The Orthogonal Colouring Game

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(Thanks to Dominique for helping with the slides!)
1. The mutually orthogonal Latin squares game
   → strategy for player 2 to force a draw

2. The orthogonal colouring game on graphs
   → complexity, graphs with a strictly matched involution, general strategy

3. Characterising graphs admitting a strictly matched involution
   → complexity, construction, bounds for counting
Given: two \((n \times n)\)-boards, each associated with a player, and \(m\) colours. 

Game: Alternately, the players colour an entry in any board such that

(Latin square condition) Every colour appears at most once in each row and column of the board.

(Orthogonality) An ordered pair of colours of the same entry from Board 1 and Board 2 appears at most once.

The player with the highest score (number of coloured entries in their board) wins.
The mutually orthogonal Latin squares game

**Given:** two \((n \times n)\)-boards, each associated with a player, and \(m\) colours.

**Game:** Alternately, the players colour an entry in any board such that

- **(Latin square condition)** Every colour appears at most once in each row and column of the board.
- **(Orthogonality)** An ordered pair of colours of the same entry from Board 1 and Board 2 appears at most once.

The player with the highest score (number of coloured entries in their board) wins.

**Example**

```
  +---+---+---+---+   +---+---+---+---+
  |   |   |   |   |   |   |   |   |
  +---+---+---+---+   +---+---+---+---+
  |   |   |   |   |   |   |   |   |
  +---+---+---+---+   +---+---+---+---+
  |   |   |   |   |   |   |   |   |
  +---+---+---+---+   +---+---+---+---+
  |   |   |   |   |   |   |   |   |
  +---+---+---+---+   +---+---+---+---+
  |   |   |   |   |   |   |   |   |
  +---+---+---+---+   +---+---+---+---+
  |   |   |   |   |   |   |   |   |
  +---+---+---+---+   +---+---+---+---+
```

Board 1  Board 2

Player 2 wins.
**The mutually orthogonal Latin squares game**

**Given:** two \((n \times n)\)-boards, each associated with a player, and \(m\) colours.

**Game:** Alternately, the players colour an entry in any board such that

*(Latin square condition)* Every colour appears at most once in each row and column of the board.

*(Orthogonality)* An ordered pair of colours of the same entry from Board 1 and Board 2 appears at most once.

The player with the highest score (number of coloured entries in their board) wins.

**Example**

```
<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
```

Board 1

```
<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
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<th></th>
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</thead>
<tbody>
<tr>
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<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
```

Board 2

Player 2 wins.
The mutually orthogonal Latin squares game

Given: two \((n \times n)\)-boards, each associated with a player, and \(m\) colours.

Game: Alternately, the players colour an entry in any board such that

(Latin square condition) Every colour appears at most once in each row and column of the board.

(Orthogonality) An ordered pair of colours of the same entry from Board 1 and Board 2 appears at most once.

The player with the highest score (number of coloured entries in their board) wins.

Example

```
1 2
\hline
\hline
\hline
```

Board 1

```
\hline
|   |   |
\hline
|   |   |
\hline
```

Board 2

Player 2 wins.
The mutually orthogonal Latin squares game

**Given:** two \((n \times n)\)-boards, each associated with a player, and \(m\) colours.

**Game:** Alternately, the players colour an entry in any board such that

*(Latin square condition)* Every colour appears at most once in each row and column of the board.

*(Orthogonality)* An ordered pair of colours of the same entry from Board 1 and Board 2 appears at most once.

The player with the highest score (number of coloured entries in their board) wins.

**Example**

```
<table>
<thead>
<tr>
<th>1</th>
<th>X</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>2</td>
</tr>
</tbody>
</table>
```

```
<table>
<thead>
<tr>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
</tbody>
</table>
```

Player 2 wins.
Given: two \((n \times n)\)-boards, each associated with a player, and \(m\) colours.

Game: Alternately, the players colour an entry in any board such that

(Latin square condition) Every colour appears at most once in each row and column of the board.

(Orthogonality) An ordered pair of colours of the same entry from Board 1 and Board 2 appears at most once.

The player with the highest score (number of coloured entries in their board) wins.

Example

\[
\begin{array}{|c|c|}
\hline
1 & 2 \\
\hline
\end{array}
\quad
\begin{array}{|c|c|}
\hline
1 & 1 \\
\hline
\end{array}
\]

Board 1 \quad Board 2
Given: two \((n \times n)\)-boards, each associated with a player, and \(m\) colours.

Game: Alternately, the players colour an entry in any board such that

(Latin square condition) *Every colour appears at most once in each row and column of the board.*

(Orthogonality) *An ordered pair of colours of the same entry from Board 1 and Board 2 appears at most once.*

The player with the highest score (number of coloured entries in their board) wins.

Example

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Board 1

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Board 2

Player 2 wins.
Given: two \((n \times n)\)-boards, each associated with a player, and \(m\) colours.

Game: Alternately, the players colour an entry in any board such that

(Latin square condition) Every colour appears at most once in each row and column of the board.

(Orthogonality) An ordered pair of colours of the same entry from Board 1 and Board 2 appears at most once.

The player with the highest score (number of coloured entries in their board) wins.

Example

\[
\begin{array}{c}
1 \quad 2 \\
\hline
\begin{array}{c}
1 \\
2
\end{array} & \\
\begin{array}{c}
2 \\
1
\end{array}
\end{array}
\]

Board 1

\[
\begin{array}{c}
1 \quad 2 \\
\hline
\begin{array}{c}
1 \\
2
\end{array} & \\
\begin{array}{c}
2 \\
1
\end{array}
\end{array}
\]

Board 2

Player 2 wins.
The mutually orthogonal Latin squares game

Given: two \((n \times n)\)-boards, each associated with a player, and \(m\) colours.

Game: Alternately, the players colour an entry in any board such that

(Latin square condition) *Every colour appears at most once in each row and column of the board.*

(Orthogonality) *An ordered pair of colours of the same entry from Board 1 and Board 2 appears at most once.*

The player with the highest score (number of coloured entries in their board) wins.

Example

\[
\begin{array}{c|c|c|c|c}
 & & & & \\
1 & 1 & & & \\
X & X & & & \\
\hline
2 & & 2 & & \\
& & & & \\
\end{array}
\]

Score:2

\[
\begin{array}{c|c|c|c|c}
 & & & & \\
1 & 2 & & & \\
& & & & \\
\hline
2 & & & 1 & \\
& & & & \\
\end{array}
\]

Score:4 !!!

Board 1

Board 2

Player 2 wins.
The columns are paired. Player 2 responds in the “matching” column of the other board.
The columns are paired. Player 2 responds in the “matching” column of the other board.

Board 1

1

Board 2

1

There is a draw.
The columns are paired. Player 2 responds in the “matching” column of the other board.
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The drawing strategy of Player 2

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Theorem. Player 2 can always force a draw.
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Andres, Dross, Huggan, Mc Inerney, Nowakowski
The Orthogonal Colouring Game
The columns are paired. Player 2 responds in the “matching” column of the other board.

```
Board 1      Pairs  
1 2           1,1  
1 2           2,2  
2 1           2,1  

Board 2
1 2           1,2  
```

There is a draw.

Theorem.
Player 2 can always force a draw.
The columns are paired. Player 2 responds in the “matching” column of the other board.

There is a draw.
The drawing strategy of Player 2

The columns are paired. Player 2 responds in the “matching” column of the other board.

Score: 6  Score: 6
Board 1 Board 2

There is a draw.

**Theorem.** Player 2 can always force a draw.
**Given:** two \((n \times n)\)-boards, each associated with a player, and \(m\) colours.

**Game:** Alternately, the players colour an entry in any board such that

*(Latin square condition)* Every colour appears at most once in each row and column of the board.

*(Orthogonality)* An ordered pair of colours of the same entry from Board 1 and Board 2 appears at most once.

The player with the highest score (number of coloured entries in their board) wins.
Given: two *isomorphic graphs*, each associated with a player, and $m$ colours.

Game: Alternately, the players colour a *vertex* in any *graph* such that

*(Proper colouring)* Adjacent vertices receive distinct colours.

*(Orthogonality)* An ordered pair of colours of the same vertex from Graph 1 and Graph 2 appears at most once.

The player with the highest score (number of coloured *vertices* in their *graph*) wins.
Examples for the three possible outcomes of the game

Player 1 wins
with $m = 1$ colour

\[ \begin{array}{c}
\bullet & \bullet \\
\text{Graph 1} & \text{Graph 2}
\end{array} \]
Examples for the three possible outcomes of the game

**Player 1 wins**
with $m = 1$ colour

![Graph 1](image1)
![Graph 2](image2)

Theorem
For all $m \geq 3$, given an instance of the orthogonal colouring game with a partial colouring, it is PSPACE-complete to decide the outcome of the game.
Examples for the three possible outcomes of the game

Player 1 wins
with $m = 1$ colour

1

1

Graph 1  Graph 2

Score: 2 !!!  Score: 1

There is a draw with $m = 1$ colour

Score: 1  Score: 1

Player 2 wins
with $m = 2$ colours

Theorem
For all $m \geq 3$, given an instance of the orthogonal colouring game with a partial colouring, it is PSPACE-complete to decide the outcome of the game.
Examples for the three possible outcomes of the game

**Player 1 wins**
with $m = 1$ colour

Score: 2 !!!
Score: 1

![Graph 1](image1.png) ![Graph 2](image2.png)

Andres, Dross, Huggan, Mc Inerney, Nowakowski

The Orthogonal Colouring Game
Examples for the three possible outcomes of the game

**Player 1 wins**
with $m = 1$ colour

**There is a draw**
with $m = 1$ colour

![Graph 1](image1.png) ![Graph 2](image2.png)

Score: 2 !!! Score: 1

Score: 1 Score: 1

**Player 2 wins**
with $m = 2$ colours

Theorem
For all $m \geq 3$, given an instance of the orthogonal colouring game with a partial colouring, it is PSPACE-complete to decide the outcome of the game.
Examples for the three possible outcomes of the game

**Player 1 wins**
with \( m = 1 \) colour

**There is a draw**
with \( m = 1 \) colour

\[ \begin{align*}
\text{Graph 1} & \quad \text{Graph 2} \\
1 & \quad 1 \\
\bullet & \quad \bullet
\end{align*} \]

\[ \begin{align*}
\text{Graph 1} & \quad \text{Graph 1} \\
1 & \quad 1 \\
\bullet & \quad \bullet
\end{align*} \]
Examples for the three possible outcomes of the game

Player 1 wins with $m = 1$ colour

<table>
<thead>
<tr>
<th>Graph 1</th>
<th>Graph 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>🟦</td>
<td>💲</td>
</tr>
</tbody>
</table>

There is a draw with $m = 1$ colour

<table>
<thead>
<tr>
<th>Graph 1</th>
<th>Graph 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>🟦</td>
<td>💲</td>
</tr>
</tbody>
</table>

Score: 1

Player 2 wins with $m = 2$ colours

<table>
<thead>
<tr>
<th>Graph 1</th>
<th>Graph 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>🟦</td>
<td>💲</td>
</tr>
</tbody>
</table>

Score: 1

Theorem

For all $m \geq 3$, given an instance of the orthogonal colouring game with a partial colouring, it is PSPACE-complete to decide the outcome of the game.
Examples for the three possible outcomes of the game:

**Player 1 wins**
with $m = 1$ colour

**There is a draw**
with $m = 1$ colour

**Player 2 wins**
with $m = 2$ colours

---

**Theorem**

For all $m \geq 3$, given an instance of the orthogonal colouring game with a partial colouring, it is PSPACE-complete to decide the outcome of the game.

Andres, Dross, Huggan, Mc Inerney, Nowakowski
Examples for the three possible outcomes of the game

Player 1 wins with $m = 1$ colour

There is a draw with $m = 1$ colour

Player 2 wins with $m = 2$ colours
Examples for the three possible outcomes of the game

**Player 1 wins**
with $m = 1$ colour

**There is a draw**
with $m = 1$ colour

**Player 2 wins**
with $m = 2$ colours

Theorem
For all $m \geq 3$, given an instance of the orthogonal colouring game with a partial colouring, it is PSPACE-complete to decide the outcome of the game.

Andres, Dross, Huggan, Mc Inerney, Nowakowski

The Orthogonal Colouring Game
Examples for the three possible outcomes of the game

Player 1 wins with \( m = 1 \) colour

There is a draw with \( m = 1 \) colour

Player 2 wins with \( m = 2 \) colours

Graph 1

Graph 2

Graph 1

Graph 2

Graph 1

Graph 2

Score: 2 !!!

Score: 1

Score: 1

Score: 4 !!!

Board 1

Board 2

Theorem

For all \( m \geq 3 \), given an instance of the orthogonal colouring game with a partial colouring, it is \text{PSPACE-complete} to decide the outcome of the game.
**Instance**

- Set of boolean variables $x_1, \ldots, x_n$.
- Boolean formula $F = C_1 \land C_2 \land \ldots \land C_p$ where each $C_i$ is a disjunction of literals.
- Expression $\phi = Q_1 x_1 Q_2 x_2 \ldots Q_n x_n F$ where each $Q_j$ is either $\exists$ or $\forall$.

**Question**

Is $\phi$ true?
Alternating QSAT (PSPACE-hard problem)

Instance

- Set of boolean variables $x_1, \ldots, x_n$.
- Boolean formula $F = C_1 \land C_2 \land \ldots C_p$ where each $C_i$ is a disjunction of literals.
- Expression $\phi = Q_1 x_1 Q_2 x_2 \ldots Q_n x_n F$ where $Q_j \equiv \exists$ for $j$ odd and $Q_j \equiv \forall$ for $j$ even.

Question

Is $\phi$ true?
Recall that a truth value is either true ($T$) or false ($F$).

What are we looking for?

Does there exist a truth value ($T$ or $F$) for $x_1$ such that for all truth values ($T$ and $F$) of $x_2$, there exists a truth value ($T$ or $F$) for $x_3$ such that for all truth values ($T$ and $F$) of $x_4$, etc. such that the formula $F$ is true?
General game equivalence

Alternating QSAT $\equiv$ 2-player game where player 1 decides truth values for variables with $\exists$ and player 2 for those with $\forall$, in order. **Player 1 wins if $\phi$ is true.**

If $\phi$ is true, then player 1 can always find a truth value for his variable that will make $\phi$ true no matter what player 2 chooses for his variables.
Reduction from Alt. QSAT to Orthogonal Col. Game

3 colours: $T$, $F$, and $X$.

$10k^7 - 1$

$m^2$

Alice's first move $F(x_1, x_2, \ldots, x_n)$
\[ x_{j,1} = \bar{x}_j \]
Alice Gadget

Andres, Dross, Huggan, Mc Inerney, Nowakowski
The Orthogonal Colouring Game
An involution of a graph $G$ is an automorphism of $G$ s.t.

$$\forall v \in V(G) : (\sigma \circ \sigma)(v) = v.$$
An involution of a graph $G$ is an automorphism of $G$ s.t.

$$\forall v \in V(G) : (\sigma \circ \sigma)(v) = v.$$  

An involution of $G$ is strictly matched if

- (SI 1) the set $F = \{ v \in V(G) | \sigma(v) = v \} \subseteq V(G)$ of fixed points of $\sigma$ induces a complete graph and

- (SI 2) for every non-fixed point $v \in V(G) \setminus F$ we have the (matching) edge $v\sigma(v) \in E(G)$.
An involution of a graph $G$ is an automorphism of $G$ s.t.

$$\forall v \in V(G) : (\sigma \circ \sigma)(v) = v.$$ 

An involution of $G$ is strictly matched if

- (SI 1) the set $F = \{v \in V(G) \mid \sigma(v) = v\} \subseteq V(G)$ of fixed points of $\sigma$ induces a complete graph and

- (SI 2) for every non-fixed point $v \in V(G) \setminus F$ we have the (matching) edge $v\sigma(v) \in E(G)$.

**Main Theorem**

For any graph $G$ that admits a strictly matched involution, Player 2 has a strategy to guarantee a draw in the Orthogonal Graph Colouring Game played on $G$. 

\[\text{Andres, Dross, Huggan, Mc Inerney, Nowakowski}\]
Graphs admitting a strictly matched involution: The generalisation of the drawing strategy

Main Theorem

For any graph $G$ that admits a strictly matched involution, Player 2 has a strategy to guarantee a draw in the Orthogonal Graph Colouring Game played on $G$. 
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For any graph $G$ that admits a strictly matched involution, Player 2 has a strategy to guarantee a draw in the Orthogonal Graph Colouring Game played on $G$.

Idea of proof.
Let $\sigma$ be a fixed strictly matched involution of $G$.

Strategy of Player 2:
If Player 1 colours vertex $v$ in some copy of $G$ with colour $c$, then Player 2 colours vertex $\sigma(v)$ in the other copy of $G$ with colour $c$. 
Some other combinatorial structures whose graphs admit strictly matched involutions

Sudoku squares

Latin rectangles

Latin squares with double diagonal condition
A graph $G$ admits a strictly matched involution if and only if the vertices $V(G)$ can be partitioned into a clique $C$ and a graph that has a perfect matching $M$ such that:

1. For any two edges $vw, xy \in M$, the graph induced by $v, w, x, y \in V(G)$ is isomorphic to:
   - $K_2$ or $C_4$

2. For any edge $vw \in M$ and any vertex $z \in C$, the graph induced by $v, w, z \in V(G)$ is isomorphic to:
   - $K_1 \cup K_2$ or $K_3$
Characterisation of graphs admitting a strictly matched involution

Theorem

A graph $G$ admits a strictly matched involution if and only if the vertices $V(G)$ can be partitioned into a clique $C$ and a graph that has a perfect matching $M$ such that:

1. for any two edges $vw, xy \in M$, the graph induced by $v, w, x, y \in V(G)$ is isomorphic to:
   - $2K_2$
   - $C_4$
   - $K_4$

2. for any edge $vw \in M$ and any vertex $z \in C$, the graph induced by $v, w, z \in V(G)$ is isomorphic to:
   - $K_1 \cup K_2$
   - $K_3$
Theorem

A graph $G$ admits a strictly matched involution if and only if the vertices $V(G)$ can be partitioned into a clique $C$ and a graph that has a perfect matching $M$ such that:

1. for any two edges $vw, xy \in M$, the graph induced by $v, w, x, y \in V(G)$ is isomorphic to:
   - $2K_2$ or $C_4$ or $K_4$
   
   ![Graphs illustrating $2K_2$, $C_4$, and $K_4$]

2. for any edge $vw \in M$ and any vertex $z \in C$, the graph induced by $v, w, z \in V(G)$ is isomorphic to:
   - $K_1 \cup K_2$ or $K_3$
   
   ![Graphs illustrating $K_1 \cup K_2$ and $K_3$]
Small graphs admitting a strictly matched involution

Graphs with up to 5 vertices admitting a strictly matched involution.
Theorem

Given a graph $G$, it is **NP-complete** to determine if it admits a strictly matched involution.
$g(n)$: number of isomorphism classes of graphs with $n$ vertices.
$A(n)$: number of isomorphism classes of graphs with $n$ vertices admitting a strictly matched involution.
\( g(n) \): number of isomorphism classes of graphs with \( n \) vertices.

\( A(n) \): number of isomorphism classes of graphs with \( n \) vertices admitting a strictly matched involution.

Each graph \( G \) with \( n \) vertices admitting a strictly matched involution can be constructed in the following way:
\[ g(n) \]: number of isomorphism classes of graphs with \( n \) vertices. \\
\[ A(n) \]: number of isomorphism classes of graphs with \( n \) vertices admitting a strictly matched involution. \\

Each graph \( G \) with \( n \) vertices admitting a strictly matched involution can be constructed in the following way: \\
- \( g(k) \) choices for \( H \)
$g(n)$: number of isomorphism classes of graphs with $n$ vertices.

$A(n)$: number of isomorphism classes of graphs with $n$ vertices admitting a strictly matched involution.

Each graph $G$ with $n$ vertices admitting a strictly matched involution can be constructed in the following way:

- $g(k)$ choices for $H$

- $2^{\binom{k}{2}}$ choices for the edges between the two copies of $H$. 
$g(n)$: number of isomorphism classes of graphs with $n$ vertices.

$A(n)$: number of isomorphism classes of graphs with $n$ vertices admitting a strictly matched involution.

Each graph $G$ with $n$ vertices admitting a strictly matched involution can be constructed in the following way:
- $g(k)$ choices for $H$
- $2^{\binom{k}{2}}$ choices for the edges between the two copies of $H$.
- $2^{(n-2k)k}$ choices for the edges between the clique and $H$
  (which determine the edges between the clique and the second copy).
Counting up to isomorphism: an upper bound

\( g(n) \): number of isomorphism classes of graphs with \( n \) vertices. 

\( A(n) \): number of isomorphism classes of graphs with \( n \) vertices admitting a strictly matched involution.

Each graph \( G \) with \( n \) vertices admitting a strictly matched involution can be constructed in the following way:

- \( g(k) \) choices for \( H \)
- \( 2^{\binom{k}{2}} \) choices for the edges between the two copies of \( H \).
- \( 2^{(n−2k)k} \) choices for the edges between the clique and \( H \)

(which determine the edges between the clique and the second copy).

Theorem

\[
A(n) \leq \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} g(k) \cdot 2^{\binom{k}{2}} \cdot 2^{(n−2k)k}
\]
A flavour of the size of the upper bound

\( g(n) \): number of isomorphism classes of graphs with \( n \) vertices.

\( A(n) \): number of isomorphism classes of graphs with \( n \) vertices admitting a strictly matched involution.

**Theorem (upper bound for \( A(n) \))**

\[
A(n) \leq \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} g(k) \cdot 2^k \cdot 2^{(n-2k)k} = \mathcal{O}(c(n)\sqrt{g(n)}) \]

with “moderate” \( c(n) \)
A flavour of the size of the upper bound

\( g(n) \): number of isomorphism classes of graphs with \( n \) vertices.

\( A(n) \): number of isomorphism classes of graphs with \( n \) vertices admitting a strictly matched involution.

**Theorem (upper bound for \( A(n) \))**

\[
A(n) \leq \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} g(k) \cdot 2^{k \choose 2} \cdot 2^{n-2k} k = O(c(n) \sqrt{g(n)})
\]

“moderate” \( c(n) \)

**Observation (trivial lower bound for \( A(n) \) for \( n \geq 3 \))**

\[
A(n) \geq \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} g(k) \geq \sqrt[4]{g(n - 1)} = \Omega(d(n) \sqrt[4]{g(n)})
\]

“moderate” \( d(n) \)
Main result: for graphs admitting a strictly matched involution,
- either Player 2 has a winning strategy
- or Player 1 and Player 2 have a drawing strategy
in the Orthogonal Colouring Game on graphs.
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Open Problem
For any $m \in \mathbb{N}$, characterise the class of graphs admitting a strictly matched involution where Player 2 has a winning strategy for the $m$-colour Orthogonal Colouring Game.
Open problems

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Sub-Problem
For which \( m, n \in \mathbb{N} \), does Player 2 have a winning strategy for the \( m \)-colour mutually orthogonal Latin squares game on an \( n \times n \) board? It is a draw when \( m = 1 \).
Almost 30 year-old open problem

Colouring construction game: two players: Alice and Bob, take turns colouring the vertices of a graph while maintaining that the colouring is proper.

Alice wins if all vertices are eventually coloured, otherwise, Bob wins.

Bodlaender 1991

What is the complexity of determining the outcome of the colouring construction game?
Thanks!