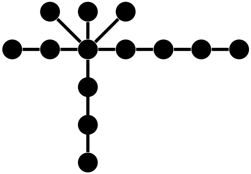

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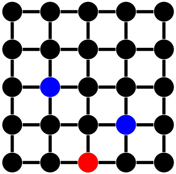

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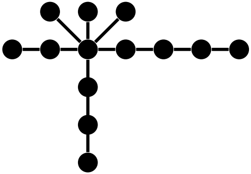

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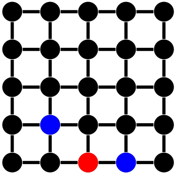

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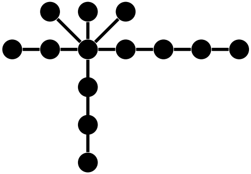

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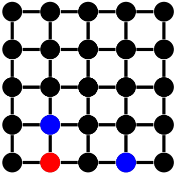

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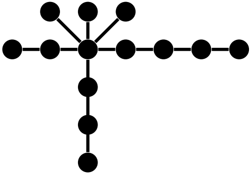

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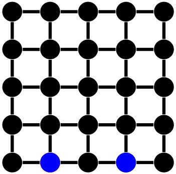

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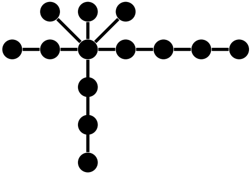

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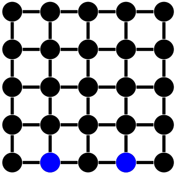

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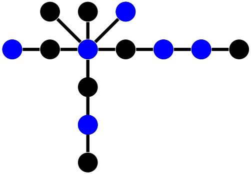

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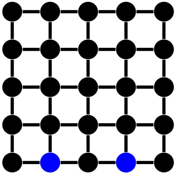

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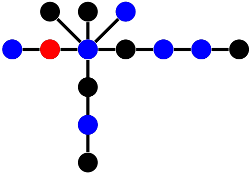

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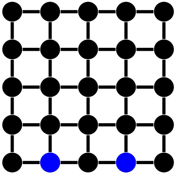

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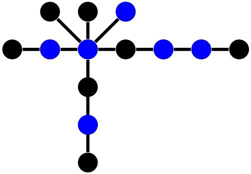

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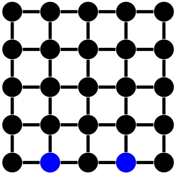

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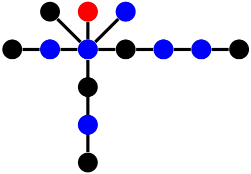

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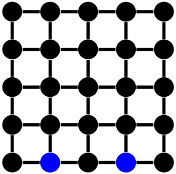

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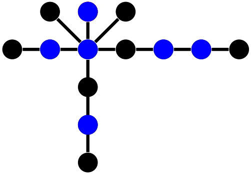

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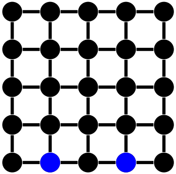

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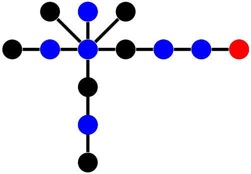

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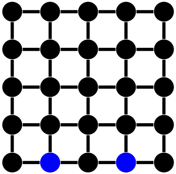

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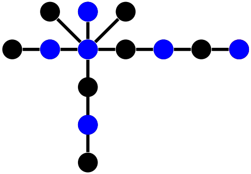

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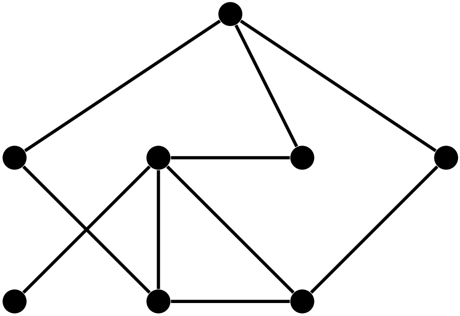
Spy (1^{st}) vs guards (2^{nd}) in a graph G.

Start : Spy placed at a vertex. Then, guards placed.

Turn-by-turn : Spy traverses up to $\mathbf{s} \geq 2$ edges. Guards traverse up to 1 edge.

Goal : Spy wants to be at least distance $\mathbf{d} + 1$ from all guards.

Ex : $\mathbf{s} = 2$ and $\mathbf{d} = 1$.



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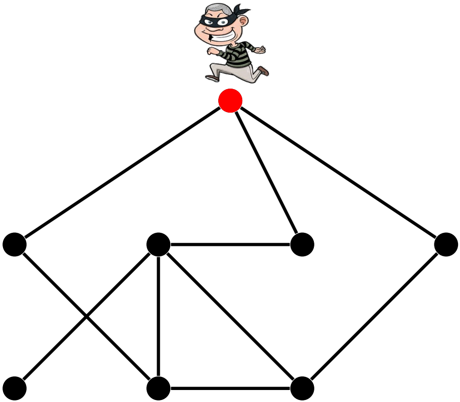

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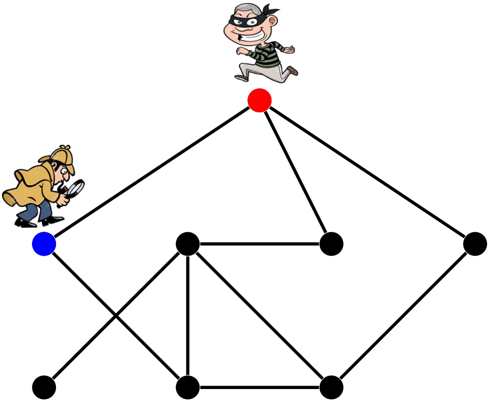

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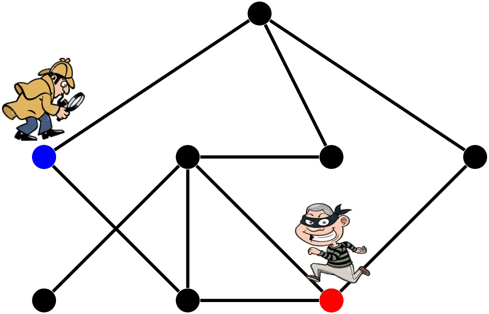

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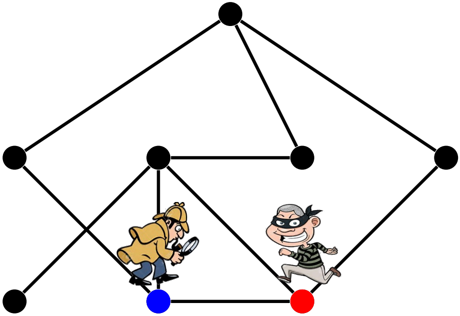

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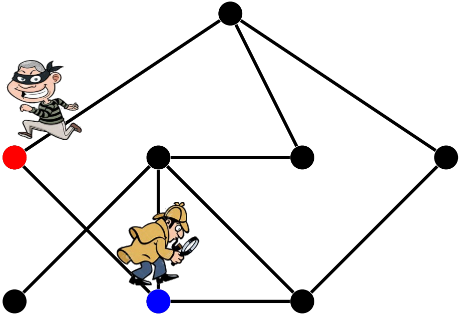

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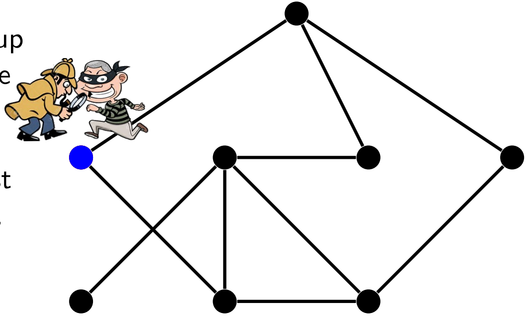

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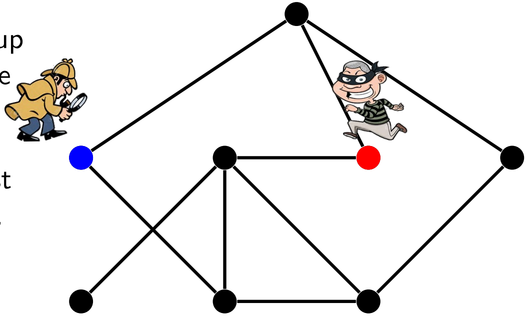

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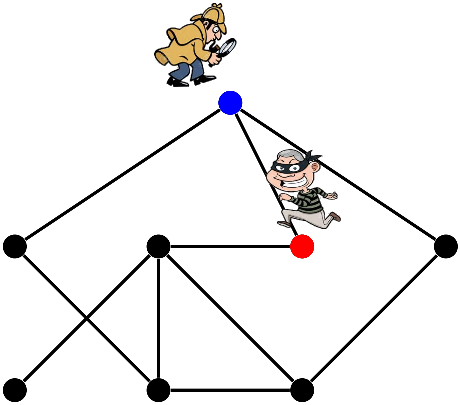

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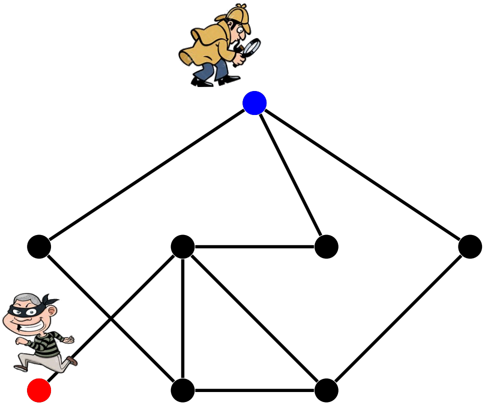

Guard Number : $gn_{s,d}(G)$

Definition

For all $s \geq 2$, $d \geq 0$ and a graph G, $gn_{s,d}(G)$ is the minimum number of guards guaranteed to win vs the spy.

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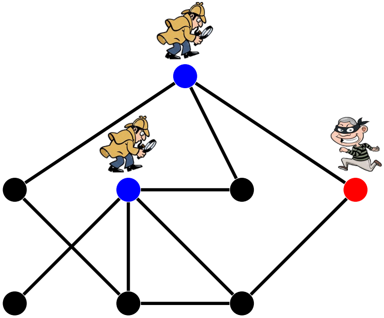

$$gn_{2,1}(G) = 2$$

$$gn_{s,1}(G) \leq \gamma(G)$$

Our Results : Computing gn

Complexity

Calculating $gn_{s,d}$ is NP-hard in general.

Tight bounds for paths

$gn_{s,d}(P_n) = \left\lceil \frac{n}{2d+2+q} \right\rceil$ where $q = \lfloor \frac{2d}{s-1} \rfloor$.

Almost tight bounds for cycles

$gn_{s,d}(C_n) \backsimeq \left\lceil \frac{n+2q}{2(d+q)+3} \right\rceil$ where $q = \lfloor \frac{2d}{s-1} \rfloor$.

Polynomial time Linear Program for trees

Can calculate $gn_{s,d}(T)$ and a corresp. strategy in polynomial time.

Grids

$\exists \beta > 0$, s.t. $\Omega(n^{1+\beta}) \leq gn_{s,d}(G_{n \times n})$.

Related Work

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Paths

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For all $s \geq 2$, $d \geq 0$, and a path P_n on n vertices,
$gn_{s,d}(P_n) = \left\lceil \frac{n}{2d+2+\lfloor \frac{2d}{s-1} \rfloor} \right\rceil$

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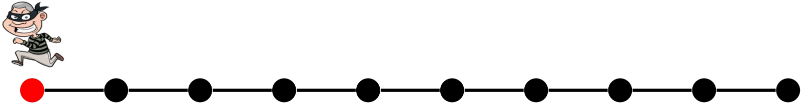

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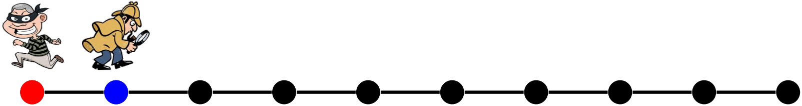

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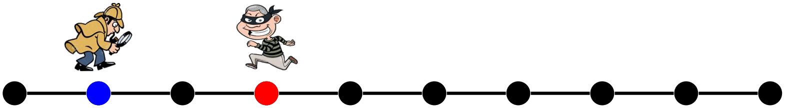

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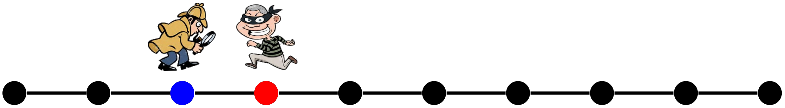

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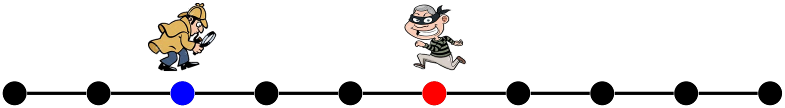

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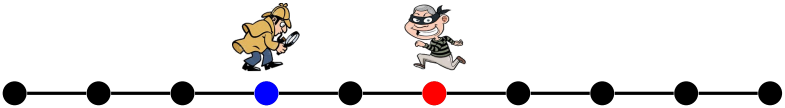

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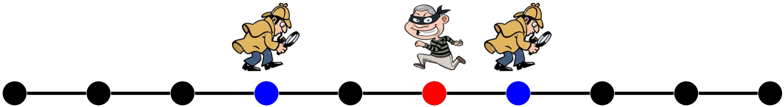

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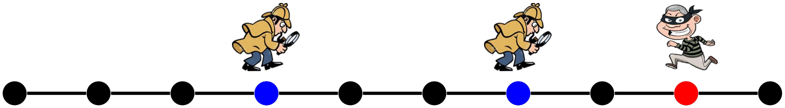

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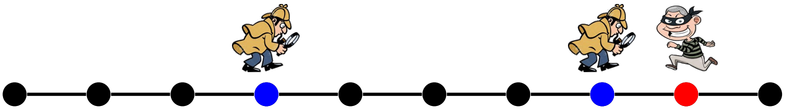

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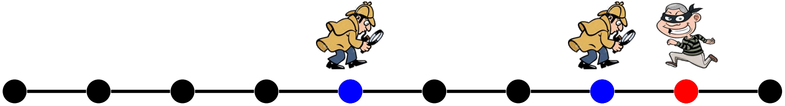

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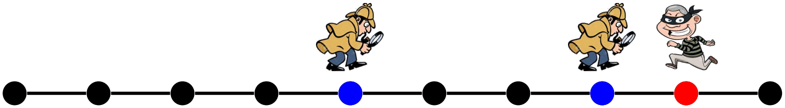

$$gn_{3,1}(P_{10}) = 2$$

Paths : Upper bound

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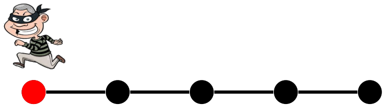

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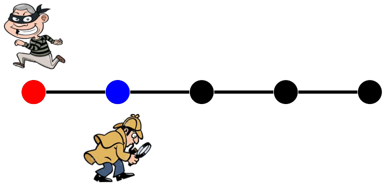

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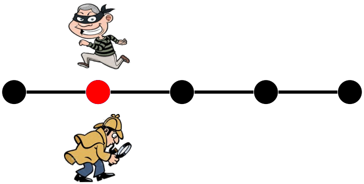

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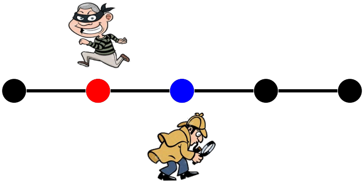

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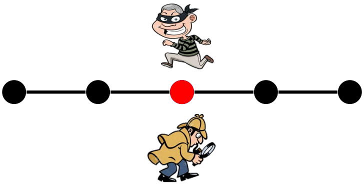

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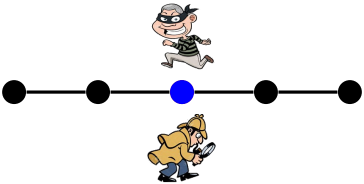

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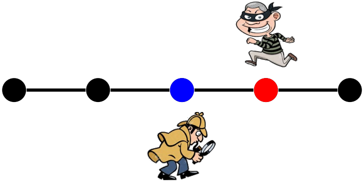

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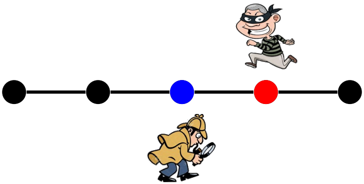

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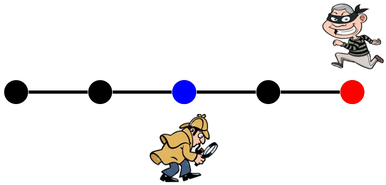

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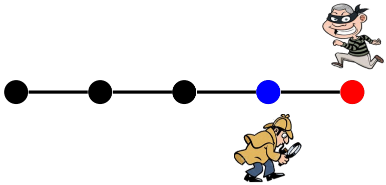

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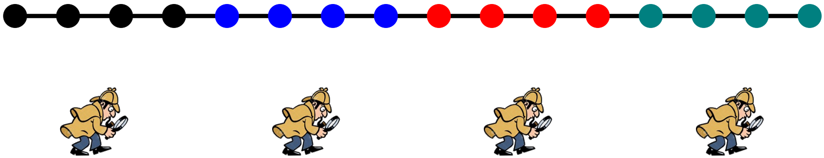

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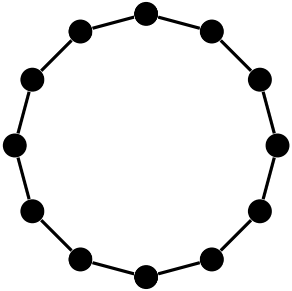
Ex : $s = 6$ and $d = 0$.

$$gn_{6,0}(C_{12}) = 4$$



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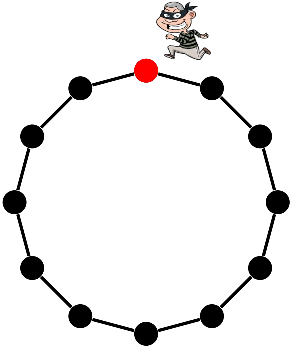

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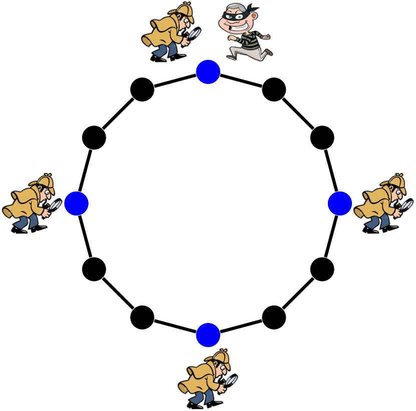

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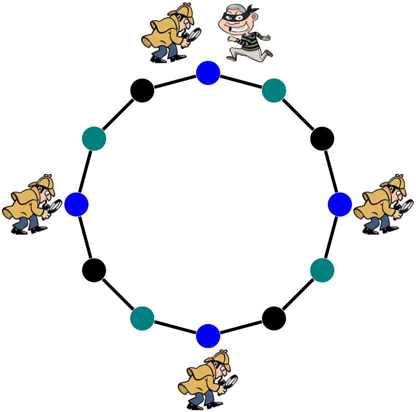

Cycles : Upper Bound Case $2d < s - 1$

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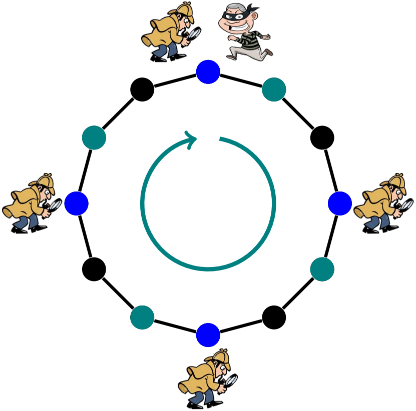

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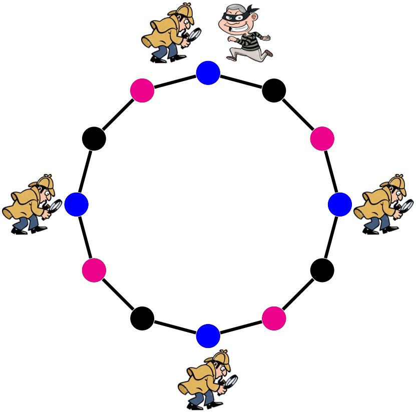

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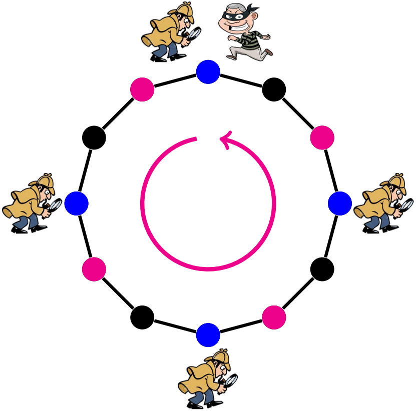


Cycles

Let $2d = q(s-1) + r$ $(0 \leq r < s-1)$ and $2d = q's + r'$ $(0 \leq r' < s)$.
Then, $q = \left\lfloor \frac{2d}{s-1} \right\rfloor$ and $q' = \left\lfloor \frac{2d}{s} \right\rfloor$.
Let $(q^*, r^*) = (q, r)$ if s is odd and $(q^*, r^*) = (q', r')$ otherwise.

Theorem

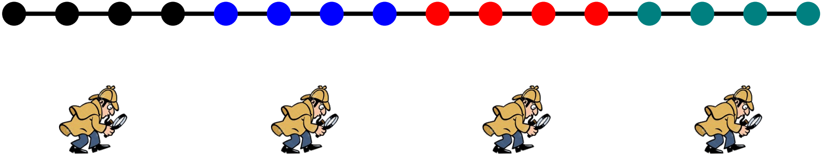
For all $s \geq 2$, $d \geq 0$ s.t. $q = 0$, and a cycle C_n on n vertices, $gn_{s,d}(C_n) = \left\lceil \frac{n}{2d+3} \right\rceil$.

Theorem

For all $s \geq 2$, $d \geq 0$ s.t. $q \neq 0$, and a cycle C_n on n vertices, $\left\lceil \frac{n+2q}{2(d+q)+3} \right\rceil \leq gn_{s,d}(C_n) \leq \left\lceil \frac{n+2q^*}{2(d+q^*)-r^*} \right\rceil$.

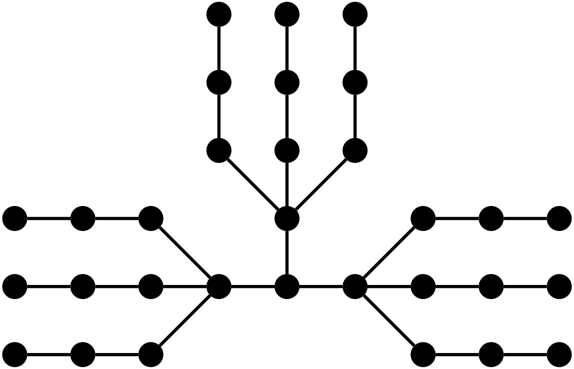
Trees are Harder

Paths : 1 guard per subpath of $2d + 2 + \lfloor \frac{2d}{s-1} \rfloor$ vertices.



Trees are Harder

Can't always divide tree into subtrees protected by a certain number of guards.



Example of a tree T where $s = 2$, $d = 1$ and $gn_{2,1}(T) = 4$.

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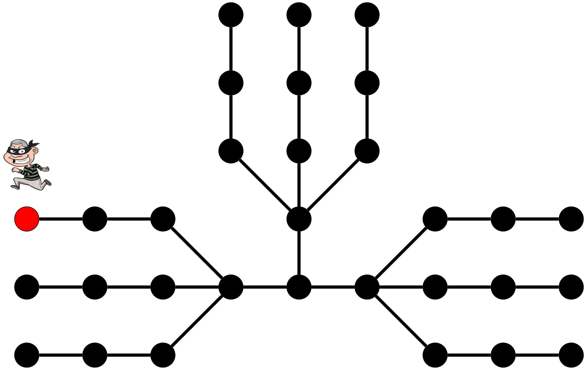

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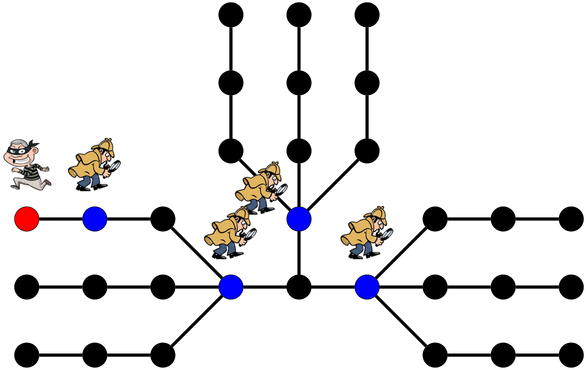

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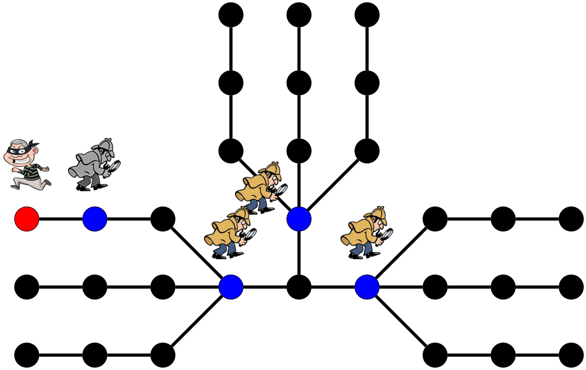

Example of a tree T where $s = 2$, $d = 1$ and $gn_{2,1}(T) = 4$.

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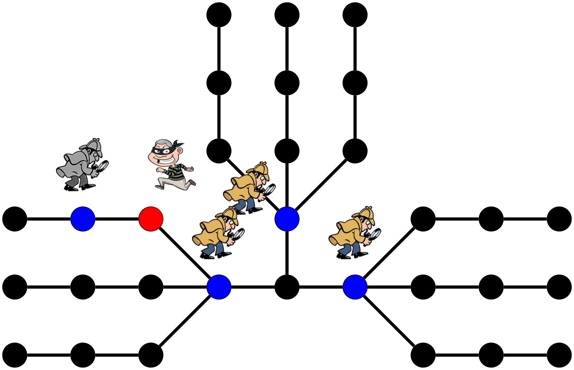

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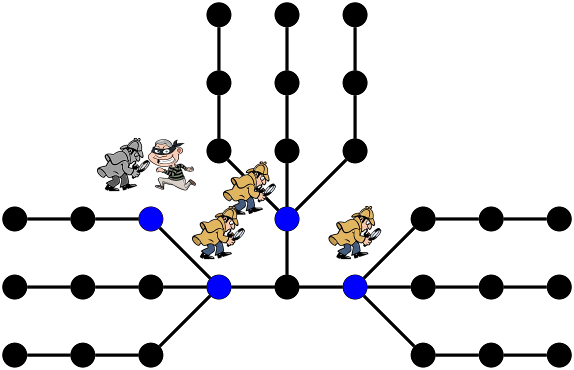

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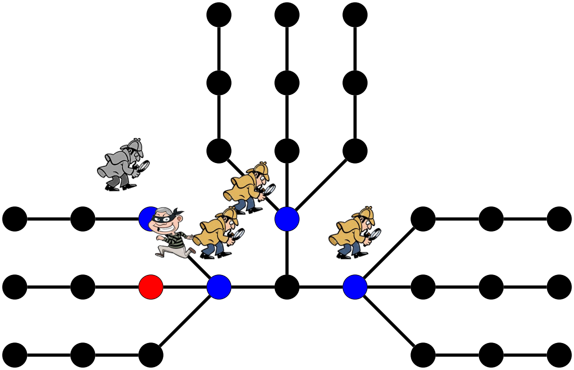

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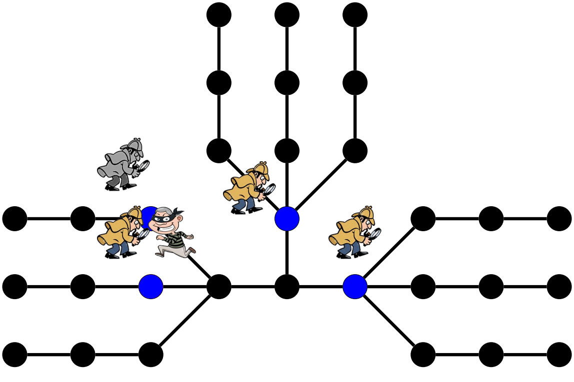

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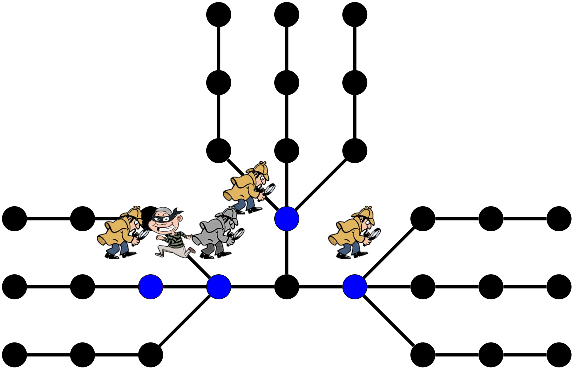

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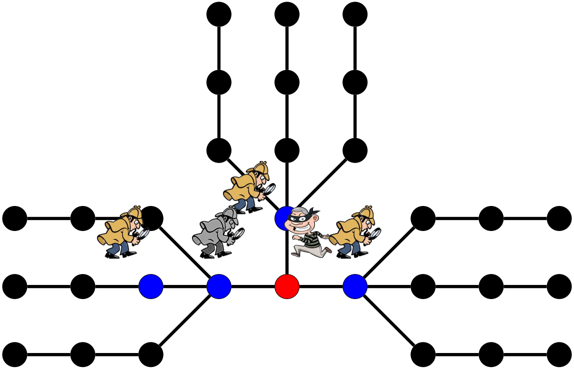

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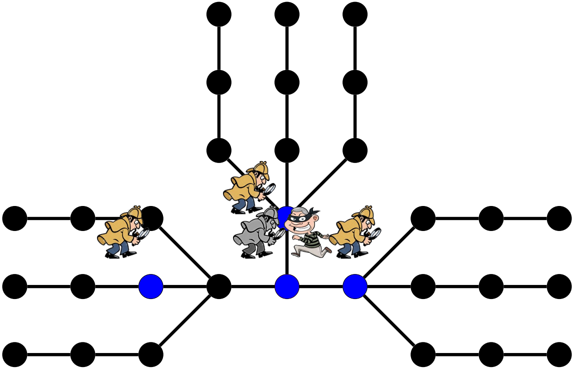

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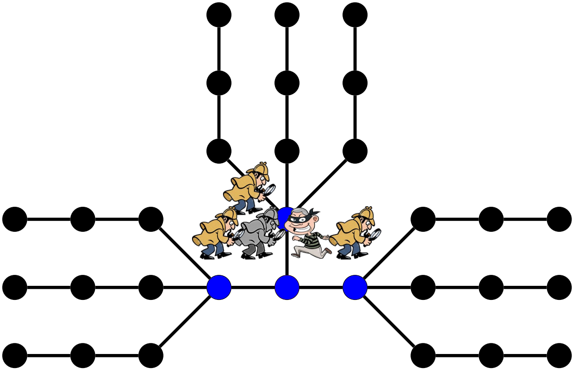

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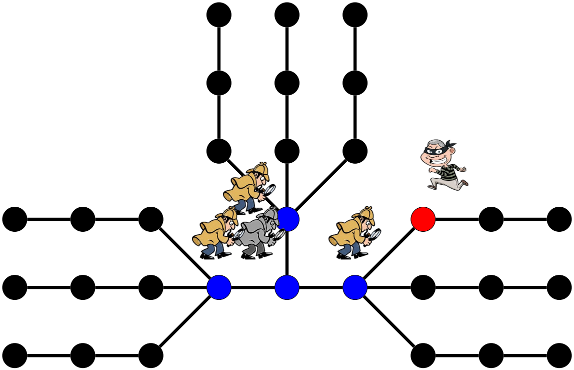

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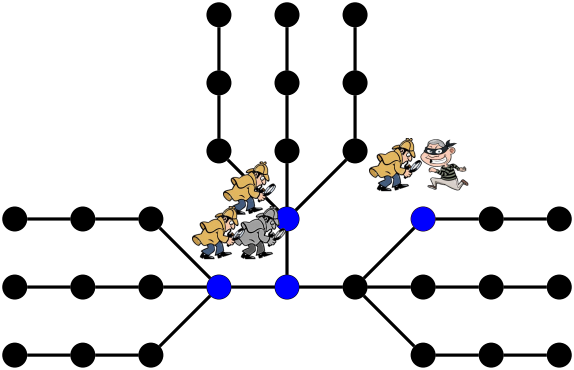

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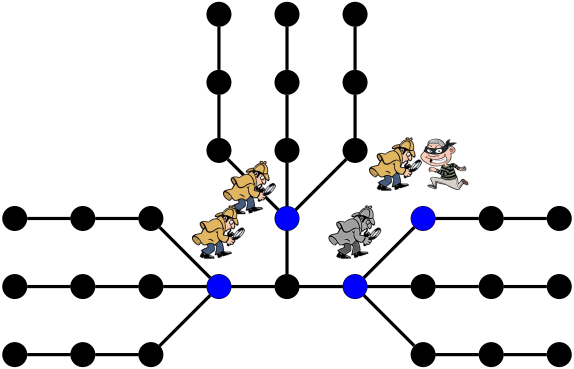

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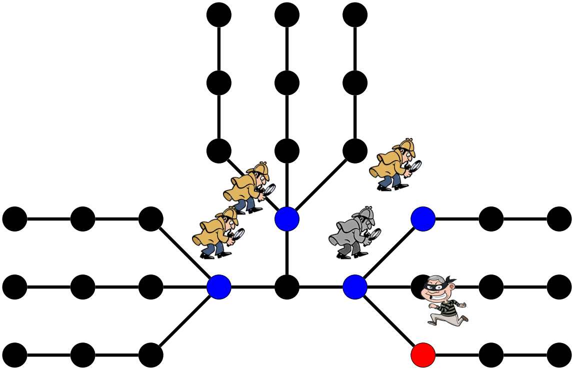

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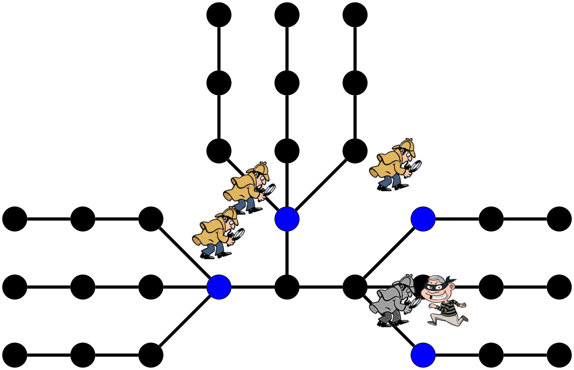

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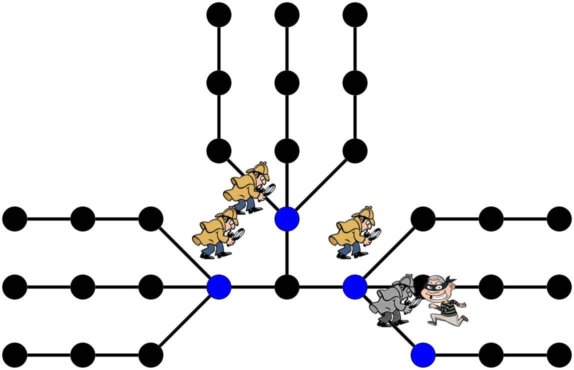

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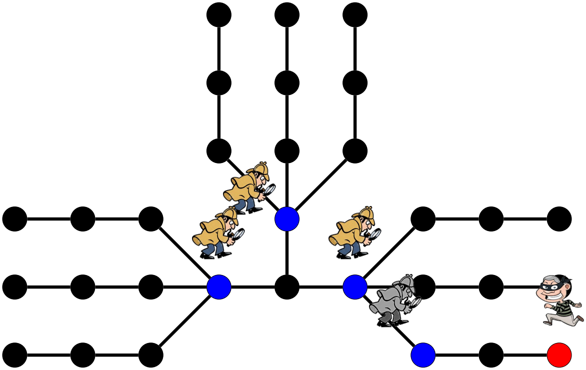

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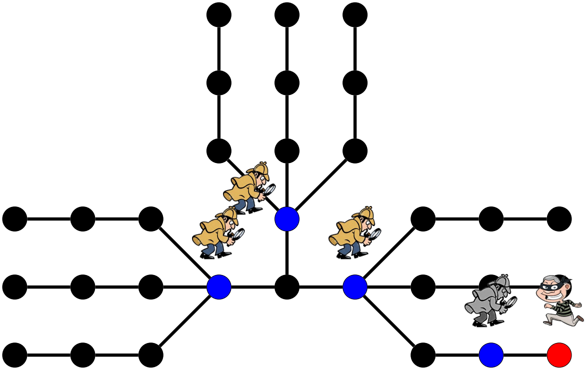

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- Guards may be fractional entities ; movements rep. by flows.
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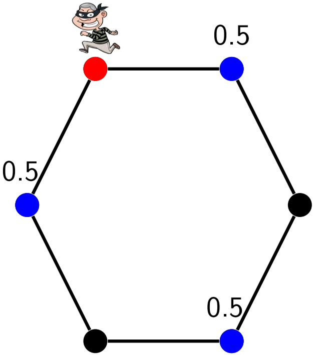

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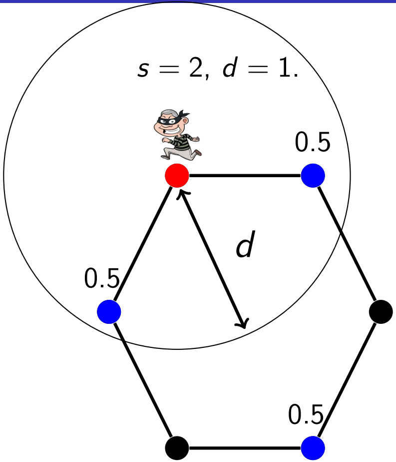

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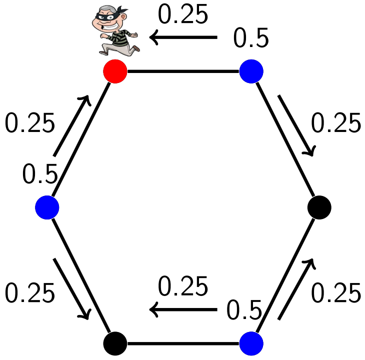

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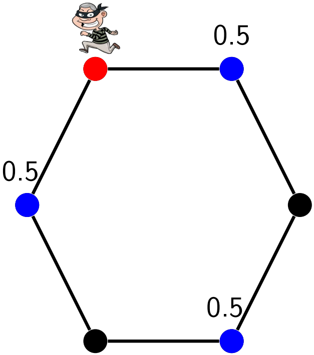

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- Optimal fractional strategy $\Rightarrow$ optimal integral strategy in trees.

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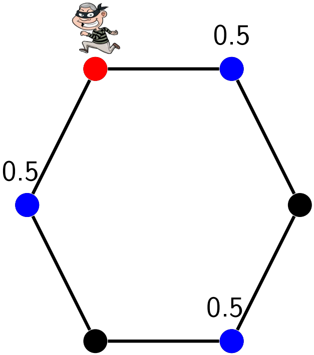

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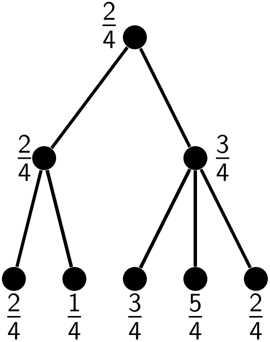
Theorem : Can transform optimal fractional strategy into optimal integral strategy in polynomial time.



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Opt. Fractional Strategy ⇒ Opt. Integral Strategy in Trees

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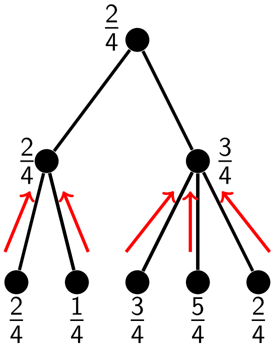

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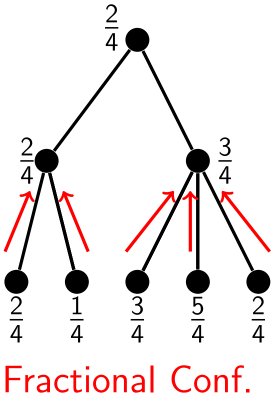

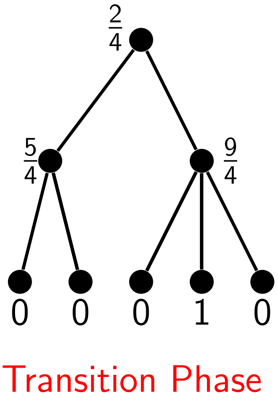

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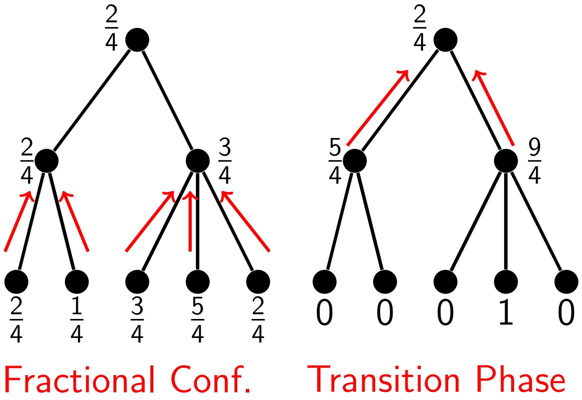

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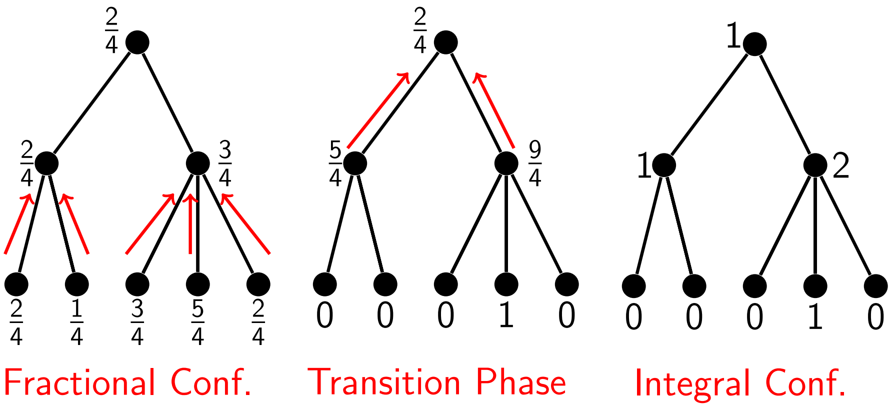

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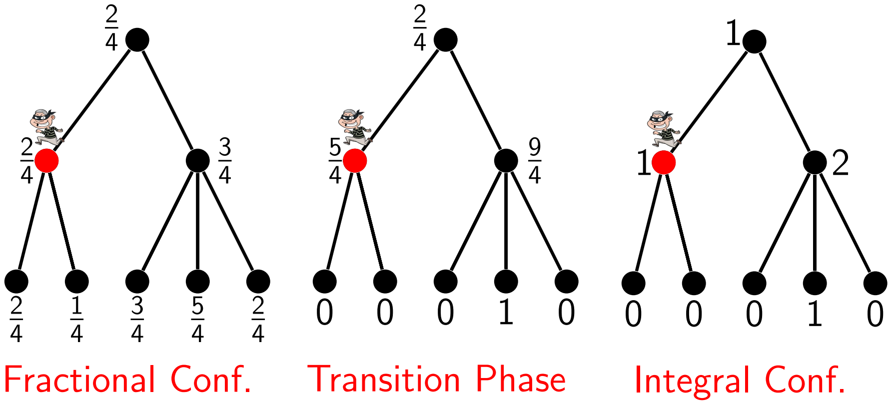

Tree's protection and guards' movements preserved.

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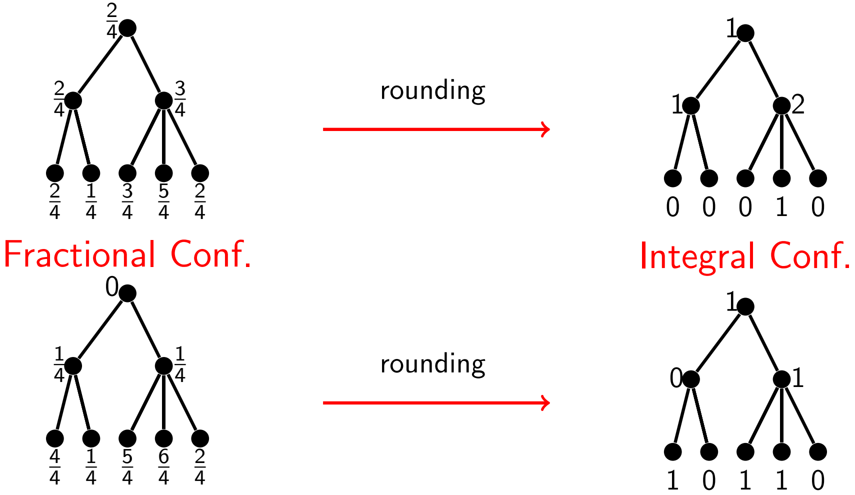

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Optimal fractional strategy $\Rightarrow$ optimal fractional restricted strategy in trees.

Can calculate optimal restricted fractional strategies with Linear Program in polynomial time.

Linear Program to Compute Restricted Strategy

Restricted strategy : $\omega : V \Rightarrow V^k$

$\omega_{x,u}$: quantity of guards on u when spy is on x.

$f_{x,x',u,u'}$: quantity of guards that go from u to u' when spy goes from x to x'.

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$$(1)\ \text{Minimize} \sum_{v \in V} \omega_{x_0,v}$$

Minimize number of guards.

Linear Program

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$$(2) \sum_{v \in N_d[x]} \omega_{x,v} \geq 1 \qquad \forall x \in V$$

Guarantees always at least 1 guard within distance d of spy.

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$$(3) \sum_{u' \in N[u]} f_{x,x',u,u'} = \omega_{x,u} \qquad \forall u \in V,\ x' \in N_s[x]$$

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$O(n^4)$ real variables and constraints.

Main Result : *gn* in Trees

Theorem

$\forall s > 1$, $d \geq 0$ and all trees T, $gn_{s,d}(T)$ and a corresponding strategy can be calculated in polynomial time.

Idea of proof : Linear Program can compute opt. frac. restr. strategy in polynomial time.

Run LP. From previous theorem, strategy is opt. frac.

Can transform opt. frac. into opt. int. in polynomial time.

Grids

Theorem

$\exists \beta > 0$, s.t. $\forall s > 1$, $d \geq 0$, $\Omega(n^{1+\beta}) \leq gn_{s,d}(G_{n \times n})$.

Idea of proof : Lower bound holds for fractional version.

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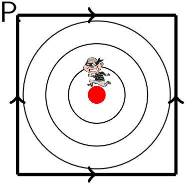
Idea of proof : Lower bound holds for fractional version.

Torus and grid have same order of number of guards.

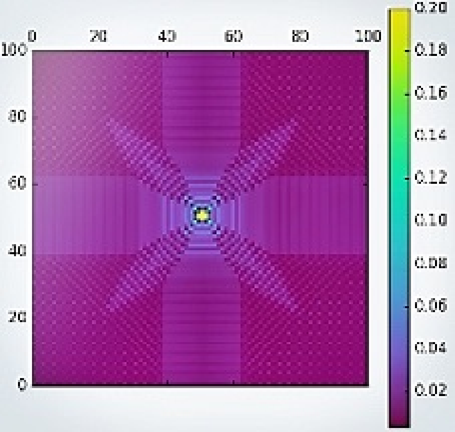
Theorem

$\exists \alpha \geq log(3/2) \approx 0.58$, s.t. $\forall s > 1$, $d \geq 0$,
$fgn_{s,d}(G_{n\times n}) \leq O(n^{2-\alpha})$.

Idea of proof : Density function $\omega^*(v) = \frac{c}{(dist(v,v_0)+1)^{log 3/2}}$ for a constant $c > 0$ satisfies LP.



Distribution of Guards in the Torus for an optimal symmetrical spy-positional strategy when $n = 100$, $m = 100$, $s = 2$ and $d = 1$



Further Work

- Determine $gn_{s,d}(G_{n\times n})$.
- Approximate $gn_{s,d}(G)$ in polynomial time in certain classes of graphs ?
- Fractional approach applied to other combinatorial games.

Thanks !