Enquêter dans les Graphes

Nathann Cohen[1] Nícolas A. Martins[2] **Fionn Mc Inerney**[3]
Nicolas Nisse[3] Stéphane Pérennes[3] Rudini Sampaio[2]

[1]CNRS, Univ Paris Sud, LRI, Orsay, France

[2]Universidade Federal do Ceará, Fortaleza, Brésil

[3]Université Côte d'Azur, Inria, CNRS, I3S, France

June 2, 2017
Time: Too early!
AlgoTel 2017

2-Player Combinatorial Games

- Mobile agents in a graph.
- Turn-by-turn with 2 players.
  - Coordination for common goal, e.g.,
    - Cops and Robbers (capture) (Quilliot, 1978 ; Nowakowski, Winkler, 1983 ; Bonato, Nowakowski, 2011)
    - Eternal Domination (protection) (Goddard et al, 2005 ; Klostermeyer, MacGillivray, 2009).

Investigations in Graphs

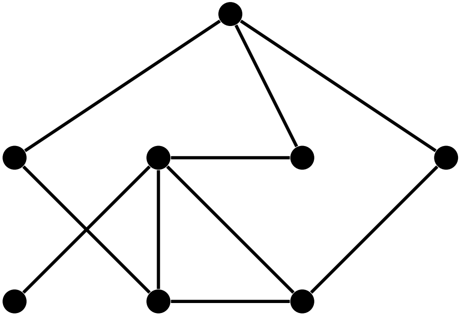
Suspect (1^{st}) vs detectives (2^{nd}) in a graph G.

Start : Suspect placed at a vertex. Then, detectives placed.

Turn-by-turn : Suspect traverses up to $\mathbf{s} \geq 2$ edges. Detectives traverse up to 1 edge.

Goal : Suspect wants to be at least distance $\mathbf{d}+1$ from all detectives.

Ex : $\mathbf{s} = 2$ and $\mathbf{d} = 1$.



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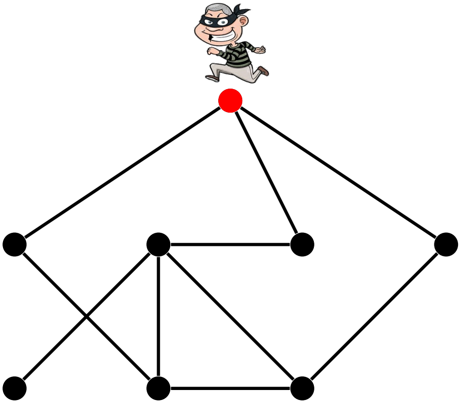

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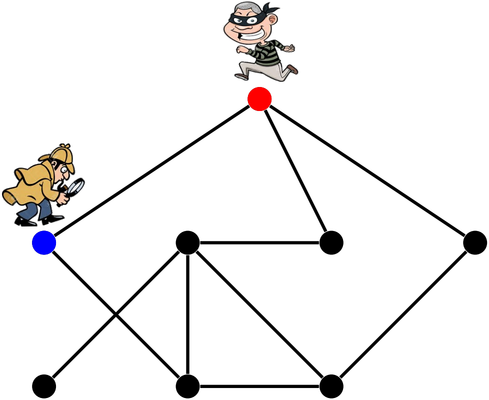

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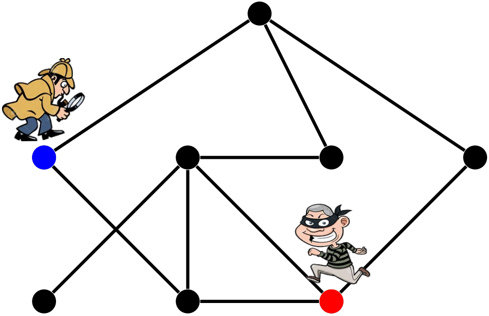

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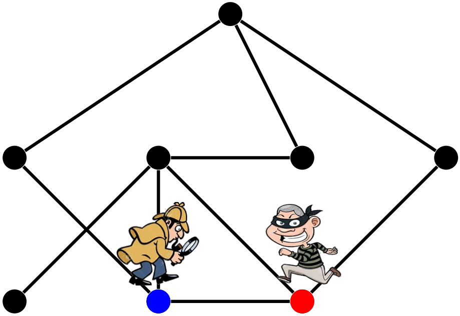

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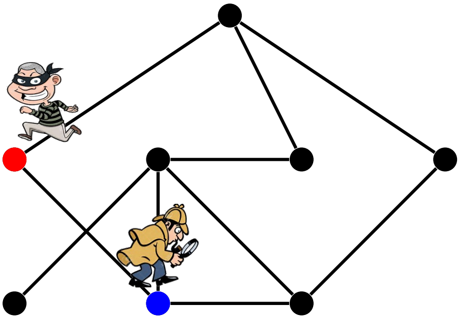

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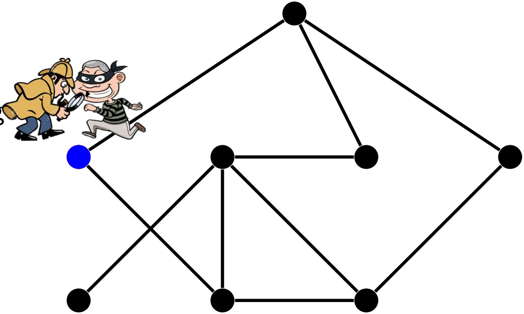

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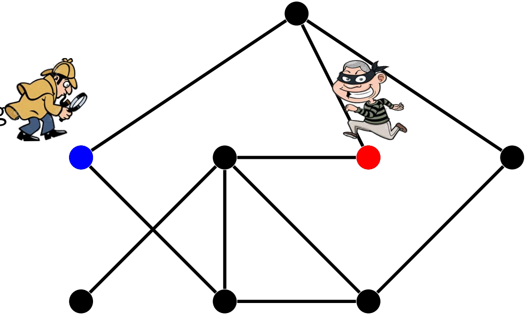

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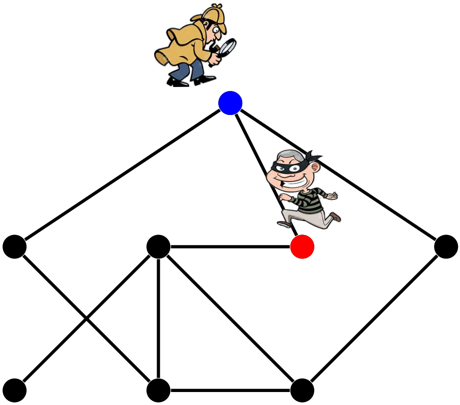

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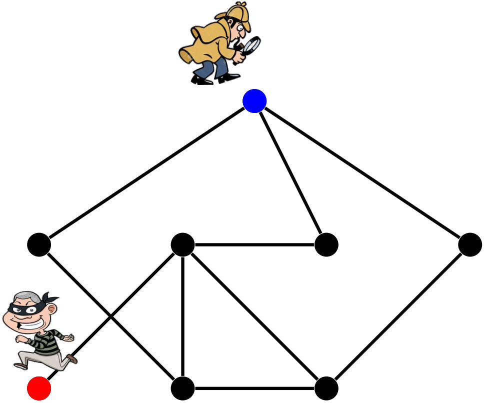

Guard Number : $gn_{s,d}(G)$

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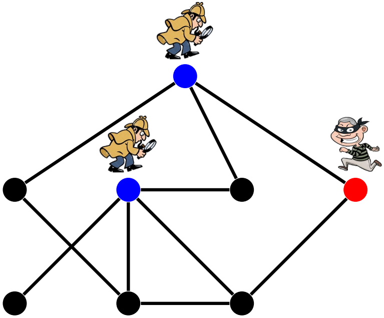

$$gn_{2,1}(G) = 2$$

$$gn_{s,1}(G) \leq \gamma(G)$$

Our Results : Computing *gn*

Complexity

Calculating $gn_{s,d}$ is NP-hard in general.

Tight bounds for paths

$gn_{s,d}(P_n) = \left\lceil \frac{n}{2d+2+q} \right\rceil$ where $q = \lfloor \frac{2d}{s-1} \rfloor$.

Almost tight bounds for cycles

$gn_{s,d}(C_n) \backsimeq \left\lceil \frac{n+2q}{2(d+q)+3} \right\rceil$ where $q = \lfloor \frac{2d}{s-1} \rfloor$.

Polynomial time Linear Program for trees

Can calculate $gn_{s,d}(T)$ and a corresp. strategy in polynomial time.

Grids

$\exists \beta > 0$, s.t. $\Omega(n^{1+\beta}) \leq gn_{s,d}(G_{n \times n})$.

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  - $\gamma^m(G) = gn_{s,d}(G)$ when $s = \infty$ and $d = 0$.

Paths

Theorem

For all $s \geq 2$, $d \geq 0$, and a path P_n on n vertices,

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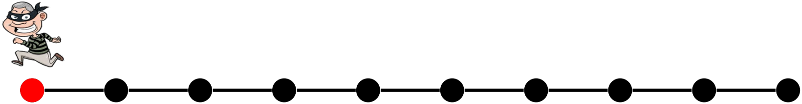

Paths : Lower bound

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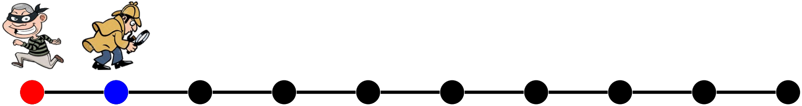

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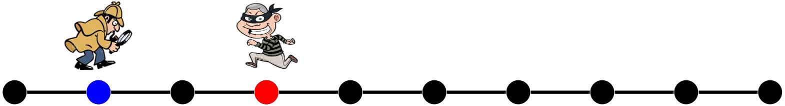

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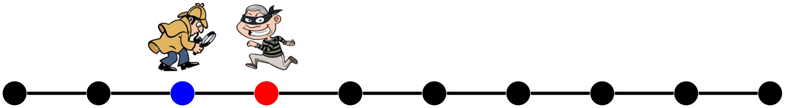

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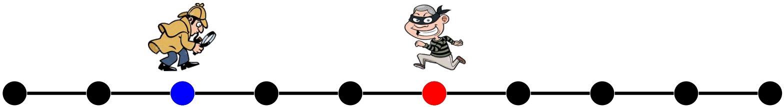

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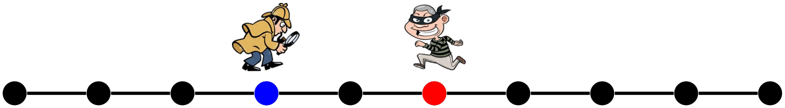

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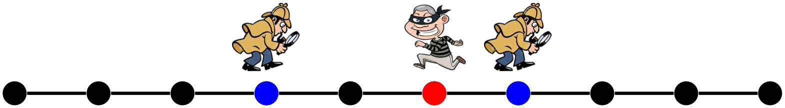

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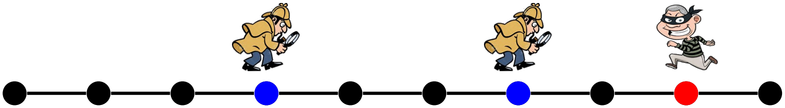

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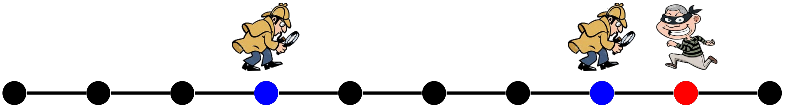

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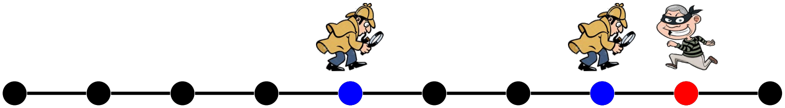

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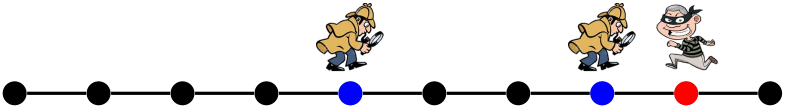

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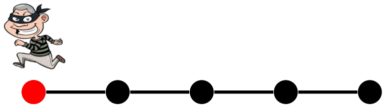

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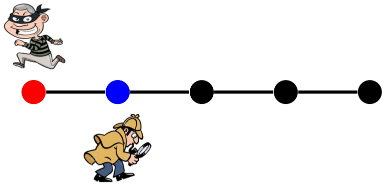

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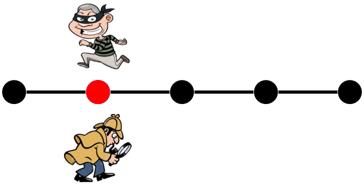

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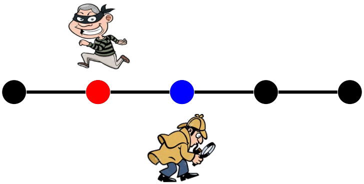

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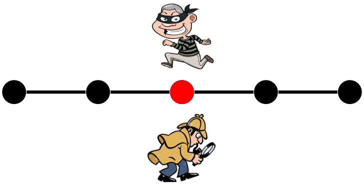

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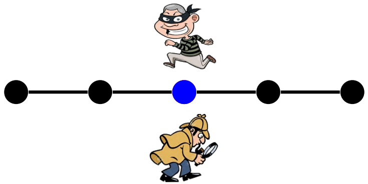

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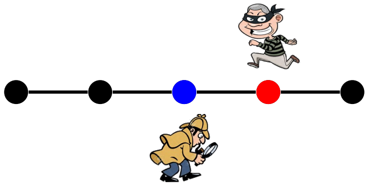

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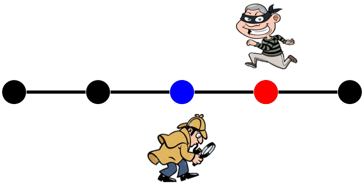

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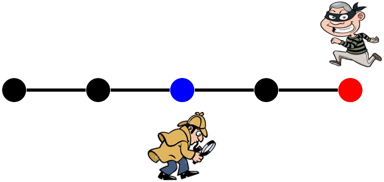

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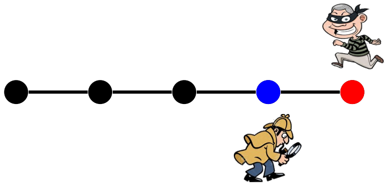
1 detective can protect subpath of $2d + 2 + \lfloor \frac{2d}{s-1} \rfloor$ vertices.



Paths : Upper bound

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For all $s \geq 2$, $d \geq 0$, and a path P_n on n vertices,

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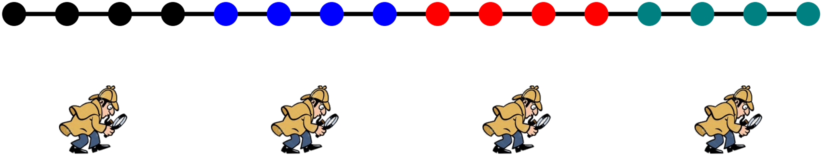

Cycles : Upper Bound Case $2d < s - 1$

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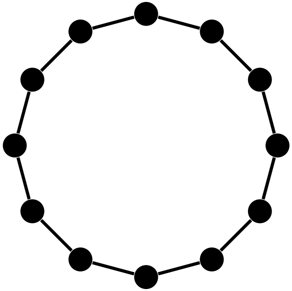
Ex : $s = 6$ and $d = 0$.

$$gn_{6,0}(C_{12}) = 4$$



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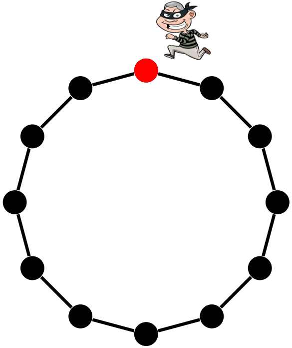

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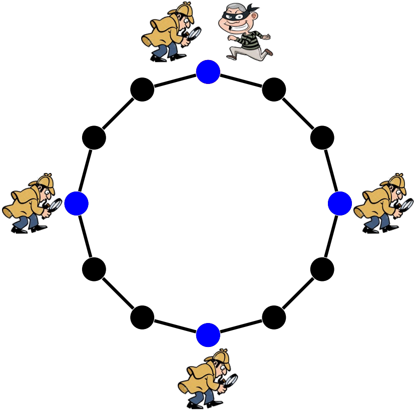

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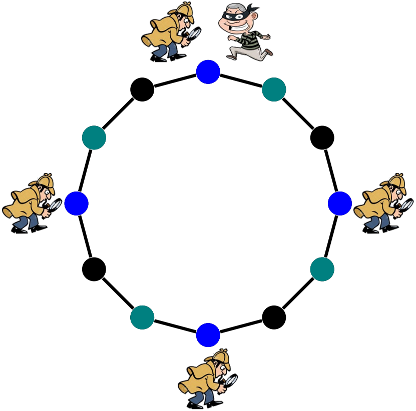

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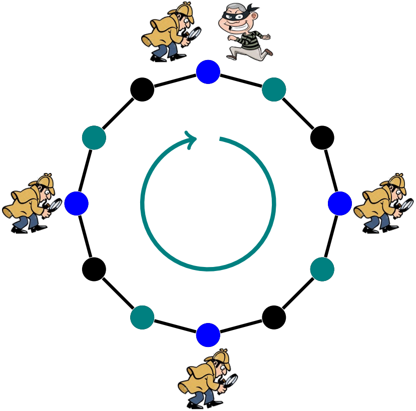

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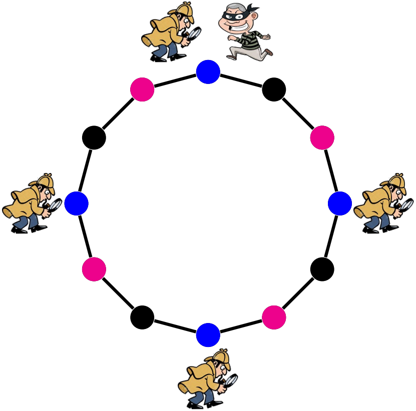

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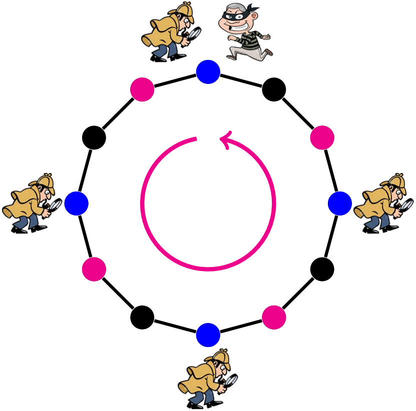


Cycles

Let $2d = q(s-1) + r$ $(0 \leq r < s-1)$ and $2d = q's + r'$ $(0 \leq r' < s)$.
Then, $q = \left\lfloor \frac{2d}{s-1} \right\rfloor$ and $q' = \left\lfloor \frac{2d}{s} \right\rfloor$.
Let $(q^*, r^*) = (q, r)$ if s is odd and $(q^*, r^*) = (q', r')$ otherwise.

Theorem

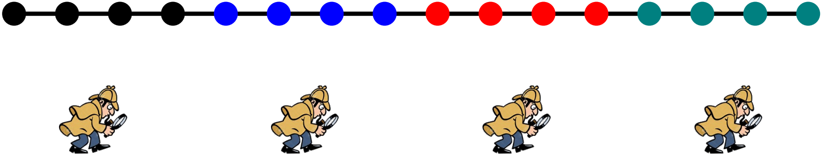
For all $s \geq 2$, $d \geq 0$ s.t. $q = 0$, and a cycle C_n on n vertices,
$gn_{s,d}(C_n) = \left\lceil \frac{n}{2d+3} \right\rceil$.

Theorem

For all $s \geq 2$, $d \geq 0$ s.t. $q \neq 0$, and a cycle C_n on n vertices,
$\left\lceil \frac{n+2q}{2(d+q)+3} \right\rceil \leq gn_{s,d}(C_n) \leq \left\lceil \frac{n+2q^*}{2(d+q^*)-r^*} \right\rceil$.

Trees are Harder

Paths : 1 detective per subpath of $2d + 2 + \lfloor \frac{2d}{s-1} \rfloor$ vertices.



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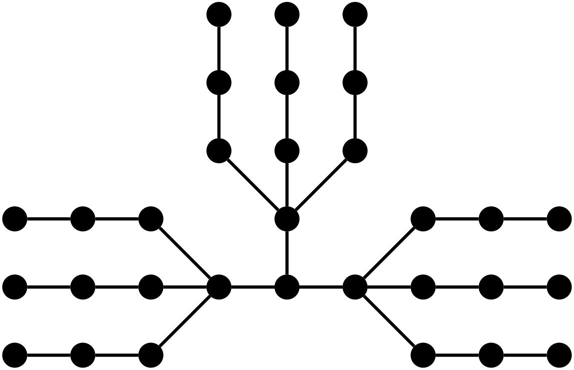
Can't always divide tree into subtrees protected by a certain number of detectives.



Example of a tree T where $s = 2$, $d = 1$ and $gn_{2,1}(T) = 4$.

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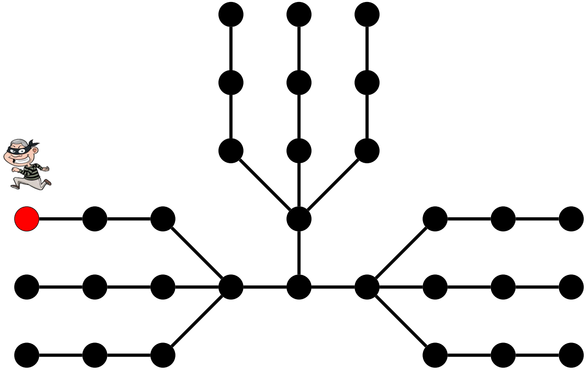

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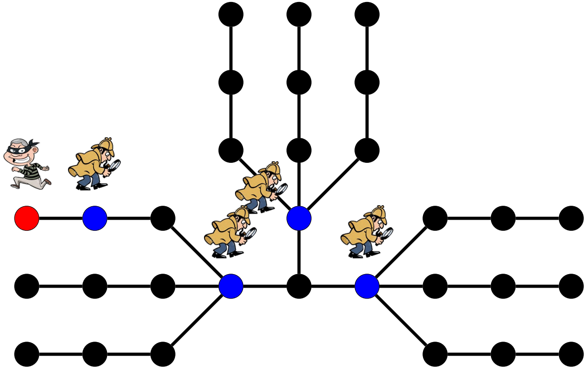

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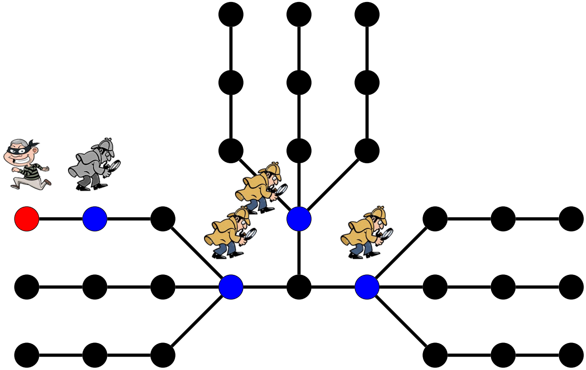

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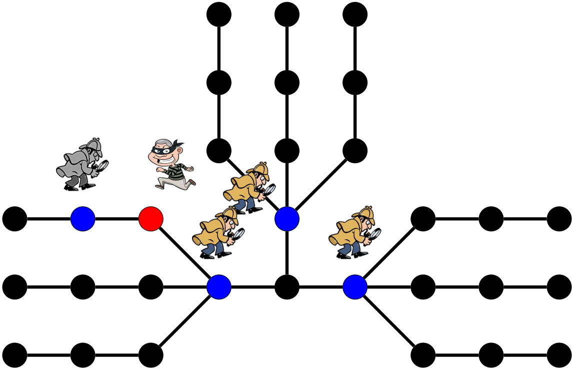

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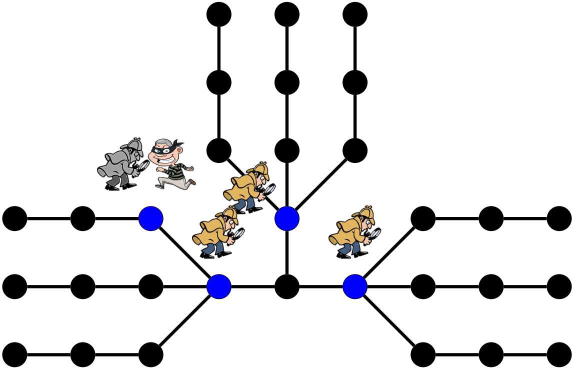

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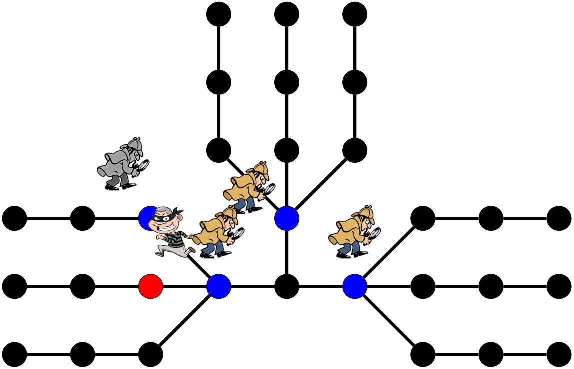

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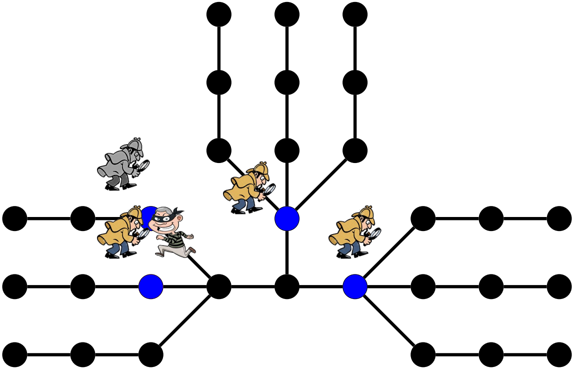

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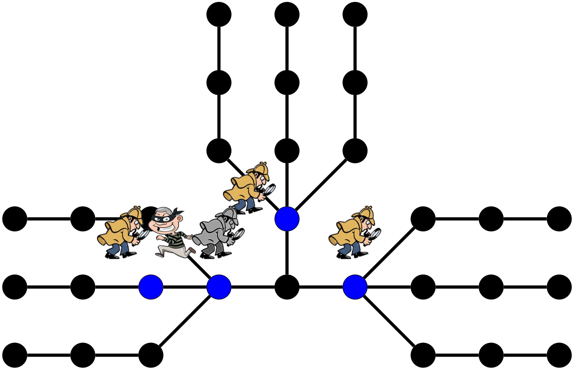

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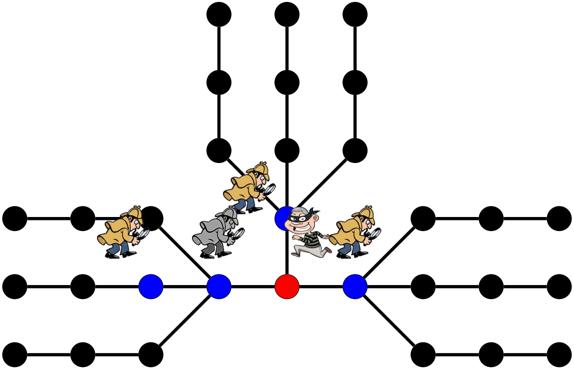

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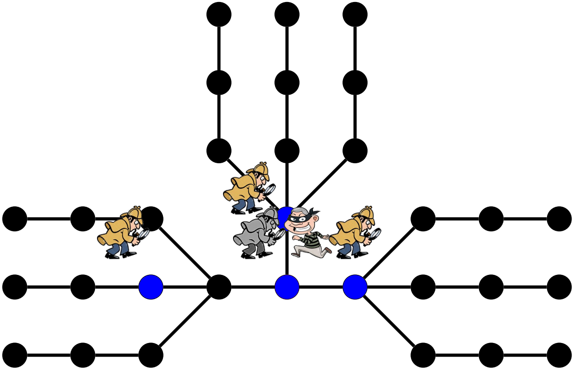

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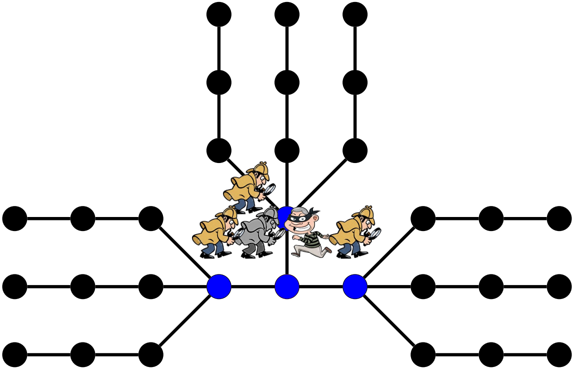

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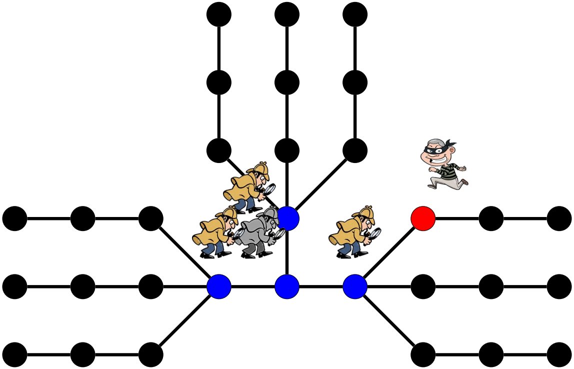

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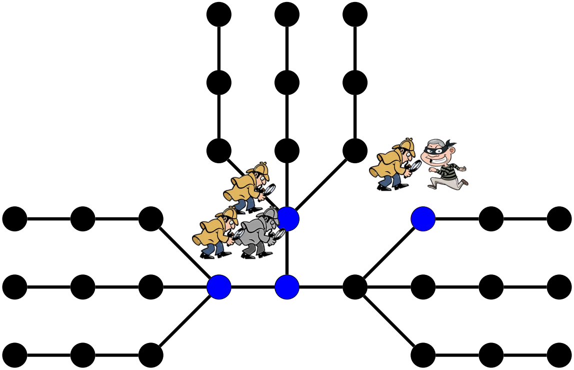

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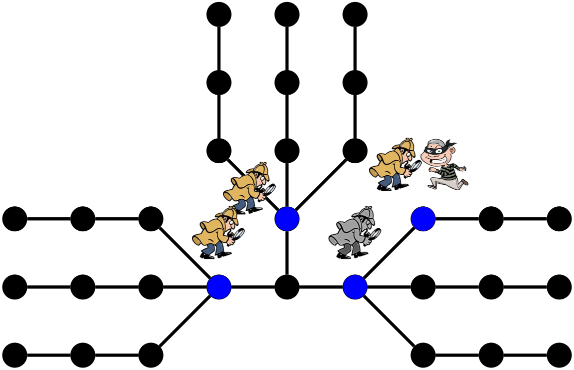

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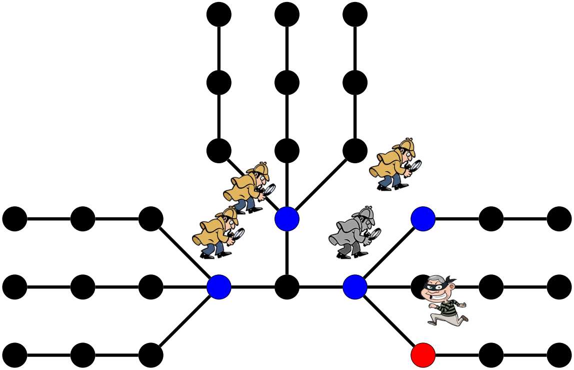

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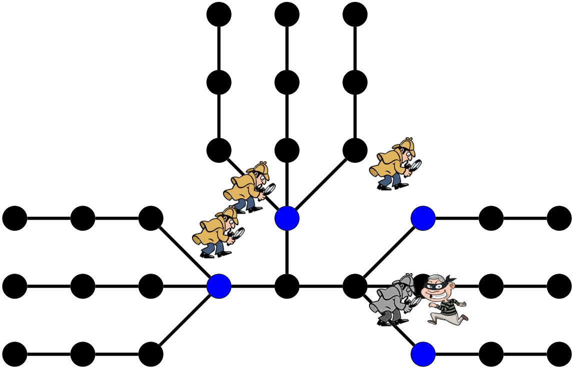

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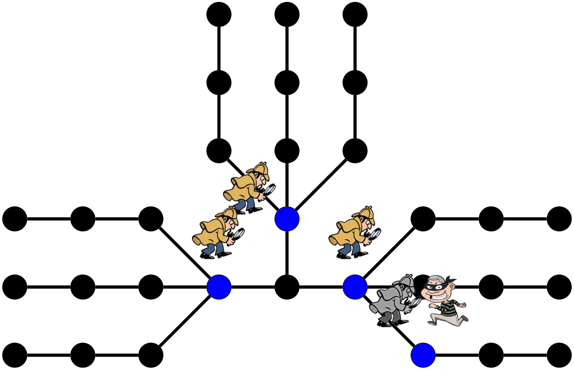

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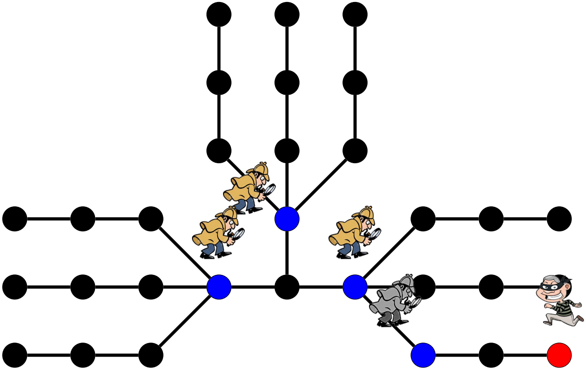

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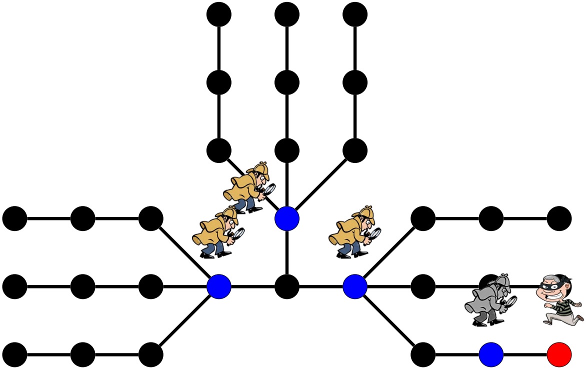

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- Detectives may be fractional entities ; movements rep. by flows.
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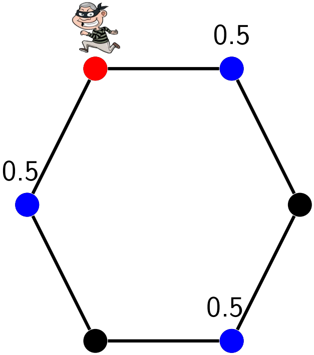

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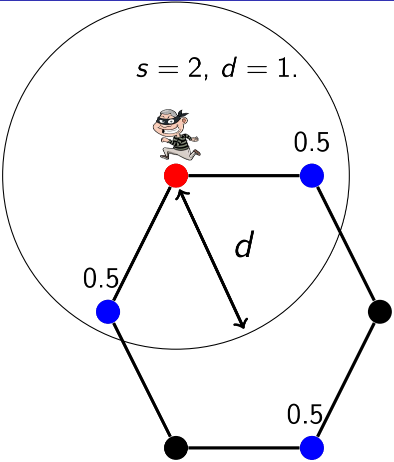

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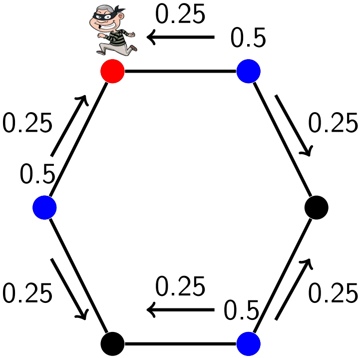

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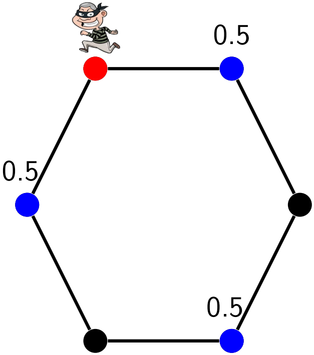

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- Optimal fractional strategy $\Rightarrow$ optimal integral strategy in trees.

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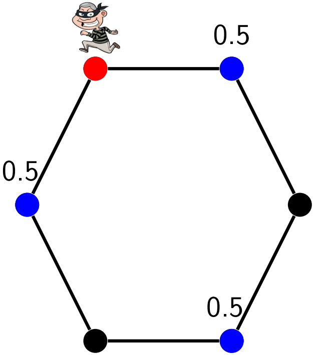

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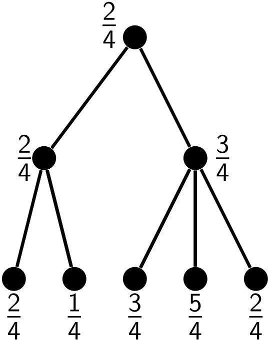

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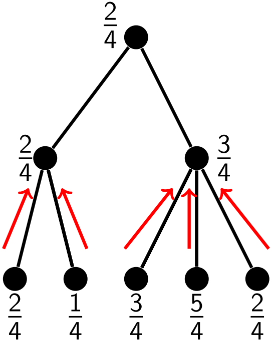

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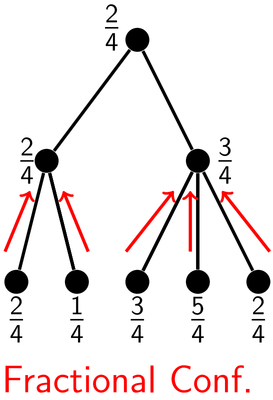

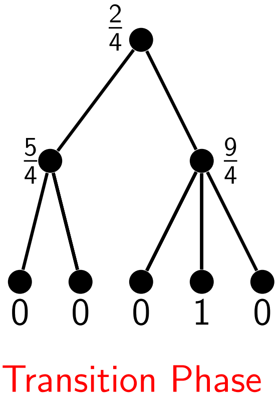

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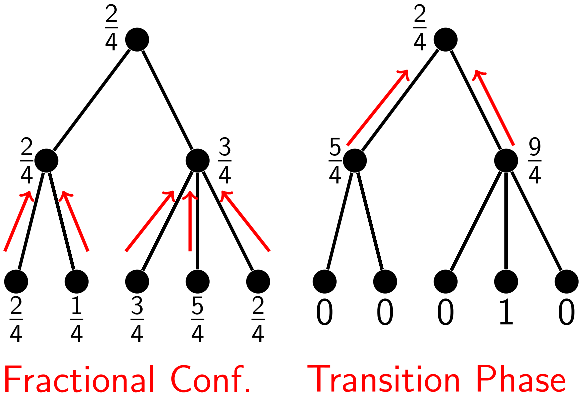

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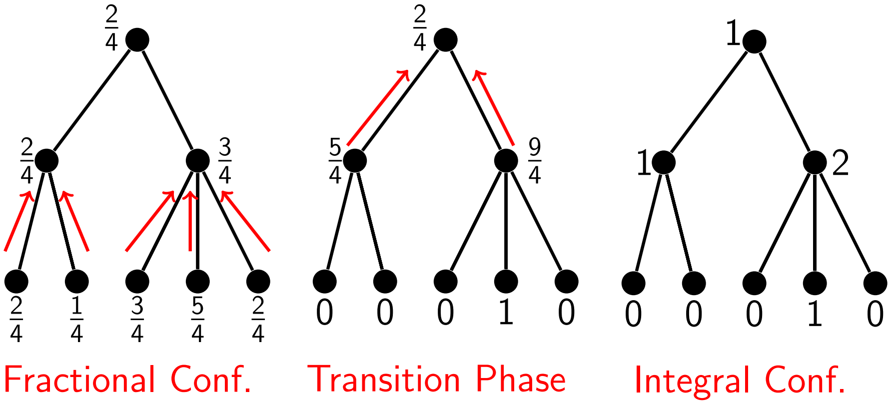

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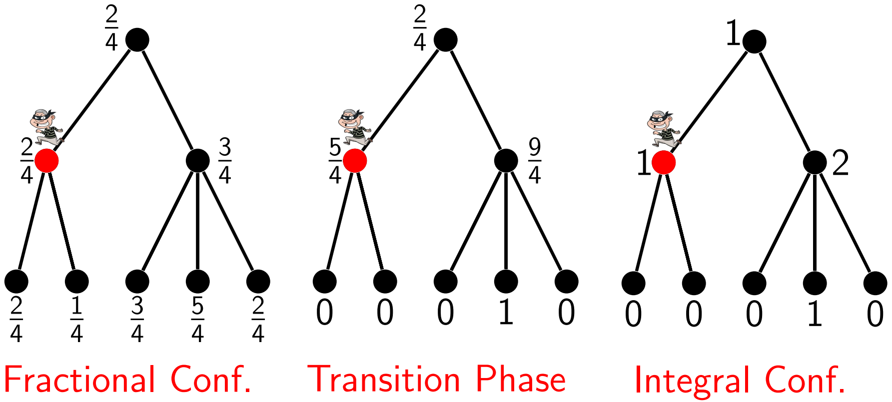

Tree's protection and detectives' movements preserved.

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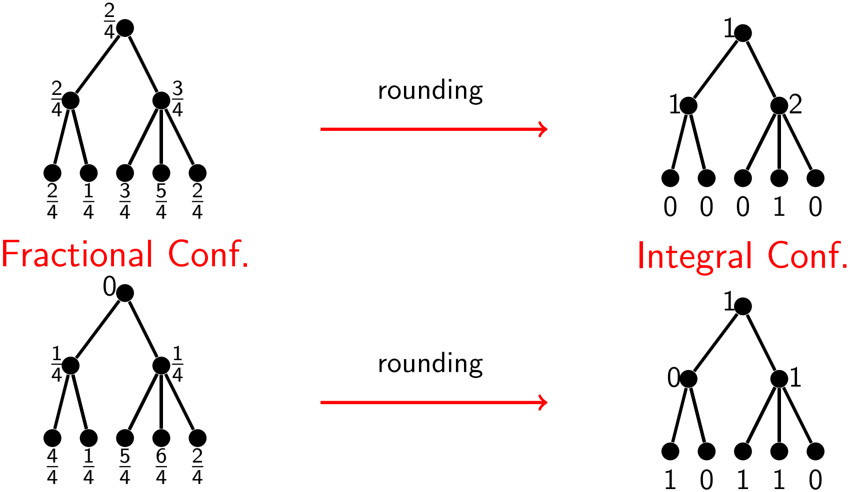

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$f : V^k \times V \Rightarrow V^k$ (Unrestricted strategy)

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Can calculate optimal restricted fractional strategies with Linear Program in polynomial time.

Linear Program to Compute Restricted Strategy

Restricted strategy : $\omega : V \Rightarrow V^k$

$\omega_{x,u}$: quantity of detectives on u when suspect is on x.

$f_{x,x',u,u'}$: quantity of detectives that go from u to u' when suspect goes from x to x'.

Linear Program to Compute Restricted Strategy

Restricted strategy : $\omega : V \Rightarrow V^k$

$\omega_{x,u}$: quantity of detectives on u when suspect is on x.

$f_{x,x',u,u'}$: quantity of detectives that go from u to u' when suspect goes from x to x'.

$$(1)\ \text{Minimize} \sum_{v \in V} \omega_{x_0,v}$$

Minimize number of detectives.

Linear Program

Restricted strategy : $\omega : V \Rightarrow V^k$

$\omega_{x,u}$: quantity of detectives on u when suspect is on x.

$f_{x,x',u,u'}$: quantity of detectives that go from u to u' when suspect goes from x to x'.

$$(2) \sum_{v \in N_d[x]} \omega_{x,v} \geq 1 \qquad \forall x \in V$$

Guarantees always at least 1 detective within distance d of suspect.

Linear Program

Restricted strategy : $\omega : V \Rightarrow V^k$

$\omega_{x,u}$: quantity of detectives on u when suspect is on x.

$f_{x,x',u,u'}$: quantity of detectives that go from u to u' when suspect goes from x to x'.

$$(3) \quad \sum_{u' \in N[u]} f_{x,x',u,u'} = \omega_{x,u} \qquad \forall u \in V,\ x' \in N_s[x]$$

$$(4) \quad \sum_{u' \in N[u]} f_{x,x',u',u} = \omega_{x',u} \qquad \forall u \in V,\ x' \in N_s[x]$$

Guarantees validity of moves of detectives when suspect moves.

Linear Program

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Guarantees validity of moves of detectives when suspect moves.

$O(n^4)$ real variables and constraints.

Main Result : *gn* in Trees

Theorem

$\forall s > 1$, $d \geq 0$ and all trees T, $gn_{s,d}(T)$ and a corresponding strategy can be calculated in polynomial time.

Idea of proof : Linear Program can compute opt. frac. restr. strategy in polynomial time.

Run LP. From previous theorem, strategy is opt. frac.

Can transform opt. frac. into opt. int. in polynomial time.

Grids

Theorem

$\exists \beta > 0$, s.t. $\forall s > 1$, $d \geq 0$, $\Omega(n^{1+\beta}) \leq gn_{s,d}(G_{n \times n})$.

Idea of proof : Lower bound holds for fractional version.

Grids

Theorem

$\exists \beta > 0$, s.t. $\forall s > 1$, $d \geq 0$, $\Omega(n^{1+\beta}) \leq gn_{s,d}(G_{n \times n})$.

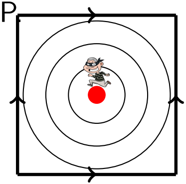
Idea of proof : Lower bound holds for fractional version.

Torus and grid have same order of number of detectives.

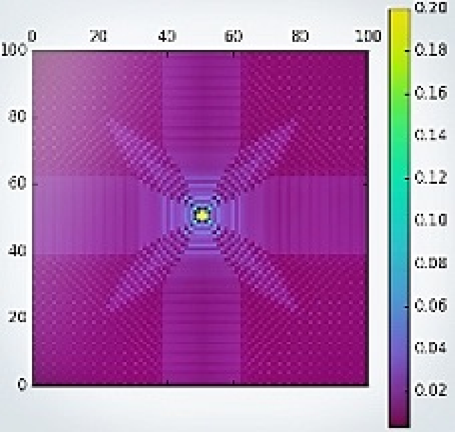
Theorem

$\exists \alpha \geq log(3/2) \approx 0.58$, s.t. $\forall s > 1$, $d \geq 0$,
$fgn_{s,d}(G_{n \times n}) \leq O(n^{2-\alpha})$.

Idea of proof : Density function $\omega^*(v) = \frac{c}{(dist(v,v_0)+1)^{log 3/2}}$ for a constant $c > 0$ satisfies LP.



Distribution of Detectives in the Torus for an optimal symmetrical suspect-positional strategy when $n = 100$, $m = 100$, $s = 2$ and $d = 1$



Further Work

- Determine $gn_{s,d}(G_{n \times n})$.
- Approximate $gn_{s,d}(G)$ in polynomial time in certain classes of graphs ?
- Fractional approach applied to other combinatorial games.

Thanks !