# The Orthogonal Colouring Game

Stephan Dominique Andres[2], François Dross[1], Melissa Huggan[3], **Fionn Mc Inerney**[1], Richard Nowakowski[3]

[1]Université Côte d'Azur, Inria, CNRS, I3S, France

[2]Faculty of Mathematics and Computer Science, FernUniversität in Hagen, Germany

[3]Dept. of Math and Stats, Dalhousie University, Canada

January 23, 2019, Lisbon, Portugal

CGT III

(Thanks to Dominique for helping with the slides!)

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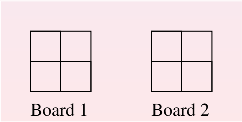

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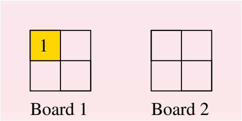

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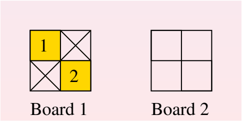

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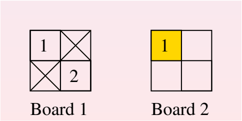

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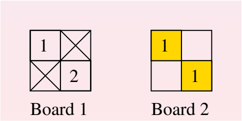

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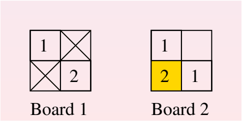

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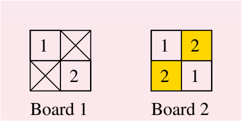

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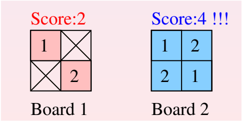
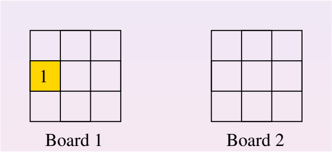



Player 2 **wins.**

# The drawing strategy of Player 2

The columns are paired. Player 2 responds in the "matching" column of the other board.



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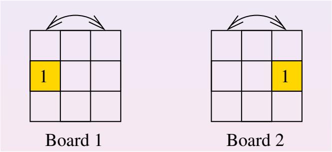

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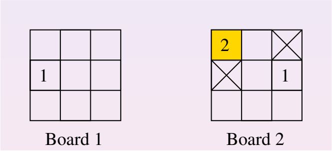

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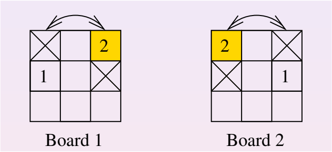

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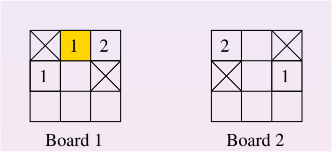

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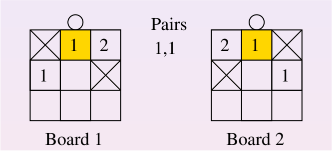

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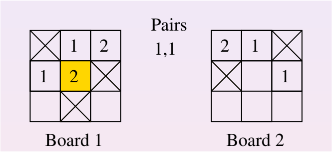

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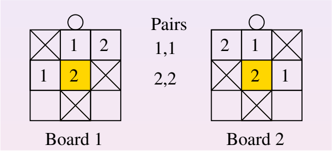

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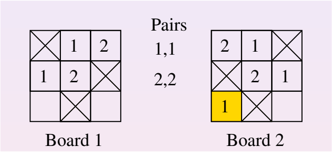

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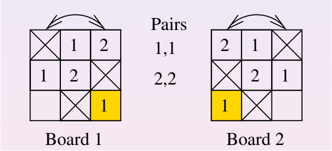

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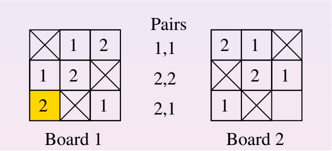

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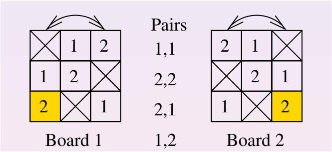

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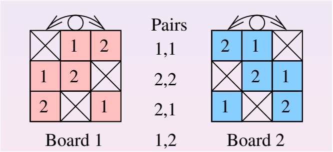

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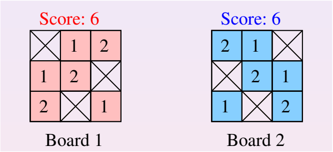


There is a **draw.**

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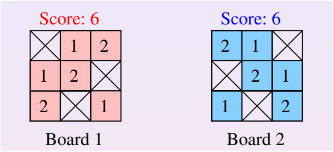


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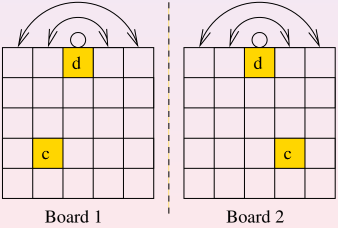
**Theorem.** Player 2 can always force a draw.



# From orthogonal Latin squares to orthogonal graphs: The mutually orthogonal Latin squares games

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The player with the highest score (number of coloured entries in their board) wins.

**Given:** two **isomorphic graphs**, each associated with a player, and *m* colours.

**Game:** Alternately, the players colour a **vertex** in any **graph** such that

> **(Proper colouring)** *Adjacent vertices receive distinct colours.*

> **(Orthogonality)** *An ordered pair of colours of the same* **vertex** *from* **Graph 1** *and* **Graph 2** *appears at most once.*

The player with the highest score (number of coloured **vertices** in their **graph**) wins.

# Examples for the three possible outcomes of the game

Player 1 wins
with $m = 1$ colour

Graph 1 Graph 2

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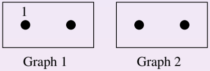

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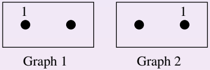

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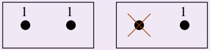


Graph 1 Graph 2

Score: 2 !!!
Score: 1

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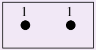


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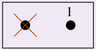


Graph 2

There is a **draw**
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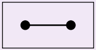

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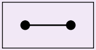

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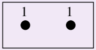


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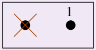


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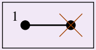


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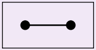

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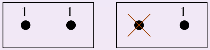


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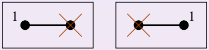


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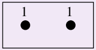

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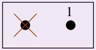

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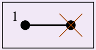

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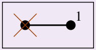
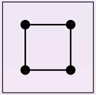


Graph 2

Player 2 wins
with $m = 2$ colours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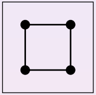
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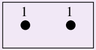

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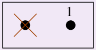

Graph 2

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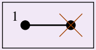

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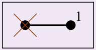

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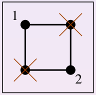

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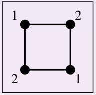

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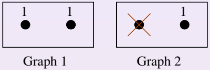


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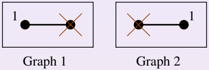


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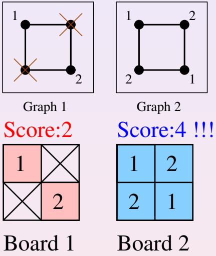

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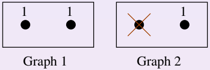


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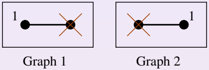
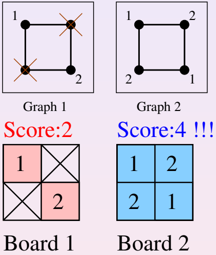



Player 2 wins
with $m = 2$ colours




**Theorem**

For all $m \geq 3$, given an instance of the orthogonal colouring game with a partial colouring, it is PSPACE-complete to decide the outcome of the game.

# Graphs admitting a strictly matched involution

An involution of a graph $G$ is an automorphism of $G$ s.t.

$$\forall v \in V(G): \quad (\sigma \circ \sigma)(v) = v.$$

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(SI 1) the set $F = \{v \in V(G) \mid \sigma(v) = v\} \subseteq V(G)$ of fixed points of $\sigma$ induces a complete graph and

(SI 2) for every non-fixed point $v \in V(G) \setminus F$ we have the (matching) edge $v\sigma(v) \in E(G)$.

# Graphs admitting a strictly matched involution: The generalisation of the drawing strategy

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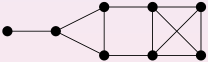
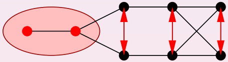
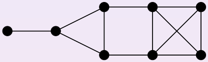


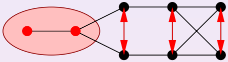



**Main Theorem**

For any graph $G$ that admits a strictly matched involution, Player 2 has a strategy to guarantee a draw in the Orthogonal Graph Colouring Game played on $G$.

# Graphs admitting a strictly matched involution: The generalisation of the drawing strategy



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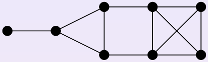

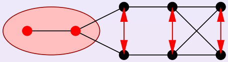


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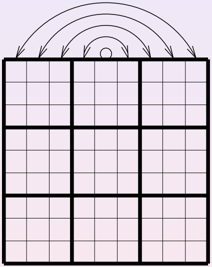
For any graph $G$ that admits a strictly matched involution, Player 2 has a strategy to guarantee a draw in the Orthogonal Graph Colouring Game played on $G$.

**Idea of proof.**
Let $\sigma$ be a fixed strictly matched involution of $G$.

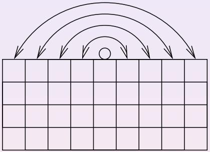
**Strategy of Player 2:**
If Player 1 colours vertex $v$ in some copy of $G$ with colour $c$, then Player 2 colours vertex $\sigma(v)$ in the other copy of $G$ with colour $c$.

# Some other combinatorial structures whose graphs admit strictly matched involutions

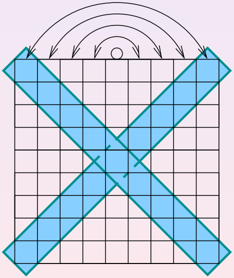
Sudoku squares



Latin rectangles



Latin squares
with double diagonal condition



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A graph $G$ admits a strictly matched involution if and only if the vertices $V(G)$ can be partitioned into a clique $C$ and a graph that has a perfect matching $M$ such that:

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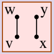
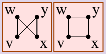
1. for any two edges $vw, xy \in M$, the graph induced by $v, w, x, y \in V(G)$ is isomorphic to:



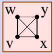

$2K_2$

or




$C_4$

or




$K_4$

# Characterisation of graphs admitting a strictly matched involution

## Theorem

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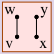


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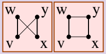


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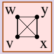
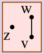
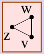



$K_4$

2. for any edge $vw \in M$ and any vertex $z \in C$, the graph induced by $v, w, z \in V(G)$ is isomorphic to:

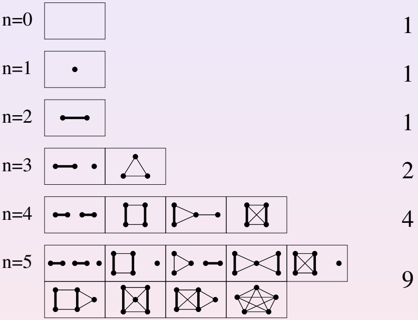



$K_1 \cup K_2$ or

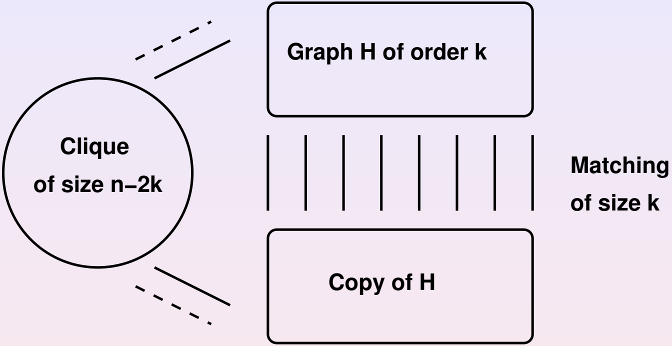



$K_3$

# Small graphs admitting a strictly matched involution




Graphs with up to 5 vertices admitting a strictly matched involution.




**Theorem**

Given a graph $G$, it is NP-complete to determine if it admits a strictly matched involution.

# Counting up to isomorphism: an upper bound

$g(n)$: number of isomorphism classes of graphs with $n$ vertices.

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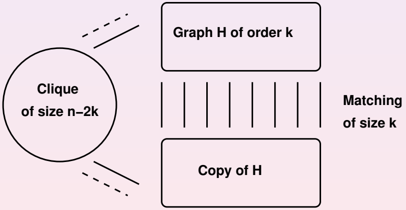

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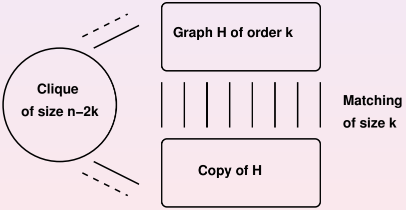

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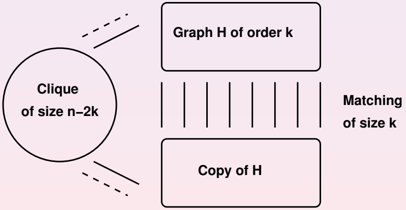

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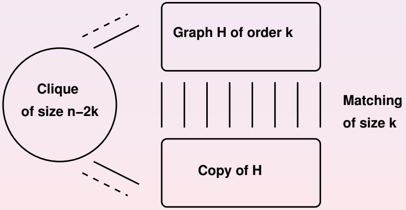

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**Theorem**

$$A(n) \leq \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} g(k) \cdot 2^{\binom{k}{2}} \cdot 2^{(n-2k)k}$$

# A flavour of the size of the upper bound

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**Observation (trivial lower bound for $A(n)$ for $n \geq 3$)**

$$A(n) \geq \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} g(k) \geq \sqrt[4]{g(n-1)} = \Omega(d(n)\sqrt[4]{g(n)}) \text{ with}$$

"moderate" $d(n)$

Main result: for graphs admitting a strictly matched involution,

- either Player 2 has a winning strategy
- or Player 1 and Player 2 have a drawing strategy

in the Orthogonal Colouring Game on graphs.

# Open problems

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For any $m \in \mathbb{N}$, characterise the class of graphs admitting a strictly matched involution where Player 2 has a winning strategy for the $m$-colour Orthogonal Colouring Game.

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## Sub-Problem

For which $m, n \in \mathbb{N}$, does Player 2 have a winning strategy for the $m$-colour mutually orthogonal Latin squares game on an $n \times n$ board? It is a draw when $m = 1$.

Thanks!