Eternal Domination in Grids

Fionn Mc Inerney, Nicolas Nisse, Stéphane Pérennes

Université Côte d'Azur, Inria, CNRS, I3S, France

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Eternal Domination Game [Burger et al, 2004; Goddard et al, 2005]

Overview

- k guards vs. 1 attacker
- **Each turn:** attacker attacks any vertex and guards may move to neighbours.
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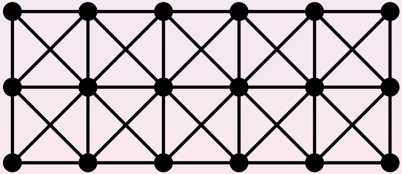
- $\gamma_{all}^{\infty}(G)$: min. # guards needed to guarantee they win in G.



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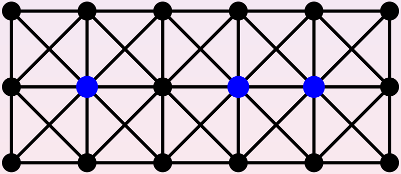

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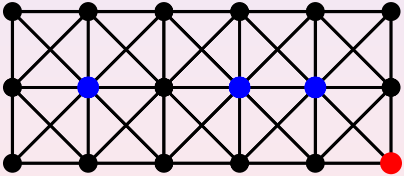

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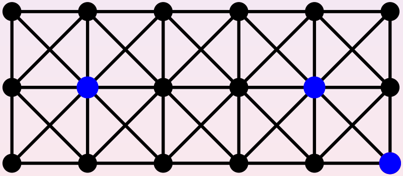

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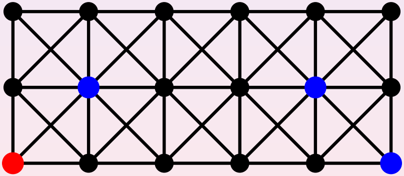

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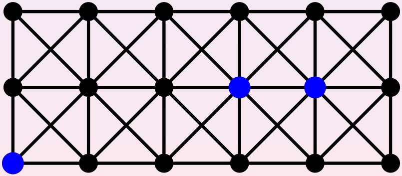

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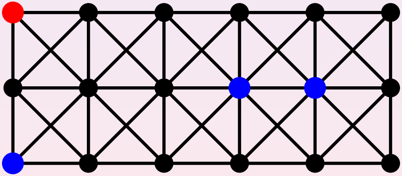

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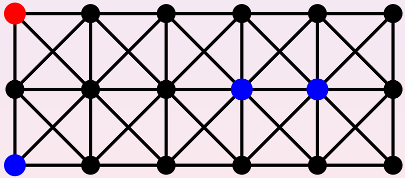

Attacker wins!

Eternal Domination Game [Burger et al, 2004; Goddard et al, 2005]

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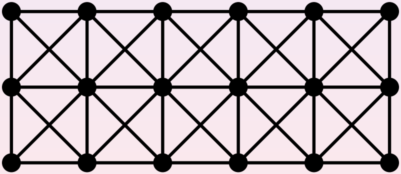

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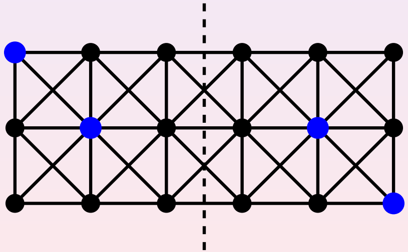

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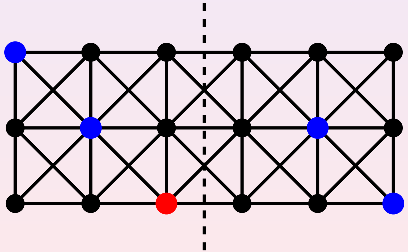

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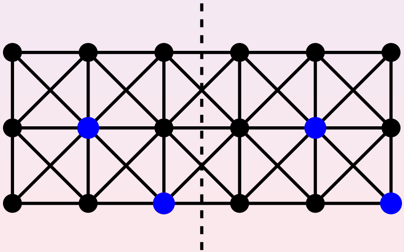

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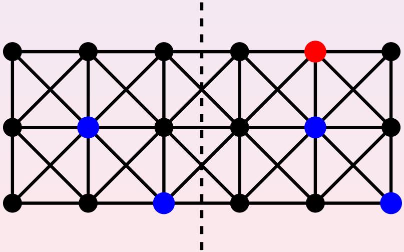

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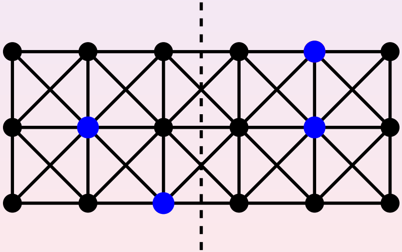

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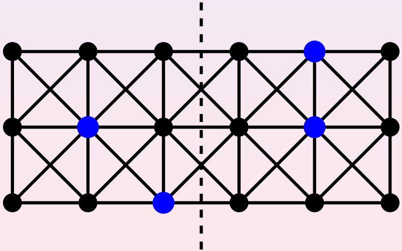

Guards win!

State of the art

- Deciding whether $\gamma_{all}^{\infty}(G) \leq k$ is NP-hard [Bard et al, 2017].
- For all G, $\gamma(G) \leq \gamma_{all}^{\infty}(G) \leq \alpha(G)$ [Goddard et al, 2005].
- Paths, cycles: $\gamma_{all}^{\infty}(P_n) = \lceil \frac{n}{2} \rceil$, $\gamma_{all}^{\infty}(C_n) = \lceil \frac{n}{3} \rceil$ [Goddard et al, 2005].
- Linear-time algorithm for trees [Klostermeyer, MacGillivray, 2009].
- $\gamma_{all}^{\infty}(G) = \alpha(G)$ for all proper interval graphs G [Braga et al, 2015].
- Linear-time algorithm for all interval graphs [Rinemberg, Soulignac, 2018].
- Recently studied in digraphs [Bagan et al, 2018].

Cartesian Grids

- $\gamma_{all}^{\infty}(P_2 \Box P_n) = \lceil \frac{2n}{3} \rceil$ [Goldwasser et al, 2013].
- $\lceil \frac{4n}{5} \rceil + 1 \leq \gamma_{all}^{\infty}(P_3 \Box P_n) \leq \lceil \frac{4n}{5} \rceil + 5$ [Messinger, 2017].
- $\gamma_{all}^{\infty}(P_4 \Box P_n)$ is known [Beaton et al, 2014].
- There are bounds for $\gamma_{all}^{\infty}(P_5 \Box P_n)$ [van Bommel et al, 2016].

Theorem [Lamprou et al, 2017]

$\gamma_{all}^{\infty}(P_n \Box P_m) = \gamma(P_n \Box P_m) + O(n+m)$.

Eternal Domination in Strong Grids [M., Nisse, Pérennes, 2018]

Note that $\gamma(P_n \boxtimes P_m) = \lceil \frac{m}{3} \rceil \lceil \frac{n}{3} \rceil$ and $\alpha(G) = \lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil$ and so

$$\left\lceil \frac{m}{3} \right\rceil \left\lceil \frac{n}{3} \right\rceil \leq \gamma_{all}^{\infty}(P_n \boxtimes P_m) \leq \left\lceil \frac{m}{2} \right\rceil \left\lceil \frac{n}{2} \right\rceil.$$

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Theorem [M., Nisse, Pérennes, 2018]

For all $m \geq n$,

$$\lfloor \frac{m}{3} \rfloor \lfloor \frac{n}{3} \rfloor + \Omega(n+m) = \gamma_{all}^{\infty}(P_n \boxtimes P_m) = \lceil \frac{m}{3} \rceil \lceil \frac{n}{3} \rceil + O(m\sqrt{n})$$

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Theorem [M., Nisse, Pérennes, 2018]

For all $m \geq n$ such that $n \mod 3 = m \mod 3 = 0$,

$\gamma(P_n \boxtimes P_m) + \Omega(n+m) = \gamma_{all}^{\infty}(P_n \boxtimes P_m) = \gamma(P_n \boxtimes P_m) + O(m\sqrt{n})$

Eternal domination in the torus $C_n \boxtimes C_m$

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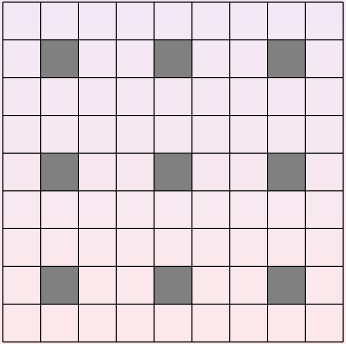

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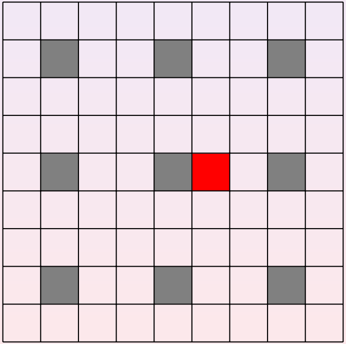

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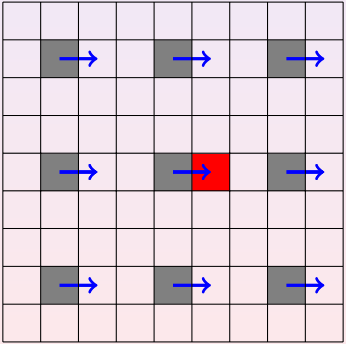

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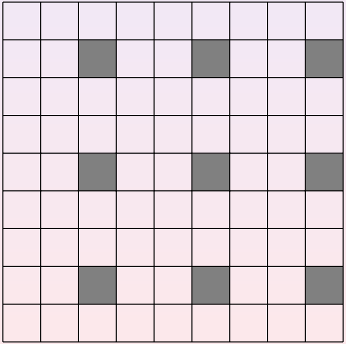

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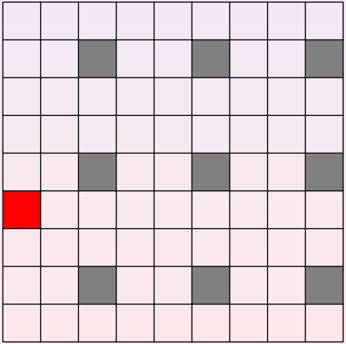

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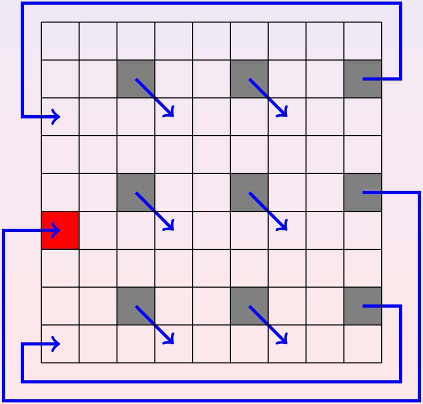

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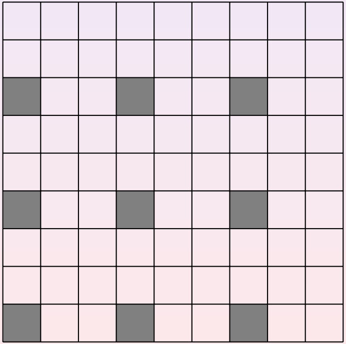

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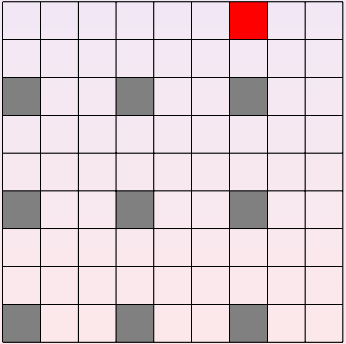

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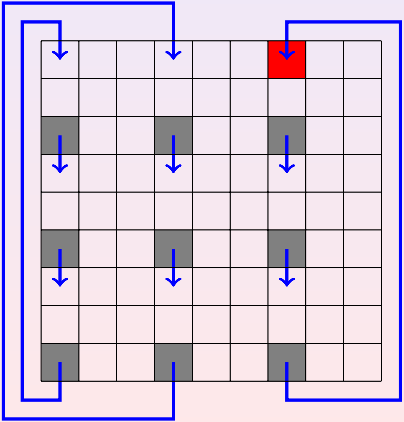

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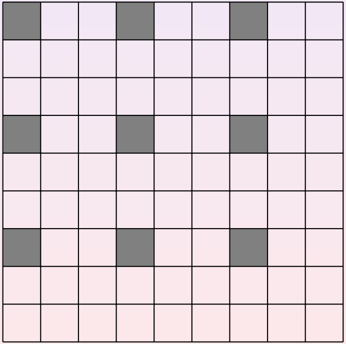
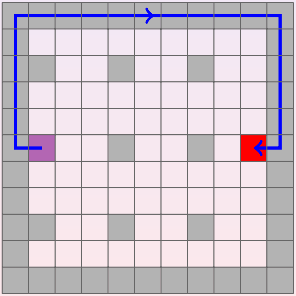


Easy in the torus because we can wrap around → impossible in the grid!

Back to the Grid: Key Lemma

Teleportation

If there are α guards on each border vertex, then $\beta \leq \alpha$ guards may teleport using the borders of the grid.



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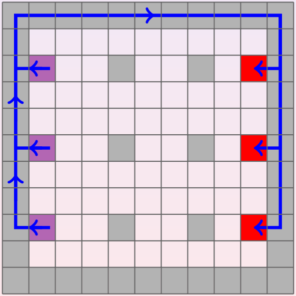


Upper Bound Overview $\gamma_{all}^{\infty}(P_n \boxtimes P_m) = \gamma(P_n \boxtimes P_m) + O(m\sqrt{n})$

Configuration

Multi-set $C = \{v_i \mid 1 \leq i \leq k\}$ giving the positions of the k guards.

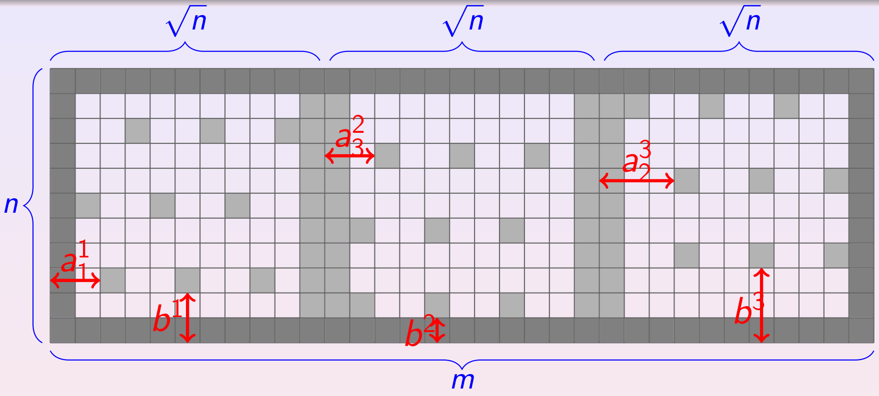
Configurations of the winning strategy: *SetWinConf*

Set of configurations that dominate the grid.

Attacks split into 3 types: Horizontal, Vertical, and Diagonal.

We show: for any attack at a vertex $v \in V(P_n \boxtimes P_m)$, the guards can move from a configuration $C \in SetWinConf$ to a configuration $C' \in SetWinConf$ where $v \in C'$.

Configuration $C \in SetWinConf$

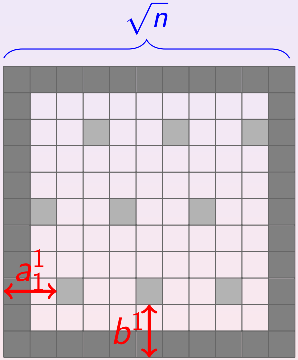


1 guard on vertices in light gray.

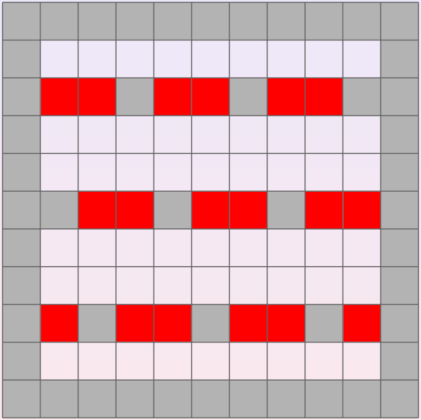
$O(\sqrt{n})$ guards on vertices in dark gray.

$\gamma(P_n \boxtimes P_m) + O(m\sqrt{n})$ guards total.

Block of Configuration $C \in SetWinConf$

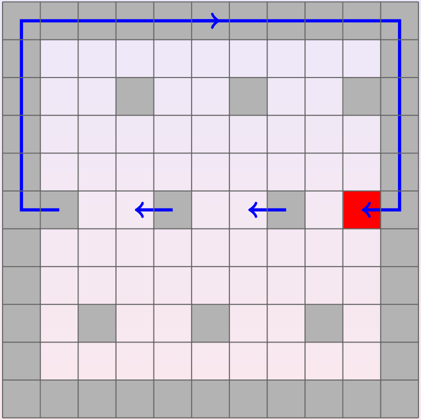


Horizontal Attacks



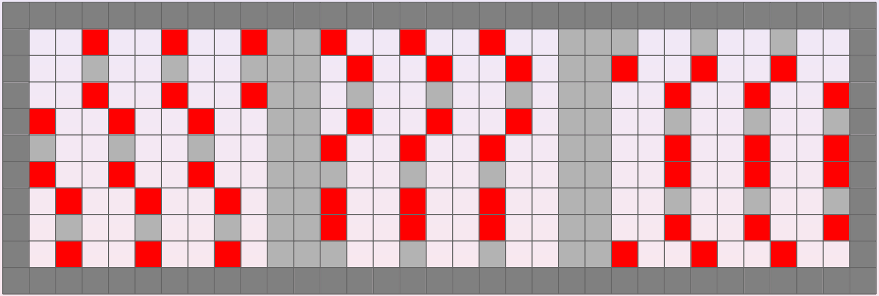
Horizontal attacks may only occur at vertices in red.

Horizontal Attacks - attack at red vertex



Only guards in same row and block move (except borders maybe).

Vertical Attacks



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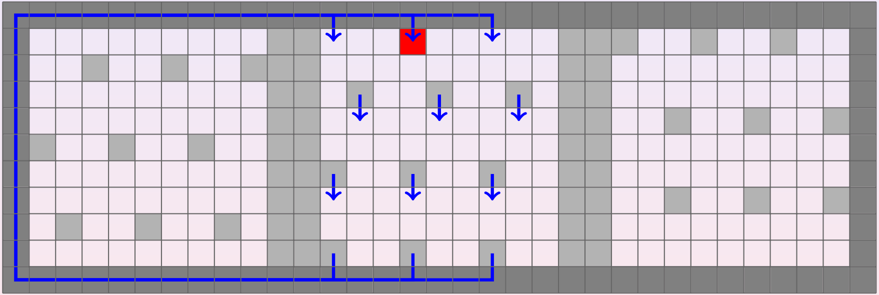

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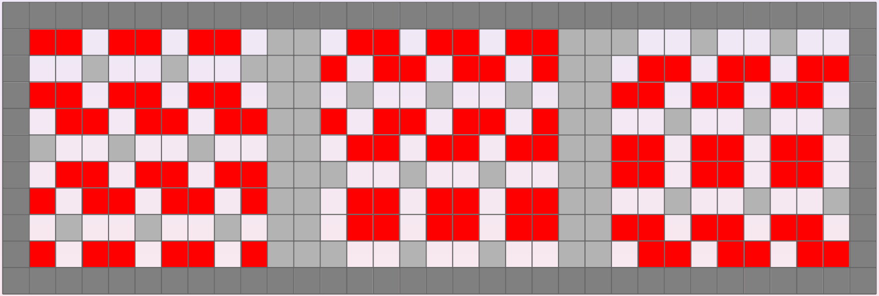
Diagonal attacks



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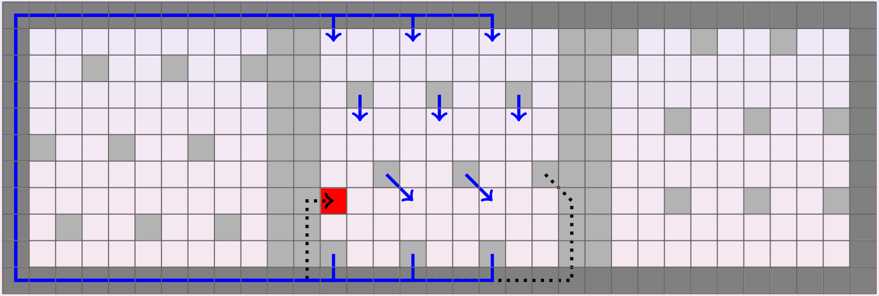
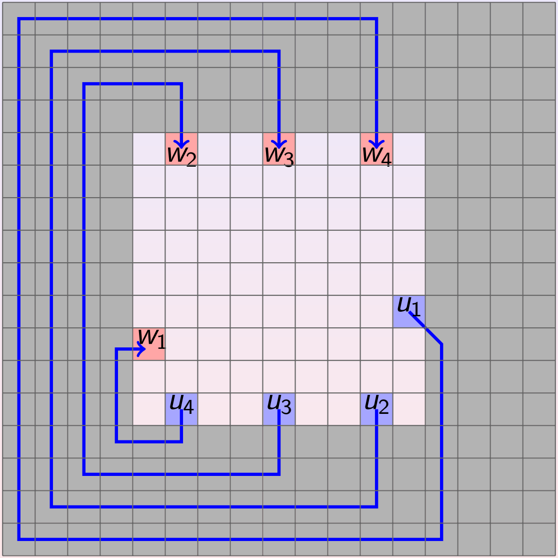
Diagonal Attacks - attack at red vertex



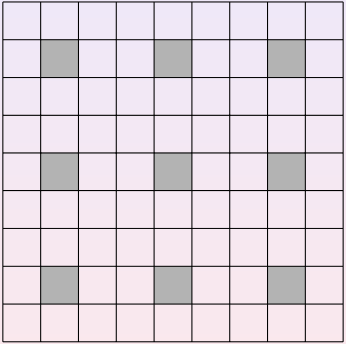
Guards in closest row (and block) move like in Horizontal and Vertical case at once and the rest in the same block move like in Vertical case.

At most 1 guard at each vertex



Lower Bound Idea of Proof $\gamma_{all}^{\infty}(P_n \boxtimes P_m) = \gamma(P_n \boxtimes P_m) + \Omega(m+n)$

At least 2 guards needed in each 4×5 subgrid on the border.



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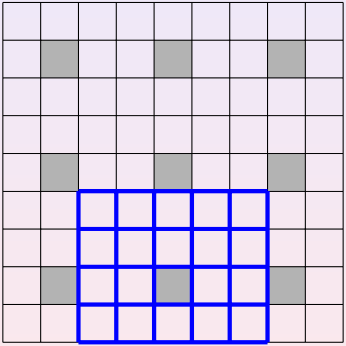

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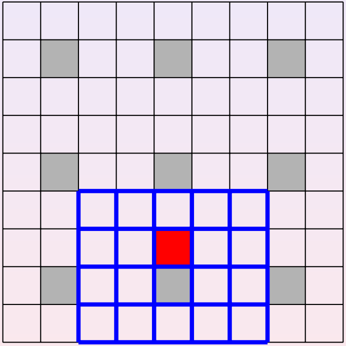

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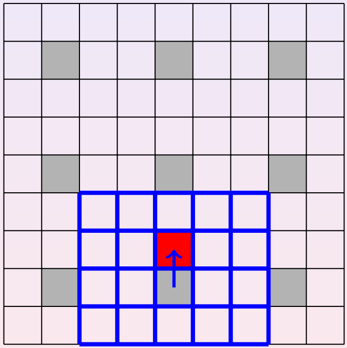

Lower Bound Idea of Proof $\gamma^{\infty}_{all}(P_n \boxtimes P_m) = \gamma(P_n \boxtimes P_m) + \Omega(m+n)$

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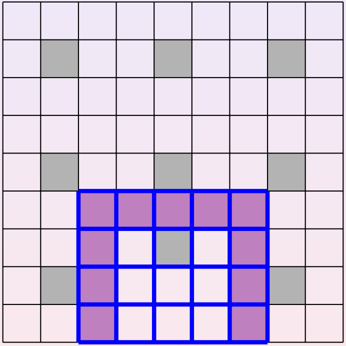

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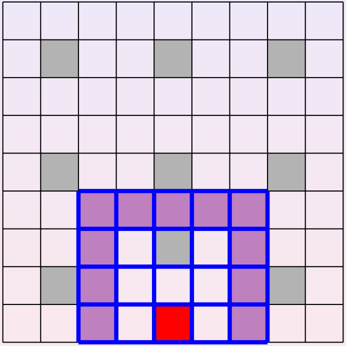
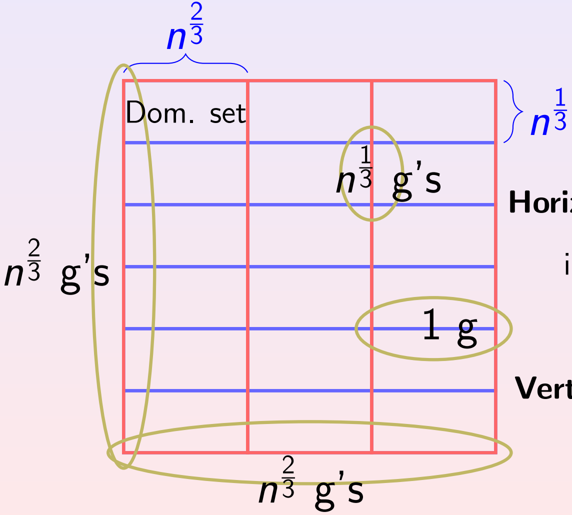


Counting argument leads to result.

Further Work

- Tighten bounds for strong grids.
- All-guards move model is *NP*-hard but unknown if in *NP*.
  - Is it *PSPACE*-complete? *EXPTIME*-complete?
- For all Cayley graphs G obtainable from abelian groups, $\gamma_{all}^{\infty}(G) = \gamma(G)$ [Goddard et al, 2005].
  - Prove $\gamma_{all}^{\infty}(H) = \gamma(H) + o(\gamma(H))$ for truncated Cayley graphs H obtained from abelian groups by generalizing our technique.



Horiz. attack: move only in small rectangle.

Vert. attack: move only in red rectangle.

Thanks!