The Orthogonal Colouring Game

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Seminar of the COATI team, Inria

(Thanks to Dominique for helping with the slides!)

Contents

1. The mutually orthogonal Latin squares game
   $\longrightarrow$ strategy for player 2 to force a draw
2. The orthogonal colouring game on graphs
   $\longrightarrow$ complexity, graphs with a strictly matched involution, general strategy
3. Characterising graphs admitting a strictly matched involution
   $\longrightarrow$ complexity, construction, bounds for counting

The mutually orthogonal Latin squares game

Given: two $(n \times n)$-boards, each associated with a player, and m colours.
Game: Alternately, the players colour an entry in any board such that

> **(Latin square condition)** *Every colour appears at most once in each row and column of the board.*
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> **(Orthogonality)** *An ordered pair of colours of the same entry from Board 1 and Board 2 appears at most once.*

The player with the highest score (number of coloured entries in their board) wins.

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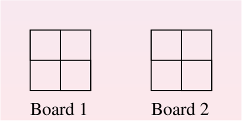

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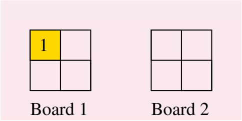

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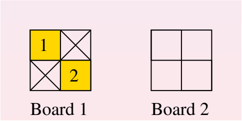

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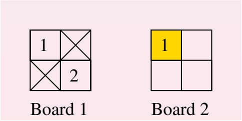

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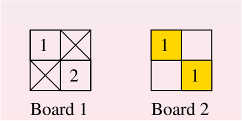

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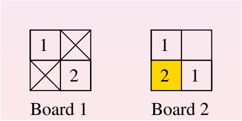

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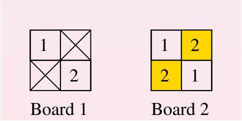

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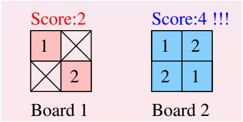
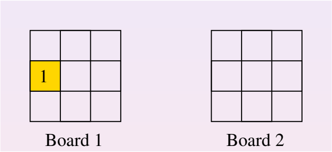


Player 2 **wins.**

The drawing strategy of Player 2

The columns are paired. Player 2 responds in the “matching” column of the other board.



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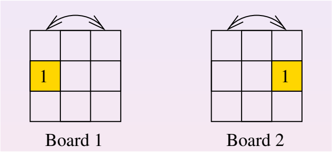

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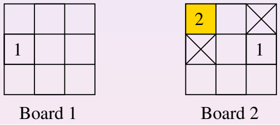

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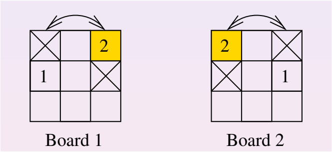

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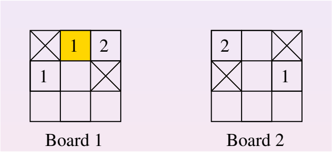

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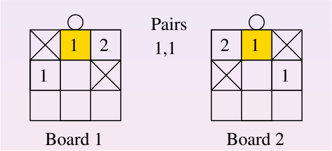

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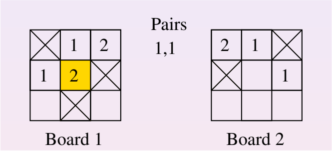

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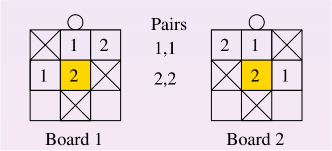

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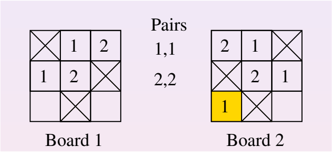

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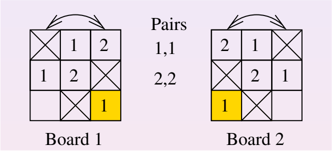

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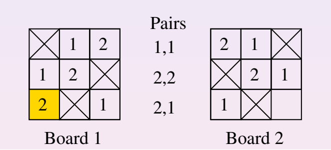

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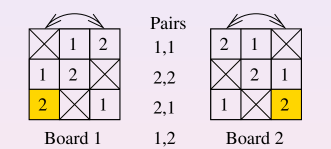

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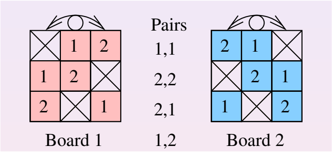

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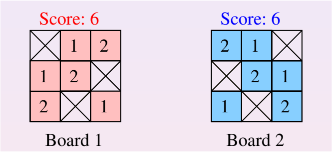

There is a **draw.**

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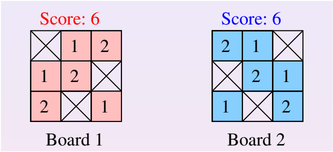

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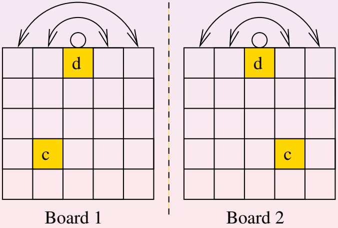
Theorem. Player 2 can always force a draw.



From orthogonal Latin squares to orthogonal graphs: The mutually orthogonal Latin squares games

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Game: Alternately, the players colour an entry in any board such that

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The player with the highest score (number of coloured entries in their board) wins.

Given: two **isomorphic graphs**, each associated with a player, and *m* colours.

Game: Alternately, the players colour a **vertex** in any **graph** such that

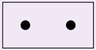
> **(Proper colouring)** *Adjacent vertices receive distinct colours.*

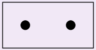
> **(Orthogonality)** *An ordered pair of colours of the same* **vertex** *from* **Graph 1** *and* **Graph 2** *appears at most once.*

The player with the highest score (number of coloured **vertices** in their **graph**) wins.

Examples for the three possible outcomes of the game

Player 1 wins
with $m = 1$ colour


Graph 1


Graph 2

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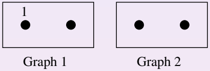

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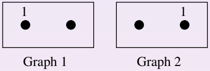

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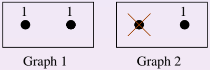

Score: 2 !!!

Score: 1

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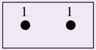

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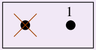

Graph 2

There is a **draw**
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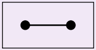

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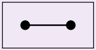

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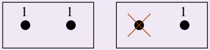

Graph 1 Graph 2

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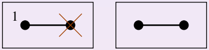

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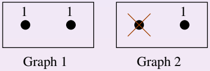

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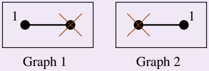

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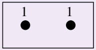

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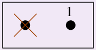

Graph 2

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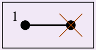

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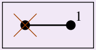
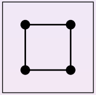


Graph 2

Player 2 wins
with $m = 2$ colours



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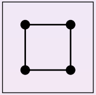

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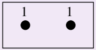

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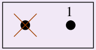

Graph 2

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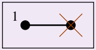

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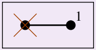

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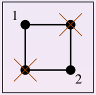

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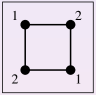

Graph 2

Examples for the three possible outcomes of the game

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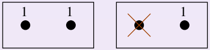

Graph 1 Graph 2

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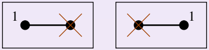

Graph 1 Graph 2

Player 2 wins
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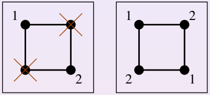
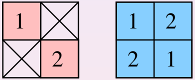


Graph 1 Graph 2

Score:2 Score:4 !!!



Board 1 Board 2

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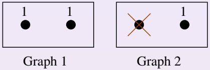
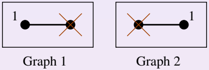
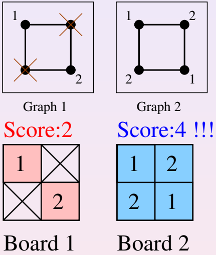


There is a draw
with $m = 1$ colour



Player 2 wins
with $m = 2$ colours



Theorem

For all $m \geq 3$, given an instance of the orthogonal colouring game with a partial colouring, it is PSPACE-complete to decide the outcome of the game.

QSAT (PSPACE-hard problem)

Instance

- Set of boolean variables $x_1, \dots, x_n$.
- Boolean formula $F = C_1 \wedge C_2 \wedge \dots C_p$ where each C_i is a disjunction of literals.
- Expression $\phi = Q_1 x_1 Q_2 x_2 \dots Q_n x_n F$ where each Q_j is either $\exists$ or $\forall$.

Question

Is ϕ true?

Alternating QSAT (PSPACE-hard problem)

Instance

- Set of boolean variables $x_1, \ldots, x_n$.
- Boolean formula $F = C_1 \wedge C_2 \wedge \ldots C_p$ where each C_i is a disjunction of literals.
- Expression $\phi = Q_1 x_1 Q_2 x_2 \ldots Q_n x_n F$ where $Q_j \equiv \exists$ for j odd and $Q_j \equiv \forall$ for j even.

Question

Is ϕ true?

Alternating QSAT in layman's terms

Recall that a truth value is either true (T) or false (F).

What are we looking for?

Does there exist a truth value (T or F) for x_1 such that

for all truth values (T and F) of x_2,

there exists a truth value (T or F) for x_3 such that

for all truth values (T and F) of x_4, etc.

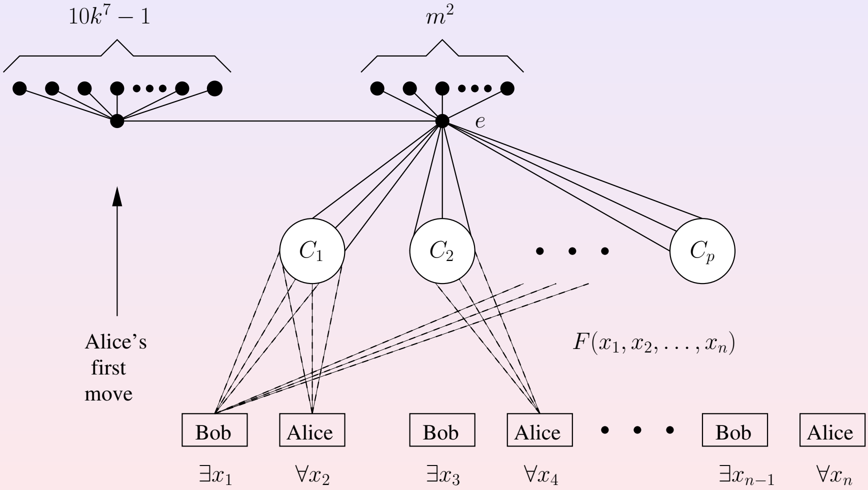
such that the formula F is true?

General game equivalence

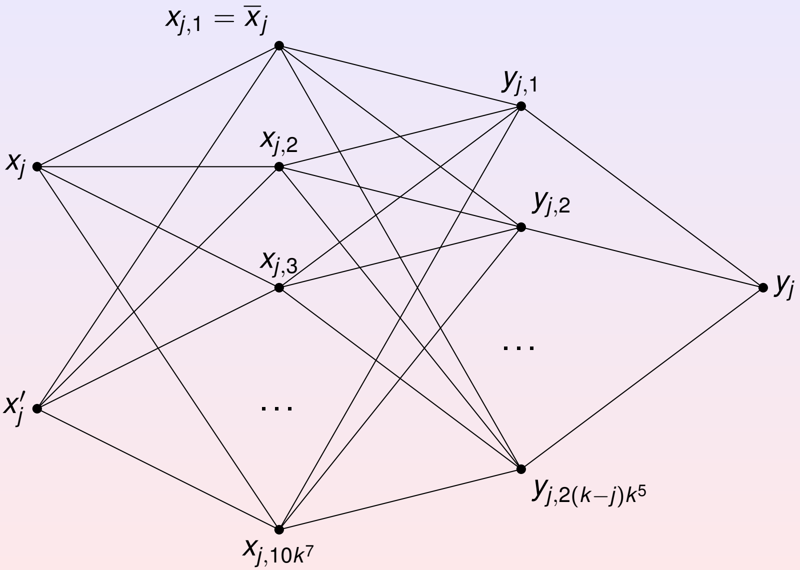
Alternating QSAT $\equiv$ 2-player game where player 1 decides truth values for variables with $\exists$ and player 2 for those with $\forall$, in order. **Player 1 wins if ϕ is true.**

If ϕ is true, then player 1 can always find a truth value for his variable that will make ϕ true no matter what player 2 chooses for his variables.

3 colours: *T*, *F*, and *X*.

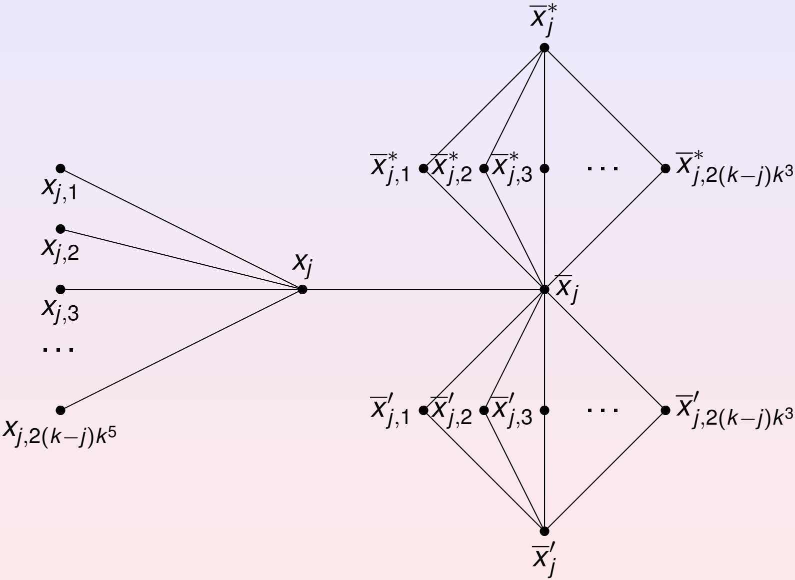


Bob Gadget



$x_{j,1} = \overline{x}_j$
$y_{j,1}$
$x_{j,2}$
x_j
$y_{j,2}$
$x_{j,3}$
y_j
$\dots$
x'_j
$\dots$
$y_{j,2(k-j)k^5}$
$x_{j,10k^7}$

Alice Gadget



Graphs admitting a strictly matched involution

An involution of a graph G is an automorphism of G s.t.

$$\forall v \in V(G): \quad (\sigma \circ \sigma)(v) = v.$$

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An involution of G is strictly matched if

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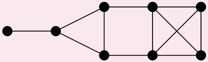
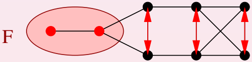
(SI 2) for every non-fixed point $v \in V(G) \setminus F$ we have the (matching) edge $v\sigma(v) \in E(G)$.





Graphs admitting a strictly matched involution: The generalisation of the drawing strategy

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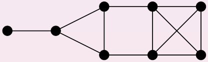

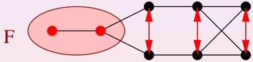

Main Theorem

For any graph G that admits a strictly matched involution, Player 2 has a strategy to guarantee a draw in the Orthogonal Graph Colouring Game played on G.

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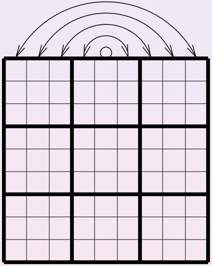
Idea of proof.
Let σ be a fixed strictly matched involution of G.

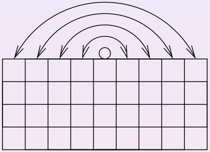
Strategy of Player 2:
If Player 1 colours vertex v in some copy of G with colour c, then Player 2 colours vertex $\sigma(v)$ in the other copy of G with colour c.

Some other combinatorial structures whose graphs admit strictly matched involutions

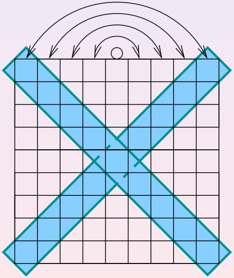
Sudoku squares



Latin rectangles



Latin squares
with double diagonal condition



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A graph G admits a strictly matched involution if and only if the vertices $V(G)$ can be partitioned into a clique C and a graph that has a perfect matching M such that:

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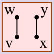
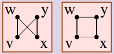
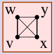
1. for any two edges $vw, xy \in M$, the graph induced by $v, w, x, y \in V(G)$ is isomorphic to:







$2K_2$ or C_4 or K_4

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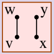
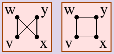


$2K_2$ or



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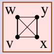
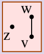
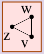


K_4

2. for any edge $vw \in M$ and any vertex $z \in C$, the graph induced by $v, w, z \in V(G)$ is isomorphic to:

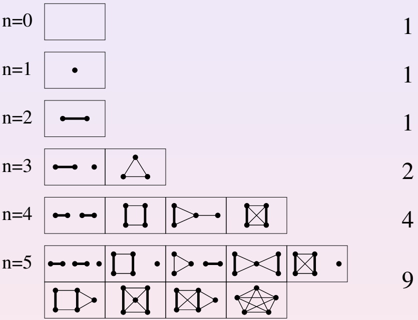


$K_1 \cup K_2$ or

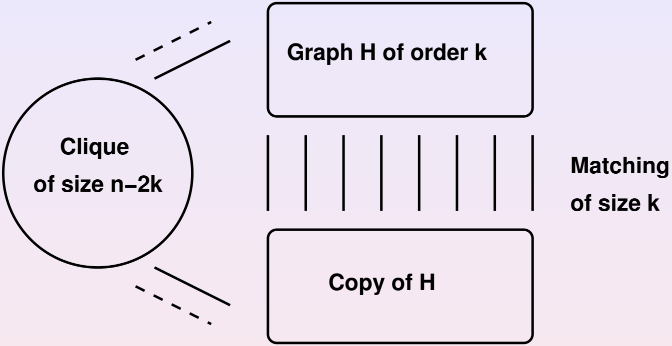


K_3

Small graphs admitting a strictly matched involution



Graphs with up to 5 vertices admitting a strictly matched involution.



Theorem

Given a graph G, it is NP-complete to determine if it admits a strictly matched involution.

Counting up to isomorphism: an upper bound

$g(n)$: number of isomorphism classes of graphs with n vertices.

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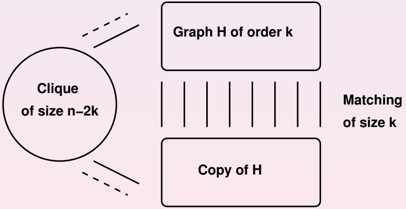
Each graph G with n vertices admitting a strictly matched involution can be constructed in the following way:



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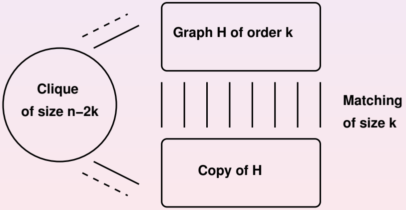

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Each graph G with n vertices admitting a strictly matched involution can be constructed in the following way:

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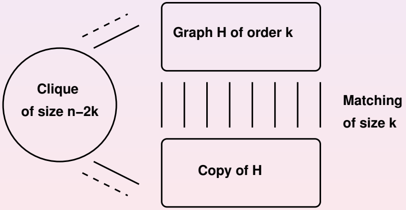

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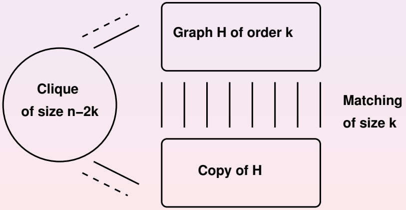

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Theorem

$$A(n) \leq \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} g(k) \cdot 2^{\binom{k}{2}} \cdot 2^{(n-2k)k}$$

A flavour of the size of the upper bound

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Theorem (upper bound for $A(n)$)

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"moderate" $c(n)$

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Observation (trivial lower bound for $A(n)$ for $n \geq 3$)

$$A(n) \geq \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} g(k) \geq \sqrt[4]{g(n-1)} = \Omega(d(n)\sqrt[4]{g(n)}) \text{ with}$$

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Open problems

Main result: for graphs admitting a strictly matched involution,

- either Player 2 has a winning strategy
- or Player 1 and Player 2 have a drawing strategy

in the Orthogonal Colouring Game on graphs.

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For any $m \in \mathbb{N}$, characterise the class of graphs admitting a strictly matched involution where Player 2 has a winning strategy for the m-colour Orthogonal Colouring Game.

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Sub-Problem

For which $m, n \in \mathbb{N}$, does Player 2 have a winning strategy for the m-colour mutually orthogonal Latin squares game on an $n \times n$ board? It is a draw when $m = 1$.

Almost 30 year-old open problem

Colouring construction game: two players: Alice and Bob, take turns colouring the vertices of a graph while maintaining that the colouring is proper.

Alice wins if all vertices are eventually coloured, otherwise, Bob wins.

Bodlaender 1991

What is the complexity of determining the outcome of the colouring construction game?

Thanks!