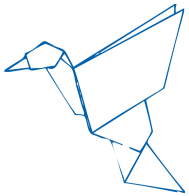
basque center for applied mathematics



Introduction
to
(Piecewise Deterministic) Markov Processes
and
Applications to Neuroscience

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"The true logic of this world is in the calculus of probabilities"
James C. Maxwell

"Faced with such a statement, most people's reaction is to smile nervously and take the author's word for it, while backing nervously away and looking for the exit.
[...]
But the trouble is that Maxwell was right."
John A. Lee (The Lancet - Oncology 2003)

motivations

noise (n.) early 13c., "loud outcry, clamor, shouting," from Old French *noise* "din, disturbance, uproar, brawl" (11c., in modern French only in phrase *chercher noise* "to pick a quarrel")

- there is noise ! e.g. neuronal noise[WS], synaptic noise[W], neuronal noise membrane noise[S] (i.e. stochastic models of ion channel gating)
  - Fatt, Katz (1950). Some observations on biological noise. Nature
  - Faisal, Selen, Wolpert (2008). Noise in the nervous system. Nature reviews. Neuroscience
  - McDonnell, Ward (2011). The benefits of noise in neural systems: bridging theory and experiment. Nat Rev Neurosci
  - Guo, Perc, Liu, Yao (2018). Functional importance of noise in neuronal information processing. EPL (Europhysics Letters)

Link's code color: "[W]" is for wikipedia, "[S]" is for scholarpedia, external links, all clickable.

references in neurosciences I

- monographes including stochastic models
  - Gabbiani, Cox (2010). Mathematics for Neuroscientists
  - Ermentrout, Terman (2010). Mathematical Foundations of Neuroscience
  - Gerstner, Kistler (2002). Spiking neuron models: single neurons, populations, plasticity
  - Gerstner, Kistler, Naud, Paninski (2014). Neuronal Dynamics: From Single Neurons to Networks and Models of Cognition (online)
  - Sterratt, Graham, Gillies, Willshaw (2001). Principles of Computational Modelling in Neuroscience
- monographes with mainly stochastic models
  - Holden (1976). Models of the Stochastic Activity of Neurones
  - Sampath, Srinivasan (1977). Stochastic Models for Spike Trains of Single Neurons
  - Tuckwell (1988). Introduction to Theoretical Neurobiology: Volume 2, Nonlinear and Stochastic Theories
  - Tuckwell (1989). Stochastic Processes in the Neurosciences
  - Greenwood, Ward (2016). Stochastic Neuron Models

references in neurosciences II

- not only stochastic and non mathematical monographes
  - Destexhe, Rudolph-Lilith (2012). Neuronal Noise
- not monographes, but with *very* interesting topics
  - Laing, Lord, eds. (2010). Stochastic Methods in Neuroscience
  - Bachar, Batzel, Ditlevsen, eds. (2013). Stochastic Biomathematical Models with Applications to Neuronal Modeling
- special probability topics in neurosciences
  - point processes: Brown (2005). Theory of point processes for neural systems. In [Chow et al., 2005]
  - PDMP: Bressloff, Maclaurin (2018). Stochastic hybrid systems in cellular neuroscience. The Journal of Mathematical Neuroscience

references in PDMP

- first formal presentation of PDMP
  - Davis (1984). Piecewise-deterministic Markov processes: A general class of non-diffusion stochastic models. Journal of the Royal Statistical Society. Series B (Methodological)
- general
  - Jacobsen (2006). Point process theory and applications: marked point and piecewise deterministic processes
  - Azaïs, Bardet, Génadot, Krell, Zitt (2014). Piecewise deterministic Markov process — Recent results. ESAIM: Proc.
  - Rudnicki, Tyran-Kamin'ska (2015). Piecewise deterministic markov process in biological models. In [Banasiak et al., 2015]
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  - Cloez, Bertrand, Dessalles, Renaud, Genadot, Alexandre, Malrieu, Florent, Marguet, Aline, Yvinec, Romain (2017). Probabilistic and piecewise deterministic models in biology. ESAIM: Procs
  - Yvinec (2015). Piecewise deterministic Markov processes, applications in biology. Laboratoire Jacques-Louis Lions, Université Paris 6

neurosciences/python I

Muller, Bednar, Diesmann, Gewaltig, Hines, Davison, eds. (2015). Python in Neuroscience

- Neo is a package for representing electrophysiology data in Python, together with support for reading a wide range of neurophysiology file formats.
  - have a look at the github repo to see if it's active (it is ☺)
  - Neo is used by a number of other software tools: • SpykeViewer data analysis and visualization • Elephant data analysis • G-node suite databasing • PyNN simulations ...
- Brian 2 spiking neural networks [github]
- TheVirtualBrain (TVB) is a neuroinformatics platform for full brain network simulations using biologically realistic connectivity
- Nengo a graphical and scripting based software package for simulating large-scale spiking and non-spiking neural systems

neurosciences/python II

- PyRates a framework for neural modeling and simulations
- exercises from Dayan, Abbott (2001). Theoretical Neuroscience: Computational And Mathematical Modeling of Neural Systems
- Github repos: neuroscience tutorials, interactive neuron model simulator,
- Github keywords: neuroscience, computational-neuroscience...

plan

plan

Poisson process and Brownian motion

- among continuous-time Markov processes
  - the Poisson process is the archetype of jump processes
  - the Brownian motion is the archetype of diffusion processes
- these two processes are the basic building blocks of jump processes and diffusion processes respectively (and also the simplest processes of their respective classes), and in a way of all continuous-time Markov processes
- if you don't understand the basics facts about these two processes, don't pretend to understand continuous-time Markov processes ☺

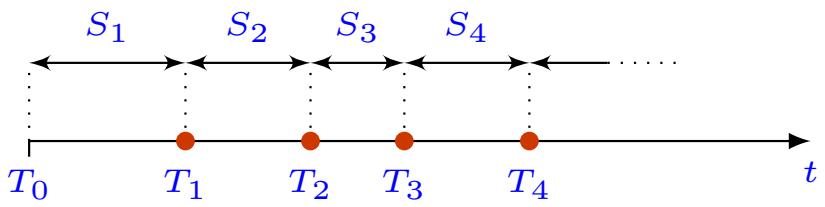
basic ingredients

- for PDMP's:
  - Poisson process
  - ODE
- for diffusion processes:
  - Brownian motion
  - Ito integral and formula
- for general Markov processes:
  - add some "martingale" [Meyer, 2009]
  - and some "linear operator theory" [Ait-Sahalia et al., 2005]

Understanding these ingredients is absolutely necessary to understand these classes of Markov processes.

section plan

point process

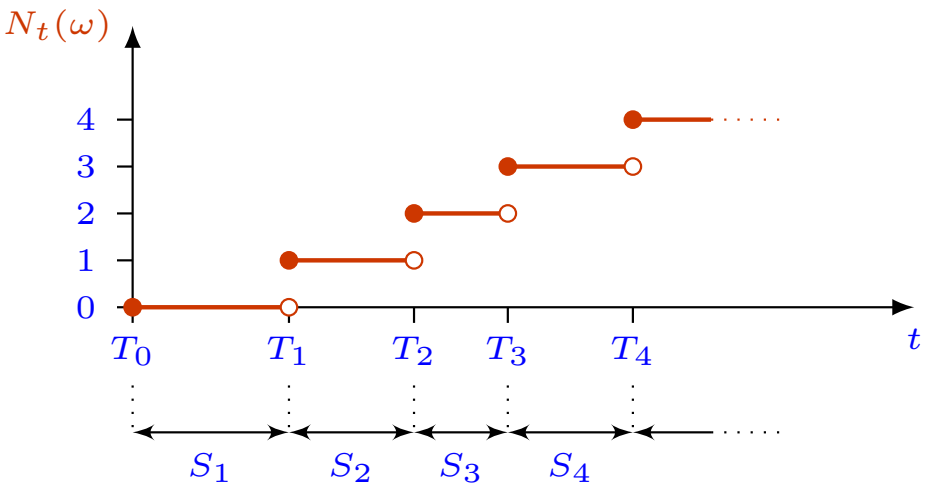


- point process on $\mathbb{R}_+$: strictly increasing sequence of positive random variables

$$0 < T_1 < T_2 < \cdots \qquad \text{(and } T_0 \stackrel{\text{def}}{=} 0\text{)}$$

  - we prefer $T_n \to \infty$, or equivalently $\#(\text{events in } [a,b]) < \infty$ for all $0 \le a < b < \infty$ (regular process, no explosion)
  - inter-event times $S_n = T_n - T_{n-1}$

counting process



- associated counting process, i.e. nb of events in $[0, t]$:

$$N_t \stackrel{\text{def}}{=} \#\{T_n \leq t\} = \max\{n\,;\, T_n \leq t\} = \sum_{n=1}^{\infty} 1_{[0,t]}(T_n)$$

randomness is only in the instants of event or equivalently the inter-events times

Renewal and Poisson processes

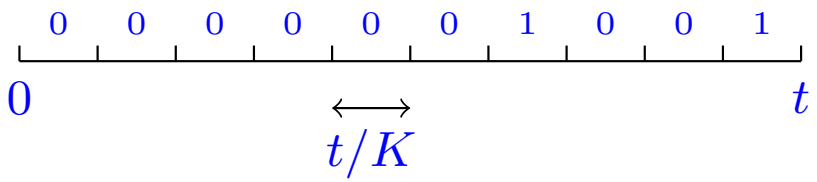
- if S_n i.i.d.: N_t is called a renewal process[w]
- if, moreover, $S_n \sim \mathrm{Exp}(\lambda)$ then N_t is called a Poisson process[w]
  - the first and simplest continuous time Markov process
  - λ is the intensity or rate of the process, i.e. the mean number of events per unit time: $\mathbb{E}(N_t - N_s) = \lambda\,(t-s)$ (for all $0 \leq s \leq t$)

The Poisson process[w] has not been developed by Siméon Poisson[w] and was even proposed before him ☺. It is believed the term *Poisson process* was coined between 1936 and 1939 at the Stockholm University where Willy Feller was working as well as Harald Cramér and and his PhD student Lundberg[w], see [Guttorp and Thorarinsdottir, 2012].

plan

section plan

construction: events in discrete time



- on each sub-interval of length t/K, and independently, we have an event (1) with proba $\lambda\, t/K$ and no event (0) with proba $1 - \lambda\, t/K$, so:
  - $0, \ldots, 0, 1$ with proba $(1 - \lambda\, t/K)^{K-1}\, \lambda\, t/K$
  - $0, \ldots, 0, 0$ with proba $(1 - \lambda\, t/K)^{K}$
- $S \stackrel{\text{def}}{=}$ time of the first event, then the proba for no event on $[0, t]$ is

$$\mathbb{P}(S > t) = \mathbb{P}(0 \cdots 0 \ K \text{ times }) = \left(1 - \frac{\lambda\, t}{K}\right)^{K} \underset{K \to \infty}{\simeq} e^{-\lambda\, t}$$

S follows a geometric distribution can be approximated by an exponential distribution of parameter λ

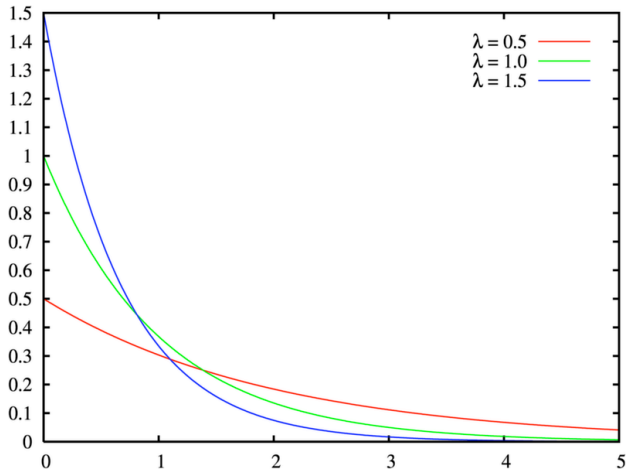
section plan

exponential distribution

- S follows an exponential distribution[w] if it admits the pdf:

$$p(t) = \lambda\, e^{-\lambda\, t}\,, \quad t > 0$$

  - a "natural" distribution defining the duration of a phenomenon or the time occurence of an event
  - $\mathbb{E}(S) = 1/\lambda$, $\mathrm{var}(S) = 1/\lambda^2$
  - λ: rate or intensity (i.e. mean number of events per unit of time)



[plan]

section plan

memoryless property

$$\mathbb{P}(S > s + t | S > t) = \mathbb{P}(S > s) \qquad (\triangle)$$

- the probability that the event will not occur before $t + s$ knowing that it did not occur before t is equal to the probability that it did not occur before s
- central result

  a positive random variable S is exponentially distributed **iff** $(\triangle)$

LLN and CTL

- Normal Approximation to Poisson: we know that $N_t \sim \mathrm{Poisson}(\lambda\, t)$ and for $\lambda\, t$ large

$$\mathrm{Poisson}(\lambda\, t) \simeq N(\lambda\, t, \lambda\, t)$$

  (the approximation is good for $\lambda\, t \geq 10$ and excellent for $\lambda\, t \geq 1000$)

- consider the number of events per unit time

$$\frac{N_t}{t} \qquad \text{or} \qquad \frac{N_t - N_s}{t - s}$$

- we already know that
  - $\mathbb{E} N_t / t = \lambda$
  - $\mathrm{var}(N_t/t) = \lambda\, t / t^2 = \lambda / t$ which tends to 0 as $t \to \infty$
- law of large numbers

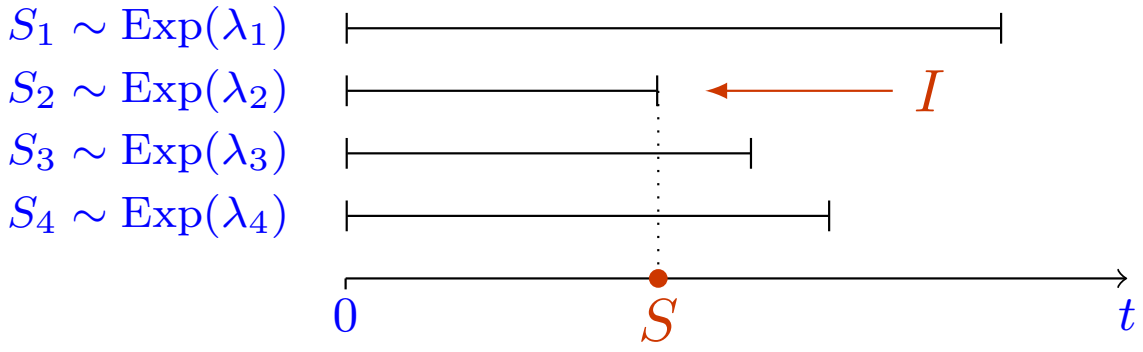
$$\frac{N_t}{t} \xrightarrow[t \to \infty]{\text{a.s.}} \lambda$$

  in fact for $\lambda\, t$ large enough [plan]

superposition I

- Let $S_i \sim \mathrm{Exp}(\lambda_i)$ and independent, $i = 1, \ldots, n$, and:

$$S = \min(S_1, \ldots, S_n) \qquad I = \mathrm{Arg}\min(S_1, \ldots, S_n)$$



then

$$S \perp\!\!\!\perp I \qquad S \sim \mathrm{Exp}(\bar{\lambda}) \qquad \mathbb{P}(I = i) = \lambda_i / \bar{\lambda}$$

with $\bar{\lambda} = \lambda_1 + \cdots + \lambda_n$

superposition II

- first key tool of all the simulation algorithms

  for N events:
  - basic simulation:
    - N simulations of Exp distribution
    - computing a minimum + arg of this minimum among N elements
  - simulation by superposition:
    - sum of N elements
    - 1 simulation of Exp distribution
    - 1 simulation of a categorical distribution[w]

  for N large, the first approach is unbearable

explosion criteria

- $S_1, S_2, \ldots$ independent, $S_n \sim \mathrm{Exp}(\lambda_n)$ with $0 < \lambda_n < \infty$

$$T_\infty \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} S_n \qquad \text{(explosion time)}$$

then there is only two cases:

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} \begin{cases} < \infty & \text{then } \mathbb{P}(T_\infty < \infty) = 1 \quad \text{(explosion)} \\ = \infty & \text{then } \mathbb{P}(T_\infty = \infty) = 1 \quad \text{(no explosion)} \end{cases}$$

- still true when $0 \leq \lambda_n \leq \infty$
- constant case $\lambda_n \equiv \lambda$: no explosion

Swiss Army Knife

- the exponential distribution allows for "natural" computations, eg:
  - at a traffic light: 3 cars and 1 motorcycle per minute (on average)
  - so they are $3 + 1 = 4$ vehicles per minute (on average)
  - next vehicle will arrive in 15s (on average)
  - this vehicle will be a car with proba $\frac{3}{4}$ and a motorcycle with proba $\frac{1}{4}$

plan

section plan

definition

- $(X_t)_{t\geq 0}$: a family of random variables indexed by $t \in \mathbb{R}_+$ (continuous time) and taking value in
  - $\mathbb{N}$ like the Poisson process

    or
  - $\mathbb{R}^d$ like the Brownian motion

    or
  - $\mathcal{M}(\mathbb{R}^d)$ like the empirical measure of N interacting processes taking values in $\mathbb{R}^d$

trajectories

- a trajectory is the function $t \mapsto X_t(\omega)$ for ω given
  - trajectories are supposed càdlàg[w] (right continuous with left limits)
  - $X_{t^-} \stackrel{\text{def}}{=} \lim_{s \uparrow t} X_s$
  - if $X_{t^-} = X_t$ then the trajectory is continuous in t
  - trajectories of the processes considered here live in $\mathcal{D}(\mathbb{R}_+; \mathbb{R}^d)$ or $\mathcal{C}(\mathbb{R}_+; \mathbb{R}^d)$ the spaces of càdlàg or continuous functions from $\mathbb{R}_+$ with values in $\mathbb{R}^d$.

These spaces can be equipped with distances that induce topologies that make that spaces Polish, i.e. complete and separable topological spaces. Separability is a property that makes it possible to avoid the difficulties of measurability of the processes indexed by non-countable sets with values in non-countable spaces. For example for two càdlàg processes X_t et Y_t, the separability of $\mathcal{D}(\mathbb{R}_+; \mathbb{R}^d)$ implies that $\{X_t = Y_t; 0 \leq t \leq T\}$ is mesurable. Completeness is a classical property that drastically simplify the analysis, like the study of the convergence in law of processes (see this discussion).

law I

- the law of the process determines how the process evolves in time
  - if X_t takes values in $\mathbb{N}$, the law is

$$\left\{\mathbb{P}(X_{t_{1:n}} = i_{1:n})\ ;\ n\,,\ 0 \leq t_1 < \cdots < t_n\,,\ i_1, \ldots, i_n \in \mathbb{N}\right\}$$

  - if X_t takes values in $\mathbb{R}^d$, the law is

$$\left\{\mathbb{P}(X_{t_{1:n}} \in B_{1:n})\ ;\ n\,,\ 0 \leq t_1 < \cdots < t_n\,,\ B_1, \ldots, B_n \in \mathcal{B}(\mathbb{R}^d)\right\}$$

where (notation) $X_{t_{1:n}}$ is $(X_{t_1}, \ldots, X_{t_n})$, $X_{t_{1:n}} = i_{1:n}$ means $X_{t_j} = i_j$ $(1 \leq j \leq n)$, $X_{t_{1:n}} \in B_{1:n}$ means $X_{t_j} \in B_j$ $(1 \leq j \leq n)$.

law II

se placer en dehors de cas pathologiques :-)
Kolmogorov extension theorem [w]
Kolmogorov extension theorem [Charpentier et al., 2007]
Kolmogorov's Heritage in Mathematics (book)
xxx pathological Markov processes
[Kendall and Reuter, 1954][Reuter, 1957]; [Doob, 1945, p. 468]:
voir le pb ; aussi [Kendall, 1956]; aussi [Chung, 1956]
XXX [van Casteren, 2011] Markov Processes, Feller Semigroups and Evolution Equations
voir Delphine Salort [Salort, 2017] Some PDE models in neuroscience
url

independence

- independence: $(X_t)_{t\geq 0} \perp\!\!\!\perp (Y_t)_{t\geq 0}$ if $\forall n, m$, $0 \leq t_1 \leq \cdots \leq t_n$, $0 \leq s_1 \leq \cdots \leq s_m$

$$X_{t_{1:n}} \perp\!\!\!\perp Y_{s_{1:m}}$$

i.e. for all $(\alpha_{1:n}, \beta_{1:m}) \in \mathbb{R}^{n+m}$

$$\varphi_{X_{t_{1:n}}, Y_{s_{1:m}}}(\alpha_{1:n}, \beta_{1:m}) = \varphi_{X_{t_{1:n}}}(\alpha_{1:n})\, \varphi_{Y_{s_{1:m}}}(\beta_{1:m})$$

where

$$\varphi_{Y_{s_{1:m}}}(\beta_{1:m}) \stackrel{\text{def}}{=} \mathbb{E} \exp\big(i \sum_k \beta_k\, Y_{s_k}\big) \quad \text{(characteristic function)}$$

increments

- increments of $(X_t)_{t\geq 0}$ is the collection of rv $X_t - X_s$ for all $0 \leq s \leq t$
  - independent increments: for all n , $0 \leq t_0 \leq \cdots \leq t_n$

$$X_{t_1} - X_{t_0}\ ,\ X_{t_2} - X_{t_1}\ ,\ X_{t_n} - X_{t_{n-1}} \text{ are } \perp\!\!\!\perp$$

  - stationary increments:

$$\mathrm{law}(X_{t+s} - X_s) = \mathrm{law}(X_t - X_0) \quad \forall s \geq 0$$

section plan

why Markov is fun

- why do we have LLN and CLT for Markov processes ?
  - because they are built for that ☺
  - a Markov process is (almost by definition) an exemple of non-independent sequence of random variables for which we expect to prove LLN and CLT

origine I

- LLN^{w}: $(X_n)_{n\in\mathbb{N}}$ i.i.d. rv with $\mathbb{E}|X_0| < \infty$,

$$\bar{X}_n \stackrel{\text{def}}{=} \frac{X_1 + \cdots + X_n}{n} \xrightarrow[n\to\infty]{} \mathbb{E}(X_0)$$

- CLT^{w}: $(X_n)_{n\in\mathbb{N}}$ i.i.d. rv with $\mathbb{E}(|X_0|^2) < \infty$,

$$\sqrt{n}\big(\bar{X}_n - \mathbb{E}X_0\big) \xrightarrow[n\to\infty]{\text{law}} N(0, \mathbb{E}(|X_0|^2))$$

origine II

- Prove such results for sequences that are not independent ?
  - LLN Markov (1906). Rasprostranenie zakona bolíshih chisel na velichiny, zavisyaschie drug ot druga (extension of the law of large numbers to dependent quantities). Izvestiya Fiziko-matematicheskogo obschestva pri Kazanskom universitete
  - CLT Markov (1913). Primer statisticheskogo issledovaniya nad tekstom "Evgeniya Onegina", illyustriruyuschij svyazí ispytanij v cepí (an example of statistical study on the text of "Eugene Onegin" illustrating the linking of events to a chain). In Izvestija Imp. Akademii nauk, serija VI, volume 3

  the extension of CLT to dependent sequences of rv's engendered a metaphysical or even theological debate [Basharin et al., 2004], for the history of the CLT see [Fischer, 2011]
  construction : Operator Methods for Continuous-Time Markov Processes

Markov property

- a process X_t is Markov when

$$\mathrm{law}(\text{futur}|\text{past}) = \mathrm{law}(\text{futur}|\text{present})$$

  i.e. to determine the futur behavior of the process knowing all its past, you just have to keep its present in mind
- or given the present: futur and past are independent
- it is an hypothesis on the structure of the law of the process X_t

section plan

case $\mathbb{N}$ I

- if X_t takes values in $\mathbb{N}$

$$\mathbb{P}(X_t = i | X_{t_{1:n}} = i_{1:n}) = \mathbb{P}(X_t = i | X_{t_n} = i_n)$$

for all n, $0 \leq t_1 < \cdots < t_n < t$, $i_1, \ldots, i_n, i$ where

$$\mathbb{P}(X_t = i | X_{t_{1:n}} = i_{1:n}) \stackrel{\text{def}}{=} \frac{\mathbb{P}(X_{t_{1:n}} = i_{1:n}, X_t = i)}{\mathbb{P}(X_{t_{1:n}} = i_{1:n})}$$

- the law of the process is entirely defined by the
  - initial law $\nu(i) = \mathbb{P}(X_0 = i)$
  - transition kernel $P_{i,j}(s,t) = \mathbb{P}(X_t = j | X_s = i)$

$$\begin{aligned}\mathbb{P}(X_{t_{1:n}} = i_{1:n}) &= \sum_{i_0} \mathbb{P}(X_{0:n} = i_{0:n}) \\ &= \sum_{i_0} \nu(i_0)\, P_{i_0,i_1}(0,t_1)\, P_{i_1,i_2}(t_1,t_1) \cdots P_{i_{n-1},i_n}(t_{n-1},t_n)\end{aligned}$$

case $\mathbb{N}$ II

- time-homogeneous: $P_{i,j}(s,t) = P_{i,j}(t-s)$
- $1/2$-group property

$$P(t+s) = P(t)\,P(s)\,, \quad P(0) = I \quad \text{(Chapman-Kolmogorov)}$$

so "$P(t) = e^{t\,Q} = I + t\,Q + \frac{t^2}{2}\,Q^2 + \frac{t^3}{3!}\,Q^3 + \cdots$" where

$$Q_{i,j} \stackrel{\text{def}}{=} \lim_{t \downarrow 0} \frac{P_{i,j}(t) - \delta_{i,j}}{t}$$

see [Anderson, 1991, Norris, 1998]

- hence all the law of the process depends on the initial law ν and a matrix Q (it's a radical simplification of the structure of the law of the process)

case $\mathbb{N}$ III

- Kolmogorov equations

$$\dot{P}(t) = P(t)\,Q\,,\ \forall t \geq 0\,,\ P(0) = I \qquad \text{(forward)}$$

$$\dot{P}(t) = Q\,P(t)\,,\ \forall t \geq 0\,,\ P(0) = I\,. \qquad \text{(backward)}$$

  - the forward/backward qualifiers are easier to understand in the time-inhomogeneous case where the forward (resp. backward) equation is obtained by differentiating $P(s,t)$ wrt t (resp. s)

case $\mathbb{N}$ IV

- for example

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}(X_t = j) = \sum_i \mathbb{P}(X_t = i)\, Q_{ij}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}(\varphi(X_t)) = \mathbb{E}(Q\varphi(X_t))$$

with $Q\varphi(i) = \sum_j Q_{i,j}\, \varphi(j)$

section plan

case $\mathbb{R}^d$ I

- X_t taking values in $\mathbb{R}^d$ is Markov if

$$\mathbb{P}(X_t \in B | X_u \,;\, u \leq s) = \mathbb{P}(X_t \in B | X_s)$$

for all $0 \leq s \leq t$ and $B \subset \mathbb{R}^d$ (Borel) where

$$\mathbb{P}(X_t \in B | X_s) = P_{s,t}(X_s, B)$$

is a transition kernel (i.e. a family of probability measure on $\mathbb{R}^d$ indexed by $\mathbb{R}^d$, i.e. $B \to P_{s,t}(x, B)$ is a probability measure on $\mathbb{R}^d$ for all $x \in \mathbb{R}^d$, for all s, t fixed)

- Notations:

$$\begin{aligned}
&P_{s,t}(x, B) = \int_B P_{s,t}(x, \mathrm{d}x') = \int_{\mathbb{R}^d} 1_B(x')\, P_{s,t}(x, \mathrm{d}x') \\
&\mathbb{E}(\varphi(X_t) | X_0 = x) = \int_{\mathbb{R}^d} \varphi(x')\, P_{s,t}(x, \mathrm{d}x') \\
&\mathbb{E}(\varphi(X_t)) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(x')\, P_{s,t}(x, \mathrm{d}x')\, \nu(\mathrm{d}x)
\end{aligned}$$

with $\nu(B) = \mathbb{P}(X_0 \in B)$ (initial law)

case $\mathbb{R}^d$ II

$$\mathbb{P}(X_{t_{0:n}} \in B_{0:n}) =$$
$$= \int \nu(\mathrm{d}x_0) \int P_{t_0,t_1}(x_0, \mathrm{d}x_1) \cdots \int P_{t_{n-1},t_n}(x_{n-1}, \mathrm{d}x_n)\, 1_{B_{0:n}}(x_{0:n})$$
$$= \underbrace{\int \cdots \int}_{n+1 \text{ times}} 1_{B_{0:n}}(x_{0:n})\, P_{t_{n-1},t_n}(x_{n-1}, \mathrm{d}x_n) \cdots P_{t_0,t_1}(x_0, \mathrm{d}x_1)\, \nu(\mathrm{d}x_0)$$

- time-homogeneous $P_{s,t} = P_{t-s}$
- functional operator $P_t : \varphi \mapsto P_t\varphi$ with

$$P_t\varphi(x) \stackrel{\text{def}}{=} E[\varphi(X_t)|X_0 = x] = \int_{\mathbb{R}^d} \varphi(y)\, P_t(x, \mathrm{d}y)$$

for all $\varphi \in \mathcal{C}_0(\mathbb{R}^d)$ (continuous and null at infinity) (test functions, also called observables in physics)

case $\mathbb{R}^d$ III

- P_t is a semi-group

$$P_{t+s} = P_t\, P_s \quad \forall s,t \quad P_0 = I$$

- so we may think that "$P_t = e^{t\,\mathcal{L}}$" where $\mathcal{L}$ the infinitesimal generator of P_t (or of X_t):

$$\mathcal{L}\varphi(x) \stackrel{\text{def}}{=} \lim_{t\downarrow 0} \frac{P_t\varphi(x) - \varphi(x)}{t} = \lim_{t\downarrow 0} \frac{\mathbb{E}(\varphi(X_t)|X_0 = x) - \varphi(x)}{t}$$

"for all φ for which the previous limite exists"
see [Lant and Thieme, 2007, Davies, 1980] also
[Bakry et al., 2014, Ethier and Kurtz, 1986]
(see discussion)

section plan

infinitesimal generator

- Markov transition kernel

$$P_t(x, A) \stackrel{\text{def}}{=} \mathbb{P}(X_{t+s} \in B | X_s = x)$$

- Markov semigroup $(P_t)_{t \geq 0}$ (we use the same notation)

$$P_t \varphi(x) \stackrel{\text{def}}{=} \mathbb{E}(\varphi(X_t) | X_0 = x) = \int_{\mathbb{R}^d} \varphi(x') \, P_t(x, \mathrm{d}x')$$

a family of operators acting on some suitable function space st

$$P_{t+s} = P_t \circ P_s \,, \ \forall t, s \geq 0 \,, \ T_0 = \mathrm{Id} \qquad \text{(semigroup)}$$

So we can think that "$T_t \varphi = e^{t \mathcal{L} \varphi}$"

It's not easy [Ethier and Kurtz, 1986, Ch. 4] [Kallenberg, 1997, Ch. 17] [Bobrowski, 2005, Ch. 8] [Bakry et al., 2014, Ch. 1] [Eberle, 2017, Eberle, 2019]. Construction of Markov processes from their generators
[Dynkin, 1965, Ethier and Kurtz, 1986, Stroock and Varadhan, 1979, Rogers and Williams, 2000]

Real semigroup I

- finding all the maps $T : \mathbb{R}_+ \mapsto \mathbb{R}$ satisfying the Cauchy functional equation:

$$T(t+s) = T(t)\,T(s)\,,\ \ s,t \in \mathbb{R}_+\,,\ \ T(0) = 1 \qquad (\Delta)$$

  (usually written in a linear form $f(x+y) = f(x) + f(y)$, but $f(x) = \log(T(x))$)
- $T(t) = e^{\alpha t}$ is solution of (Δ) for all $\alpha \in \mathbb{R}$
- Reciprocally, one can prove that if $T(t)$ satisfies any of the conditions:
  - T right-continuous
  - T right-continuous in 0 or in any other point
  - T monotonic on an interval of positive length
  - T integrable
  - T Lebesgue measurable.

  then $T(t) = e^{\alpha t}$ for some $\alpha \in \mathbb{R}$.

Real semigroup II

- concept of Feller Markov semigroup/process (the nice one)
- [Norris, 1998, p. 70] gives a simple proof in the case of monotonicity, [Aczél and Dhombres, 1989, Ch. 2] give a detailed presentation.
- Exhibiting a discontinuous function that satisfies (Δ) is not very simple as it is nowhere continuous and even not measurable !

  See [Kharazishvili, 2018, p. 179] for a description of such functions, the proof is based upon Hamel's basis named after Georg Hamel[w] who studied discontinuous solutions of (Δ) with the help of the axiom of choice[w] of Ernst Zermelo[w] [Hamel, 1905] (Hamel was less inspired when he evoked the spiritual bond between mathematics and the "Third Reich", see his MacTutor biography).

section plan

ODE ☺

$$\dot{x}(t) = f(x(t)) \qquad (\Delta)$$

- flow: $\Phi(t,x)$ solution of (Δ) at time t and starting from x ie

$$\partial_t \Phi(t,x) = f(\Phi(t,x))\,,\ t \geq 0\,,\ \Phi(0,x) = x$$

- it's a Markov process with transition kernel

$$P_t(x,B) = 1_B(\Phi(t,x)) \quad \text{i.e. } P_t(x,\mathrm{d}x') = \delta_{\Phi(t,x)}(\mathrm{d}x')$$

and generator $\mathcal{L} : f \mapsto \mathcal{L}\varphi$ with

$$\mathcal{L}\varphi(x) = \lim_{t\downarrow 0} \frac{P_t\varphi(x) - \varphi(x)}{t} = \lim_{t\downarrow 0} \frac{\mathbb{E}(\varphi(x(t))) - \varphi(x)}{t}$$

$$= \lim_{t\downarrow 0} \frac{\varphi(x(t)) - \varphi(x)}{t} = f(x)\cdot\nabla\varphi(x) \qquad (f(x)\,\varphi'(x) \text{ in dim 1})$$

- the Newtonian mechanics is Markovian
- ODE is a sub-branch of Markov processes ☺

Poisson process

- as $N_t - N_s \sim \mathrm{Poisson}(\lambda\,(t-s))$, the transition kernel is

$$P_t(i,j) = \mathbb{P}(N_{t+s} = j | N_s = i) = e^{-\lambda\,t}\,\frac{(\lambda\,t)^{j-i}}{(j-i)!}$$

  so for t small:

$$P_t(i,j) = \begin{cases} \lambda\,t + o(t) & \text{if } j = i+1 \\ 1 - \lambda\,t + o(t) & \text{if } j = i \\ o(t) & \text{if } j > i+1 \end{cases}$$

  hence $P_t(i,j) - \delta_{i,j} = \lambda\,t$ if $j = i+1$, $-\lambda\,t$ if $j = i$, 0 other cases
- generator

$$Q = \begin{pmatrix} -\lambda & \lambda & & \\ & -\lambda & \lambda & \\ & & \ddots & \ddots \\ & & & \end{pmatrix}$$

  or $Q\varphi(x) = \lambda\,\big(\varphi(x+1) - \varphi(x)\big) \qquad x \in \mathbb{N}$

Brownian motion (Wiener process) in $\mathbb{R}$ I

- $P_t(x, \mathrm{d}x')$ Gaussian distribution of mean x and variance t, ie

$$P_t(x, B) = \frac{1}{\sqrt{2\pi\, t}} \int_B \exp\left(-\frac{(x'-x)^2}{2\, t}\right) \mathrm{d}x'$$

- infinitesimal generator

$$\mathcal{L}\varphi(x) = \frac{1}{2}\triangle\varphi(x) \quad \forall \varphi \in \mathcal{C}_0^2(\mathbb{R}) \qquad \text{(Laplacian)}$$

- B_t has independent increments and $B_t - B_s \sim N(0, t-s)$

$$\mathbb{E}\big(\varphi(B_t)|B_0 = x\big) \simeq \mathbb{E}\big(\varphi(x) + \varphi'(x)\,(B_t - x) + \frac{1}{2}\varphi''(x)\,(B_t - x)^2|B_0 = x\big)$$

$$= \varphi(x) + \varphi'(x)\,\mathbb{E}(B_t - x|B_0 = x) + \frac{1}{2}\varphi''(x)\,\mathbb{E}[(B_t - x)^2|B_0 = x]$$

$$= \varphi(x) + \frac{1}{2}\varphi''(x)\,t$$

Brownian motion (Wiener process) in $\mathbb{R}$ II

- Fokker Planck (weak form)

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}(\varphi(B_t)) = \tfrac{1}{2}\mathbb{E}\varphi''(B_t)$$

if $p_t(x)$ is the density of B_t, then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int \varphi(x)\,p_t(x)\,\mathrm{d}x &= \frac{1}{2}\int \varphi''(x)\,p_t(x)\,\mathrm{d}x \\ &= -\frac{1}{2}\int \varphi'(x)\,p_t'(x)\,\mathrm{d}x = \frac{1}{2}\int \varphi(x)\,p_t''(x)\,\mathrm{d}x\end{aligned}$$

for all test function φ hence we get the:

- Fokker Planck (strong form)

$$\frac{\partial}{\partial t}p_t(x) = \frac{1}{2}\,\Delta p_t(x) = \mathcal{L}^* p_t(x)$$

section plan

continuous processes

- Bownian motion:

$$\mathcal{L}\varphi(x) = \tfrac{1}{2}\Delta\varphi(x),\ \varphi \in \mathcal{C}_0^2(\mathbb{R}^d)$$

- diffusion process:

$$\mathrm{d}X_t = f(X_t)\,\mathrm{d}t + g(X_t)\,\mathrm{d}B_t$$
$$\mathcal{L}\varphi(x) = f(x)\cdot\nabla\varphi(x) + \tfrac{1}{2}\textstyle\sum_{i,j}[g(x)\,g(x)^*]_{i,j}\partial^2_{x_i,x_j}\varphi(x),$$
$$\varphi \in \mathcal{C}_0^2(\mathbb{R}^d)$$

- ODE

$$\dot{X}_t = f(X_t)$$
$$\mathcal{L}\varphi(x) = f(x)\cdot\nabla\varphi(x),\ \varphi \in \mathcal{C}_0^1(\mathbb{R}^d)$$

pure jump processes I

- Poisson process:

$$\mathcal{L}\varphi(x) = \lambda\,[\varphi(x+1) - \varphi(x)]$$
$$\varphi \in \mathcal{C}_0^1(\mathbb{R})$$

- linear birth and death process:

$$\mathcal{L}\varphi(x) = \lambda\,x\,[\varphi(x+1) - \varphi(x)] + \mu\,x\,[\varphi(x-1) - \varphi(x)] \qquad \varphi \in \mathcal{C}_0^1(\mathbb{R})$$

- linear birth and death process with immigration:

$$\mathcal{L}\varphi(x) = \lambda\,x\,[\varphi(x+1) - \varphi(x)] + \mu\,x\,[\varphi(x-1) - \varphi(x)] + \alpha\,[\varphi(x+1) - \varphi(x)] \qquad \varphi \in \mathcal{C}_0^1(\mathbb{R})$$

- logistic process:

$$\mathcal{L}\varphi(x) = (\lambda\,x)\,[\varphi(x+1) - \varphi(x)] + (\mu\,x + \varepsilon\,x^2)\,[\varphi(x-1) - \varphi(x)] \qquad \varphi \in \mathcal{C}_0^1(\mathbb{R})$$

pure jump processes II

- general pure jump process:

$$\mathcal{L}\varphi(x) = \lambda(x) \int_{\mathbb{R}_d} \big(\varphi(y) - \varphi(x)\big)\, \rho(x, \mathrm{d}y)$$
$$\varphi \in \mathcal{C}_0^1(\mathbb{R}^d)$$

section plan

Poisson process

- generator

$$\mathcal{L}\varphi(x) = \lambda\,\big(\varphi(x+1) - \varphi(x)\big)$$

- Taylor expansion

$$\varphi(x+1) \simeq \varphi(x) + 1\,\varphi'(x) + \frac{1}{2}\,1^2\,\varphi''(x)$$

- generator approximation

$$\mathcal{L}\varphi(x) = \lambda\,\big(\varphi(x+1) - \varphi(x)\big) \simeq \lambda\,\varphi'(x) + \frac{1}{2}\,\lambda\,\varphi''(x)$$

which corresponds to the following diffusion process

$$\mathrm{d}X_t = \lambda\,\mathrm{d}t + \sqrt{\lambda}\,\mathrm{d}B_t$$

i.e. $\dot{X}_t = \lambda + \sqrt{\lambda}$ "white noise"

linear birth and death

$$\mathcal{L}\varphi(x) = \lambda\, x\, [\varphi(x+1) - \varphi(x)] + \mu\, x\, [\varphi(x-1) - \varphi(x)]$$

$$\begin{aligned}\mathcal{L}\varphi(x) &\simeq \lambda\, x\, [\varphi'(x) + \tfrac{1}{2}\, \varphi''(x)] + \mu\, x\, [(-1)\, \varphi'(x) + \tfrac{1}{2}\, (-1)^2\, \varphi''(x)] \\ &\simeq (\lambda - \mu)\, x\, \varphi'(x) + \tfrac{1}{2}\, (\lambda + \mu)\, x\, \varphi''(x)\end{aligned}$$

$$\mathrm{d}X_t = (\lambda - \mu)\, X_t\, \mathrm{d}t + \sqrt{(\lambda + \mu)\, X_t}\, \mathrm{d}B_t$$

ie

$$\dot{X}_t = (\lambda - \mu)\, X_t + \sqrt{(\lambda + \mu)\, X_t} \text{ “white noise”}$$

the ODE model account only for $\lambda - \mu$, the stochastic model accounts for $\lambda - \mu$ and $\lambda + \mu$

piecewise deterministic Markov process (PDMP)

- piecewise deterministic Markov process (PDMP):

$$\mathcal{L}\varphi(x) = f(x) \cdot \nabla\varphi(x) + \lambda(x) \int_{\mathbb{R}_d} \big(\varphi(y) - \varphi(x)\big)\, \rho(x, \mathrm{d}y),$$
$$\varphi \in \mathcal{C}_0^1(\mathbb{R}^d)$$

- exemple

$$\mathcal{L}\varphi(x) = -\alpha\, \varphi'(x) + \lambda\, [\varphi(x+1) - \varphi(x)]$$

  - what is that ?
  - how to simulate that ? (simple ☺)

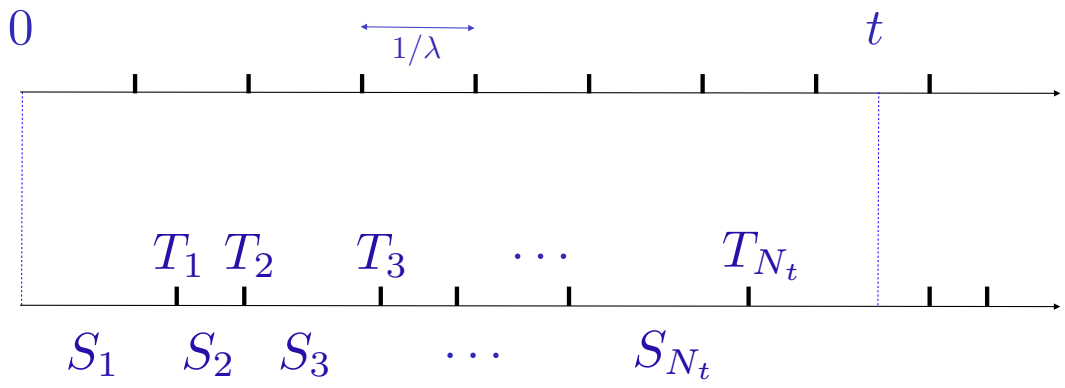
plan

principle

- events occur at rate λ
  - cars at a crossroad
  - customers at a counter
  - capture of animals
  - immigration in an ecosystem
  - spikes
  - bursts
  - ...

section plan

deterministic model/stochastic model I



- λ: nb of events per unit of time (frequency $1/\lambda$)
- deterministic model:
  - nb of events at t: $\lambda\, t$ (or $\lfloor \lambda\, t \rfloor$)
- stochastic model:
  - nb of events at t: $N_t = \sum_{n=1}^{\infty} 1_{\{T_n \leq t\}}$
  - we want $\mathbb{E}(S_n) = \frac{1}{\lambda}$

  $\rightarrow$ Poisson process

Poisson process

- a Poisson process with intensity λ is a couting process $(N_t)_{t\geq 0}$ with S_n i.i.d. $\mathrm{Exp}(\lambda)$
- intensity λ is the mean rate event as the mean time between two events is $\mathbb{E}(S_n) = 1/\lambda$.

Markov property of the Poisson process

- $(N_t)_{t\geq 0}$ Poisson process of intensity λ
- $s > 0$ given, consider the process

$$\tilde{N}_t \stackrel{\text{def}}{=} N_{t+s} - N_s$$

then $(\tilde{N}_t)_{t\geq 0}$ is also a Poisson process of intensity λ independent of $(N_r)_{0\leq r\leq s}$.

ma doc I

ddddd
voir [Buckwar and Riedler, 2011]
voir [Pitman, 2003]
faisable [Anderson et al., 2014] ?
voir [Caumel, 2011] !!! il fait bien les mesures de poisson et les processus de renouvellement
hawkes [Laub et al., 2015]
voir [Cadre, 2014]
voir [CLOEZ, 2011]

ma doc II

hawkes: general [Zhu, 2013], in neurosciences [Chevallier, 2016] neuroscience, see Chornoboy et al. [25], – [Johnson, 1996] used various point processes, including Poisson process, renewal process and the Hawkes process and showed that neural discharges patterns convey time-varying information intermingled with the neuron's response characteristics. —
[Pernice et al., 2011, Pernice et al., 2012] have used Hawkes process to model the spike train dynamics in the studies of neuronal networks — Hawkes' point process theory allows the treatment of correlations on the level of spike trains as well as the understanding of the relation of complex connectivity patterns to the statistics of pairwise cor- relations. — stats : [Reynaud-Bouret et al., 2014] showed that the homogeneous Poisson process hypothesis is always re- jected and that the inhomogeneous Poisson process hypothesis is rarely accepted.

ma doc III

(pour moi : [Eberle, 2017] fait TOUT ???)
faut reprendre [Ross, 2010] qui généraliser Poisson
[Stannat, 2017] fait des choses biens, notamment en comportement long, decomposition de martingale etc
what could you read from : [Tuckwell, 1981] ???? Nothing ?
sympa [Keeler, 2016]
nonhomogeneous [Pasupathy, 2010] (provient de [Lewis and Shedler, 1979])

section plan

what is a law ?

c'est quoi une loi ? markov : c'est la loi ?
Un processus $(X_t)_{t\geq 0}$ à valeurs dans $\mathbb{N}$ est appelé processus de Markov lorsqu'il vérifie:

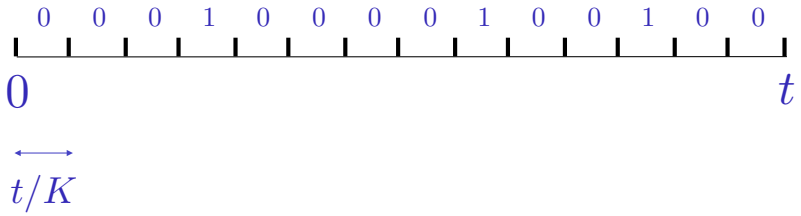
$$\mathbb{P}(X_{t_{n+1}} = i_{n+1} | X_{t_n} = i_n, \ldots, X_{t_0} = i_0) = \mathbb{P}(X_{t_{n+1}} = i_{n+1} | X_{t_n} = i_n)$$

pour tout $n \geq 1$, $0 \leq t_0 \leq t_1 \leq \cdots \leq t_{n+1}$, $i_0, \ldots, i_{n+1} \in \mathbb{N}$. On peut aussi noter cette propriété de la façon suivante:

$$\mathrm{law}(X_t | X_r,\, 0 \leq r \leq s) = \mathrm{law}(X_t | X_s)\,,\ \forall 0 \leq s \leq t\,.$$

Proposition ?? implique propriété de Markov.

number of events I



- binomial distribution with the discretization of $[0, t]$: the probability that k events occur among K Bernoulli trials is:

$$\mathbb{P}(N(t) = k) \simeq \binom{K}{k} (\lambda \, \tfrac{t}{K})^k \, (1 - \lambda \, \tfrac{t}{K})^{K-k}$$

  i.e. choose k places among K and in each place draw 1 with proba $\lambda t/K$ and 0 with proba $1 - \lambda t/K$
- letting $K \to \infty$, we get [Feller, 1968, Ch. VI]:

$$\mathbb{P}(N(t) = k) = \frac{(\lambda \, t)^k}{k!} \, e^{-\lambda \, t}$$

  Poisson distribution $\mathcal{P}(\lambda \, t)$ with parameter $\lambda \, t$, $\mathbb{E}(N(t)) = \mathrm{var}(N(t)) = \lambda \, t$

[plan]

number of events II

- mean & variance

$$\mathbb{E}\big(N_{t+\delta} - N_t\big) = \delta\,\lambda \qquad \mathbb{E}\big((N_{t+\delta} - N_t)^2\big) = \delta\,\lambda$$

- variability in the number of spikes / the mean number of spikes

$$F \stackrel{\text{def}}{=} \frac{\text{variance on an interval}}{\text{mean on an interval}} = 1 \qquad \text{(Fano factor}^{\text{w}}\text{)}$$

(remember: the mean is **never** enough)
[Gabbiani and Cox, 2010, Ch. 15]

section plan

superposition

- $k \perp\!\!\!\perp$ Poisson processes $(N_t^\ell)_{t\geq 0}$, $1 \leq \ell \leq k$
- $(N_t^\ell)_{t\geq 0}$ intensity λ_ℓ,
- the superposition process

$$N_t \stackrel{\text{def}}{=} N_t^1 + \cdots + N_t^k$$

  - N_t is a Poisson processes $\lambda \stackrel{\text{def}}{=} \lambda_1 + \cdots + \lambda_k$
  - $\mathrm{law}(N_t^\ell | N_t = n) = \mathrm{Binom}(n, \lambda_\ell/\lambda)$ ([w])

decomposition

- $(N_t)_{t\geq 0}$ Poisson λ
- each event is labeled (independently from N_t)
  - type 1 with probability p
  - type 2 with probability $q = 1 - p$
- let $N_t^i \stackrel{\text{def}}{=} \#$events of type i in $[0, t]$, $i = 1, 2$
- then
  - N_t^1 (resp. N_t^2) is Poisson of intensity $p\,\lambda$ (resp. $q\,\lambda$)
  - $N_t^1 \perp\!\!\!\perp N_t^2$
- It's obviously what we want !

  If at a crossroads passes 3 cars and 1 motorcycle on average per minute, then 4 vehicles passes on average per minute. Conversely, if 4 vehicles pass per minute and there is a probability of $\frac{3}{4}$ whether it is a car and a probability $\frac{1}{4}$ that it is a motorcycle, then on average 3 cars and 1 motorcycle pass through this crossroads every minute ☺

increments

$(N_t)_{t\geq 0}$ a counting is a λ-Poisson process iff:

- **Independent increments**:
  $N_{t_1} - N_0,\, N_{t_2} - N_{t_1}, \ldots, N_{t_n} - N_{t_{n-1}}$ are independent for all n and $0 < t_1 < \cdots < t_n$
- **Poissonian increments**:
  $N_{t+s} - N_s \sim \mathrm{Poisson}(\lambda\, t)$ for all $t, s > 0$

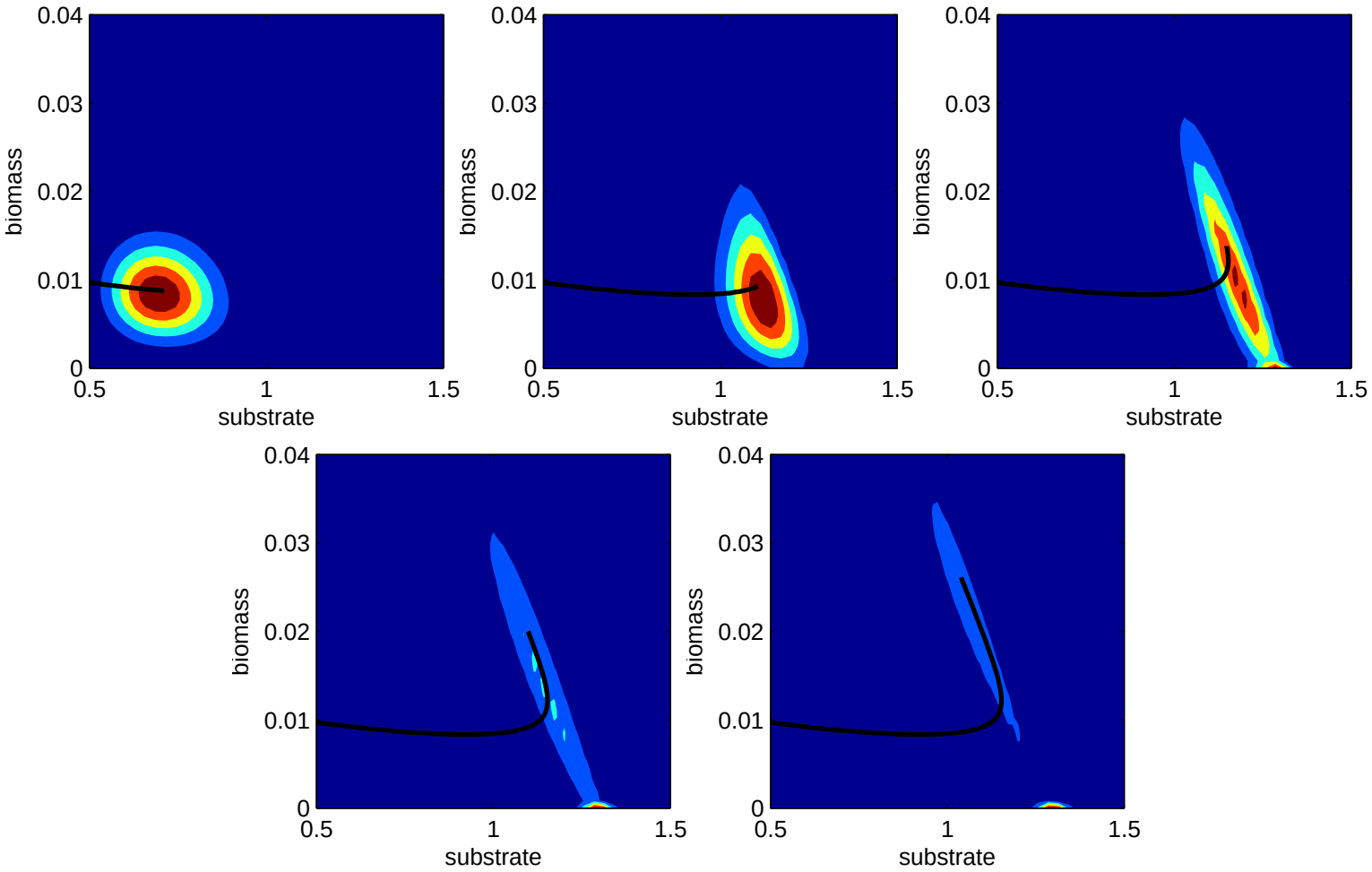
section plan

computing law vs trajectories simulations I

- to "understand" a Markov process we can
  - solve numerically the Kolmogorov or Fokker-Planck equations
  - simulate trajectories (Monte Carlo)
- Lagrangian vs Eulerian
  - "Lagrangian specification of the flow field is a way of looking at fluid motion where the observer follows an individual fluid parcel as it moves through space and time."[w]
  - "The Eulerian specification of the flow field is a way of looking at fluid motion that focuses on specific locations in the space through which the fluid flows as time passes."[w]

computing law vs trajectories simulations II

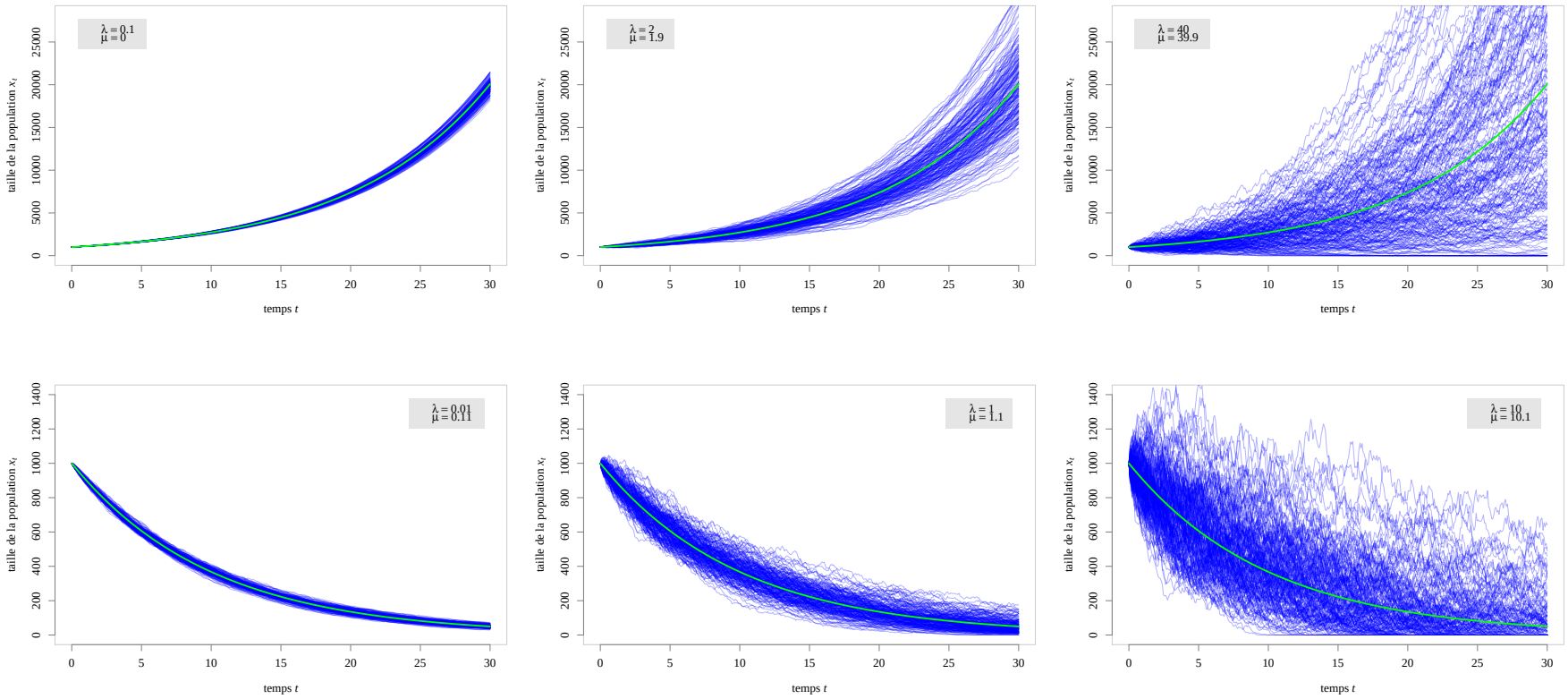
- Fokker Planck in 2D



- Curse of dimensionality [plan]

computing law vs trajectories simulations III

- Monte Carlo simulation of a birth and death pure jump process



simulation I

How to simulate a trajectory $t \to N(t)$ *?*

- method 1: sampling

  $t = 0$, T given, n given
  $\delta = T/n$
  $N = 0$
  save (t, N)
  while $t < T$ **do**
  $\quad \bar{N} \sim \mathrm{Poisson}(\lambda\,\delta)$
  $\quad t \leftarrow t + \delta$
  $\quad N \leftarrow N + \bar{N}$
  $\quad$ save (t, N)
  end while

  - it's not an approximation (even if the trajectories doesn't seem like Poisson trajectories: it jumps only at δt's)
  - generalizable in all cases

simulation II

- method 2: simulation of the waiting times

  $t = 0$, $N = 0$
  enregistrer (t, N)
  while $t < T$ **do**
  $\Delta \sim \mathcal{E}(\lambda)$
  $t \leftarrow t + \Delta$
  $N \leftarrow N + 1$
  enregistrer (t, N)
  end while

  - exact
  - generalizable to some extend

simulation III

- method 3: specific to the Poisson process

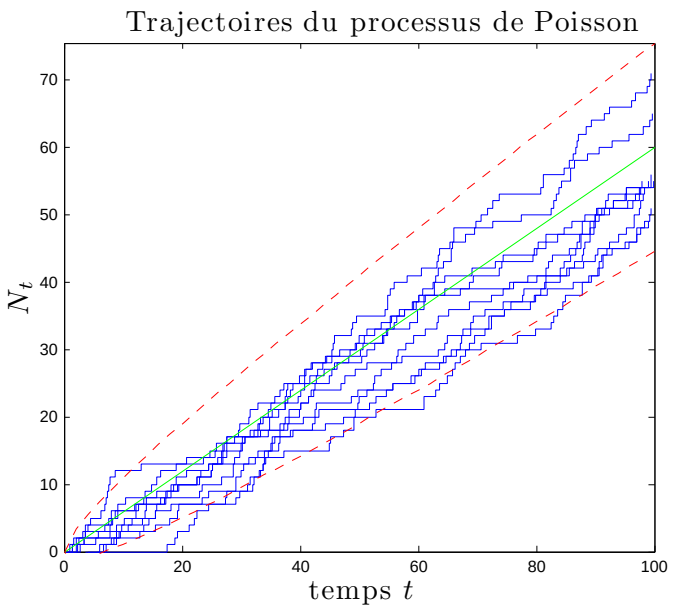
  T given
  $k \sim \mathcal{P}(\lambda T)$
  $U_1, \ldots, U_k \overset{\text{iid}}{\sim} U[0, T]$
  $(U_1, \ldots, U_k) \leftarrow \text{sort}(U_1, \ldots, U_k)$

  - exact
  - very efficient
  - not generalizable
  - non recursive

  [Norris, 1998, p. 78]

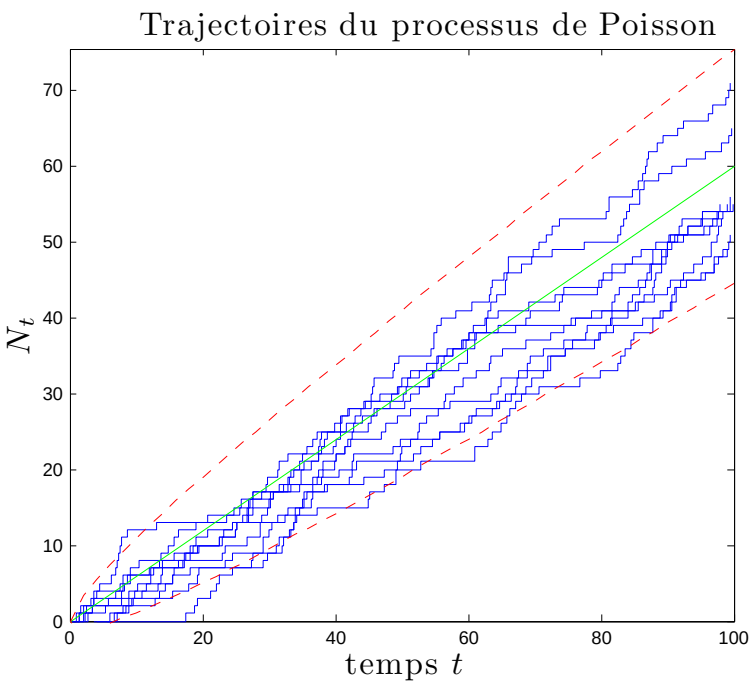
simulation IV

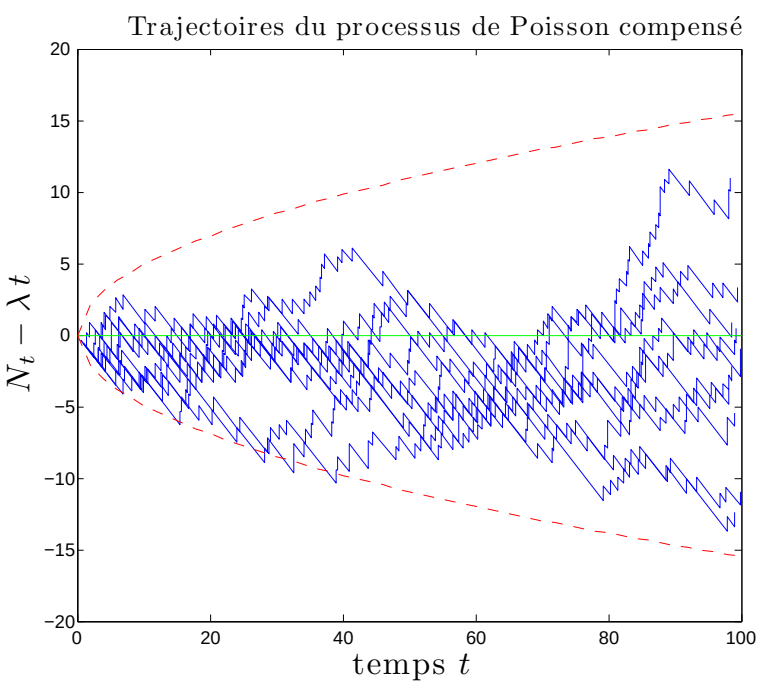
```
lambda = 0.6;  % taux
Tmax   = 100;  % instant final
nb_mc  = 10 ;  % nb de simu de Monte Carlo
for n=1:nb_mc
    nb_evenements = random('Poisson',lambda*Tmax) ;
    instants      = random('Uniform',0,Tmax,1,nb_evenements);
    instants      = sort(instants);
    stairs(instants,(0:length(instants)-1));
end
```



section plan

decomposition I





$$N_t = \underbrace{\mathbb{E}(N_t)}_{\lambda\, t} + \sqrt{\lambda}\, \underbrace{\frac{N_t - \mathbb{E}(N_t)}{\sqrt{\lambda}}}_{\mathcal{M}_t} = \lambda\, t + \sqrt{\lambda}\, \mathcal{M}_t$$

decomposition II

$$N_t = \lambda\, t + \sqrt{\lambda}\, \mathcal{M}_t$$

- drift: $\lambda\, t$
- martingal: $\mathbb{E}(\mathcal{M}_t) = 0$, $\mathrm{var}(\mathcal{M}_t) = t$
- "the drift $\beta_t = \dot{\mathcal{M}}_t$" is a noise

$$\text{“}\dot{N}_t = \lambda + \sqrt{\lambda} \times \beta_t\text{”}$$

- the drift contains information about λ
  but the noise intensity too

decomposition III

- in stochastic we like to write:

$$\mathrm{d}N_t = \lambda\,\mathrm{d}t + \sqrt{\lambda}\,\mathrm{d}\mathcal{M}_t$$

(to be sure that nobody will understand ? ☺) for :

$$N_t = N_0 + \int_0^t \lambda\,\mathrm{d}s + \int_0^t \sqrt{\lambda}\,\mathrm{d}\mathcal{M}_s = N_0 + \lambda\,t + \sqrt{\lambda}\,\mathcal{M}_t$$

(in stochastic we usually write N_t instead of $N(t)$, it's cultural ☺)

- N_t is not a "noise", $\mathcal{M}_t$ is

section plan

rescaling I

- number of events: N_t, $t \in [0, T]$
  $\sim$frequency of events: $X_T(t) = N_{t\,T}/T$, $t \in [0, 1]$
- for a given range of $X_T(t)$ there is a difference whether T is large or not
- rescaling: to make $T \to \infty$ the rate λ should depend on T (unless $X_T(t) \to 0$), we replace λ by $\lambda\,T$:

$$X_T(t) = \frac{\lambda\,T}{T}\,t + \frac{\sqrt{\lambda\,T}}{T}\,\mathcal{M}_T(t) = \lambda\,t + \frac{1}{\sqrt{T}}\sqrt{\lambda}\,\mathcal{M}_T(t)$$

rescaling II

$$X_T(t) = x(t) + \frac{1}{\sqrt{T}}\sqrt{\lambda}\,\mathcal{M}_T(t) \quad \text{avec} \quad x(t) = \lambda\, t$$

as $T \to \infty$:

$$X_T(t) \to x(t) \qquad \text{(law of large numbers)}$$

$$\sqrt{T}\,(X_T(t) - x(t)) \to \sqrt{\lambda}\,W(t) \qquad \text{(central limit theorem)}$$

- $W(t)$ is the standard Brownian Motion/Wiener process
  - $W(t)$ continuous process, $W(0) = 0$
  - independent increments $W(t) - W(s) \perp\!\!\!\perp W(s)\,,\ s < t$
  - Gaussian increments $W(t) - W(s) \sim N(0, t-s)$

Donsker invariance principle (1951) I

ubiquity of Gaussian distibution (static) and of the Brownian motion (dynamic):

- $X_1, X_2, X_3, \ldots$ i.i.d. centered with variance 1
- random walk: $S_n \stackrel{\text{def}}{=} \sum_{i=1}^n X_i$:

$$\frac{S_n}{\sqrt{n}} \xrightarrow[n\to\infty]{\text{law}} N(0,1) \qquad \text{(CLT)}$$

- random walk after diffusive rescaling:

$$\left(\frac{S_{\lfloor nt \rfloor}}{\sqrt{n}}\right)_{t\in[0,1]} \xrightarrow[n\to\infty]{\text{law}} (W(t))_{t\in[0,1]} \qquad \text{(Donsker)}$$

functional CLT (in Skorokhod space^{w}):

section plan

Poisson random measure in $\mathbb{R}$ I

We can view the Poisson process as a "random measure" on $\mathbb{R}_+$:

$$N_t = N([0,t])$$

or

$$N(B) = \#(k \,:\, T_k \in B)\,, \quad \forall B \text{ Borel set in } \mathbb{R}$$

Poisson random measure in $\mathbb{R}$ II

write it $N(\mathrm{d}s)$ or $\mathrm{d}N(s)$ (it's the same)
we can use $N(\mathrm{d}s)$ as an integrator, with φ continuous:

$$\int_0^t \varphi(s)\, N(\mathrm{d}s) = \sum_k \varphi(T_k)\, 1_{[0,t]}(T_k)$$

$$\int_B \varphi(s)\, N(\mathrm{d}s) = \sum_k \varphi(T_k)\, 1_B(T_k)$$

Poisson random measure in $\mathbb{R}$ III

$N(\mathrm{d}s)$ is a random measure:

$$\mathbb{E}\int_0^t \varphi(s)\, N(\mathrm{d}s) = \int_0^t \varphi(s)\, \lambda\, \mathrm{d}s$$

$\lambda\, \mathrm{d}s$ is it intensity measure
we can consider a Poisson random measure $N(\mathrm{d}s)$ with intensity measure $\lambda(s)\, \mathrm{d}s$

$$\mathbb{E}\int_0^t \varphi(s)\, N(\mathrm{d}s) = \int_0^t \varphi(s)\, \lambda(s)\, \mathrm{d}s$$

Poisson measure in $\mathbb{R}^n$

- spatial Poisson processes make it possible to account for uniform distributions of points in unbounded spaces like $\mathbb{R}^n$
- example: distribution of stars in certain regions of space, plants and animals of a given species in a given area
- homogeneous spatial Poisson process of intensity λ
  - for all Borelian B: $N(B)$ is a random variable of law $\mathrm{Poisson}(\lambda\,|B|)$ where $|B|$ is the volume of B
  - if B_i are disjoints then $N(B_i)$ are $\perp\!\!\!\perp$

$$\int_0^t \int_0^s \varphi(u,v)\, N(\mathrm{d}u, \mathrm{d}v) = \sum_{k=1}^{K} \varphi(\xi_k)\, 1_{[0,t]\times[0,s]}(\xi_k)$$

where $K \sim \mathrm{Poisson}(\lambda\, s\, t)$ and $\xi_k \overset{\text{iid}}{\sim} U([0,t]\times[0,s])$

section plan

drunkard's walk

- $X_t \in \mathbb{Z}$ starting in n (or any other point), it stays there according to $\mathrm{Exp}(\lambda)$ and then jump to $n \pm 1$ with probabilitu $1/2$ and $1/2$, so:

$$\begin{aligned}\mathcal{L}\varphi(x) &= \frac{\lambda}{2}\left[\varphi(x+1) - \varphi(x)\right] + \frac{\lambda}{2}\left[\varphi(x-1) - \varphi(x)\right] \\ &= \frac{\lambda}{2}\left[\varphi(x+1) + \varphi(x-1) - 2\,\varphi(x)\right]\end{aligned}$$

- diffusion Approximation

$$\mathcal{L}\varphi(x) \simeq \tfrac{\lambda}{2}\left[\varphi'(x) + \tfrac{1}{2}\,\varphi''(x)\right] + \tfrac{\lambda}{2}\left[-\varphi'(x) + \tfrac{1}{2}\,\varphi''(x)\right] = \tfrac{\lambda}{2}\,\Delta\varphi(x)$$

$$X_t = \sqrt{\lambda}\,B_t$$

(Standard Brownian motion with variance λ)

section plan

compound Poisson process I

$$X_t = \sum_{i=1}^{N_t} Y_i$$

- N_t Poisson process with intensity λ
- Y_i i.i.d. discrete
- $N_t \perp\!\!\!\perp Y_i$

burst model
[Brown, 2005]
[Muller et al., 2015]

renewal process I

- renewal process:

$$N_t \stackrel{\text{def}}{=} \max\{n\,;\, T_n \leq t\} = \sum_{n=1}^{\infty} 1_{[0,t]}(T_n)$$

  - $(S_n)_{n\geq 1}$ positive and i.i.d.
  - cumulative distribution function $F(s) = \mathbb{P}(S_n \leq s)$
  - $T_n = S_1 + \cdots + S_n$

- N_t is Markov iff it is a Poisson process iff $F = \mathrm{Exp}(\lambda)$

renewal process II

see [Sughiyama and Kobayashi, 2018]
xxx [Daley and Vere-Jones, 2003]: generalization of Poisson process
[Daley and Vere-Jones, 2003, p. 13]: On the queueing theory side, a paper of fundamental importance is Con- nie Palm's (1943) study of intensity fluctuations in traffic theory [...]. In Palm's terminology, the Poisson process is characterized by the property that every instant is a regeneration point, whereas for a general renewal process only those instants at which a new interval is started form regeneration points. Hence, he called a Poisson process a process without aftereffects and a renewal process a process with limited aftereffects.
xxx Poisson process in deep [Last and Penrose, 2017]
xxxx [Haugen, 1995] about Conny Palm
xxx Erlang[w]

renewal process III

Queueing theory[w] has its origins in research by Agner Krarup Erlang when he created models to describe the Copenhagen telephone exchange; SEE ALSO: Mean field limits[w] AND Fluid limits[w] in this context.
xxxx [Jabin, 2014] mean field Vlasov PDE

section plan

inter-spike intervals (ISI) I

- action potentials sequence of event is highly irregular, due to

  "stochastic forces" or (vs) "precise coincidences of presynaptic events"

  see Heeger (2000). Poisson model of spike generation. Handout, University of Standford
- "real randomness" or "highly complex deterministic process": in both cases stochastic models are pertinent (and deterministic too !)

inter-spike intervals (ISI) II

- Poisson hypothesis wrong:
  - absolute refractory period: no spike
  - (longer) refractory period: almost no spike
  - bursting: cluster of spikes

cluster of spikes: on tire les instants de burst et le nombre de spike par burst
precise statistical procedures should be used to reject (or not) the (homogeneous) Poisson hypothesis [Reynaud-Bouret et al., 2014]
carefull dead salmon

instantaneous firing rate I

$$\mathbb{P}(N_{t+\delta t} = n + \ell | N_t = n) = \begin{cases} \lambda\,\delta t + o(\delta t) & \text{if } \ell = 1 \\ 1 - \lambda\,\delta t + o(\delta t) & \text{if } \ell = 0 \\ o(\delta t) & \text{otherwise} \end{cases}$$

or, conditionally on $N_t = n$

$$\begin{cases} N_{t+\delta t} = n + 1 & \text{with proba } \lambda\,\delta t + o(\delta t) \\ N_{t+\delta t} = n & \text{with proba } 1 - \lambda\,\delta t + o(\delta t) \\ N_{t+\delta t} = n + \ell & \text{with proba } o(\delta t) \text{ when } \ell \geq 2 \end{cases}$$

instantaneous firing rate II

$r(t)$ instantaneous firing rate: the probability that the neuron will fire during $[t, t+\delta t]$ divided by δt, in the limit $\delta t \to 0$

$$
\begin{aligned}
r(t) &= (1/\delta t)\, \mathbb{E}(N_{t+\delta t} | N_t = n) \\
&= (1/\delta t) \Big(0\,(1 - \lambda\,\delta t) + 1\,\lambda\,\delta t + \text{“}o(\delta t)\text{”} \Big) \\
&\to \lambda
\end{aligned}
$$

section plan

Poisson process as a single neuron spike train

[Barlow et al., 1957, Gerstein and Kiang, 1960, Grossman and Viernstein, 1961, Perkel et al., 1967, Fienberg, 1974, Holden, 1976, Sampath and Srinivasan, 1977, Tuckwell, 1988]

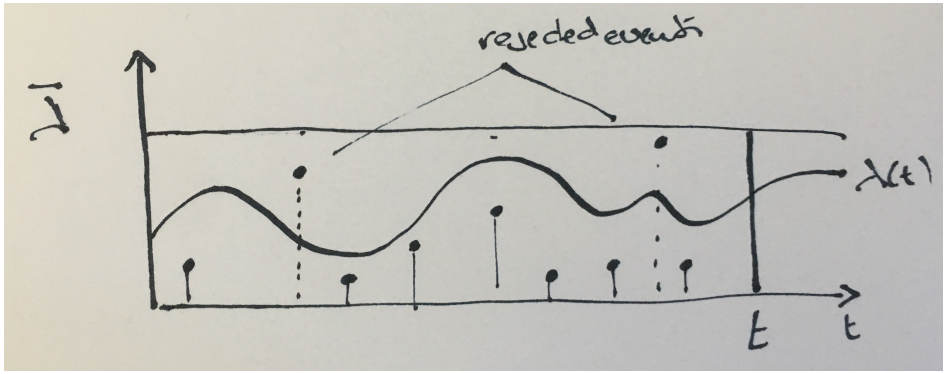
a single neuron spike train as a time point process, i.e. a process whose realizations consist of a ordered sequences of point events $< T_1 < T_2 < T_3 < \cdots$. Hence we reduce a neuronal spike to a single time event, abusively called "spike", ignoring all other spike characteristics like shape or duration. Many works had adopted this formalism [Gerstein and Kiang, 1960] but, as noted by [Gerstein and Mandelbrot, 1964], only from the statistical point of view proposing purely descriptive model that does not aim at directly understanding the mechanisms involved in the firing of a neuron. Here we propose to understand how this type of stochastic processes can be used as a modeling tool.

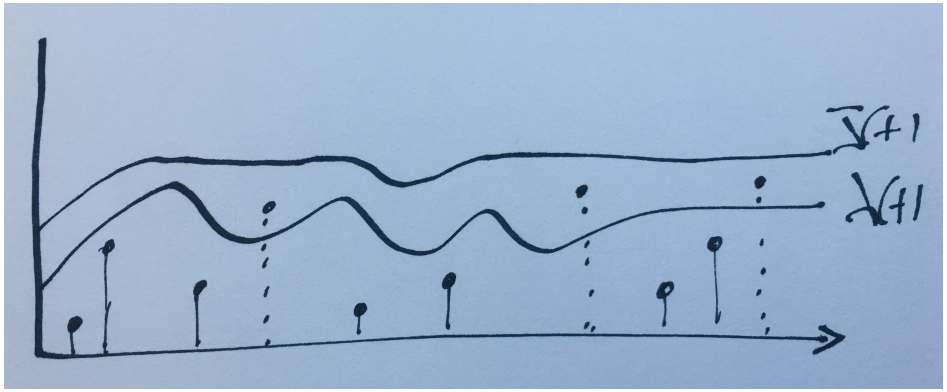
"The simple Poisson type of model has been used previously in neurophysiology. Unfortunately, it is a purely descriptive model that does not contribute much to our understanding of the detailed mechanisms that may be involved in the firing of a neuron. It is

section plan

Spike train and point processes

inhomogeneous Poisson process





counting process II

Cox processes[w]
semi-Markov processes
Hawkes processes [Reynaud-Bouret, 2015]
state-space models

xxxx I

dddd

,

xxxx

xxxx

xxxx

plan

ddd I

références simples [Graham and Talay, 2013]

section plan

ddd I

section plan

section plan

Markov processes with values in space of measures I

plan

section plan

Brownian motion

a nice guide by 2 great specialists [Pitman and Yor, 2018] (in French [Kahane, 1998])

section plan

infinitesimal generator

$$\mathcal{L}\varphi(x) = \sum_i f_i(x)\,\frac{\partial\varphi(x)}{\partial x_i} + \frac{1}{2}\sum_{ij}(g(x)\,g(x)^*)_{ij}\,\frac{\partial^2\varphi(x)}{\partial x_i\,\partial x_j}$$

$$\mathrm{d}X_t = f(X_t)\,\mathrm{d}t + g(X_t)\,\mathrm{d}B_t$$

$$\text{ie} \qquad X_t = X_0 + \underbrace{\int_0^t f(X_s)\,\mathrm{d}s}_{\text{drift term}} + \underbrace{\int_0^t g(X_s)\,\mathrm{d}B_s}_{\text{diffusion term}}$$

$$\mathbb{E}(X_{t+h}|X_t = x) = f(x) + o(h)$$
$$\mathsf{cov}(X_{t+h}|X_t = x) = g(x)\,g(x)^* + o(h)$$

i.e. $\mathrm{law}(X_{t+h}|X_t = x) = \mathcal{N}(f(x), g(x)\,g(x)^*) + o(h)$

- $f(x)$ drift coefficient, $g(x)$ diffusion coefficient
- $g(x)\,g(x)^*$ diffusion matrix

[plan]

section plan

Gaussian approximation of the Poisson distribution

$$\mathrm{Poisson}(\lambda\, t) \simeq \mathcal{N}(\lambda\, t, \lambda\, t)$$

for $\lambda\, t$ large (in practice $\lambda\, t \geq 10$ or 20)

diffusion approximation of the nonlinear birth and death model I

- via a crude approximation of the infinitesimal generator

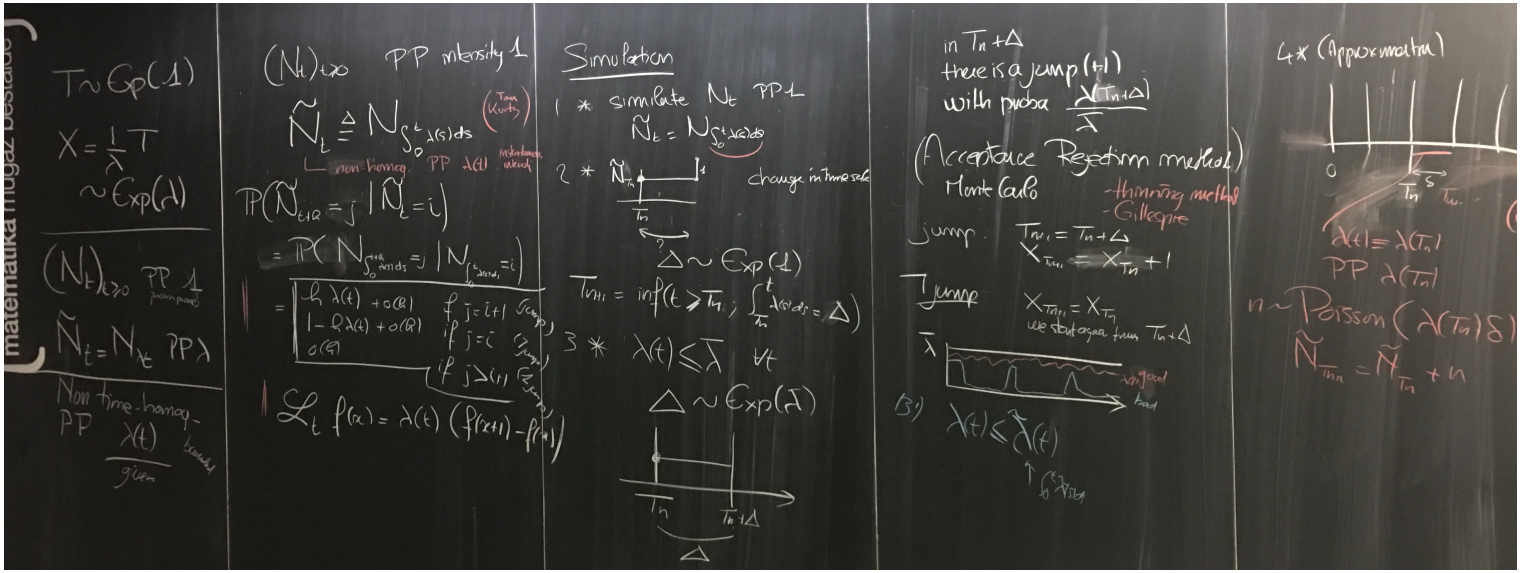
$$\begin{aligned}\mathcal{L}\varphi(x) &= \lambda(x)\,\big(\varphi(x+1)-\varphi(x)\big)+\mu(x)\,\big(\varphi(x-1)-\varphi(x)\big)\\ &\simeq \lambda(x)\,\big(1\,\varphi'(x)+\tfrac{1^2}{2}\,\varphi''(x)\big)\\ &\qquad\qquad +\mu(x)\,\big(-1\,\varphi'(x)+\tfrac{(-1)^2}{2}\,\varphi''(x)\big)\\ &= \big(\lambda(x)-\mu(x)\big)\,\varphi'(x)+\tfrac{1}{2}\,\big(\lambda(x)+\mu(x)\big)\,\varphi''(x)\end{aligned}$$

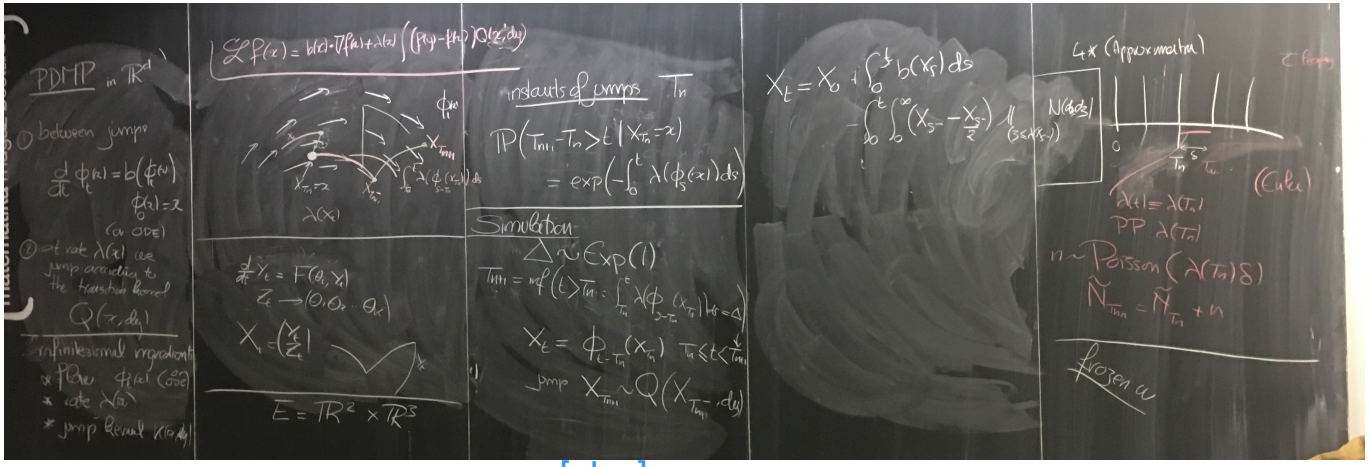
corresponding to

$$\mathrm{d}X_t = \big(\lambda(X_t)-\mu(X_t)\big)\,\mathrm{d}t+\sqrt{\lambda(X_t)+\mu(X_t)}\,\mathrm{d}B_t$$

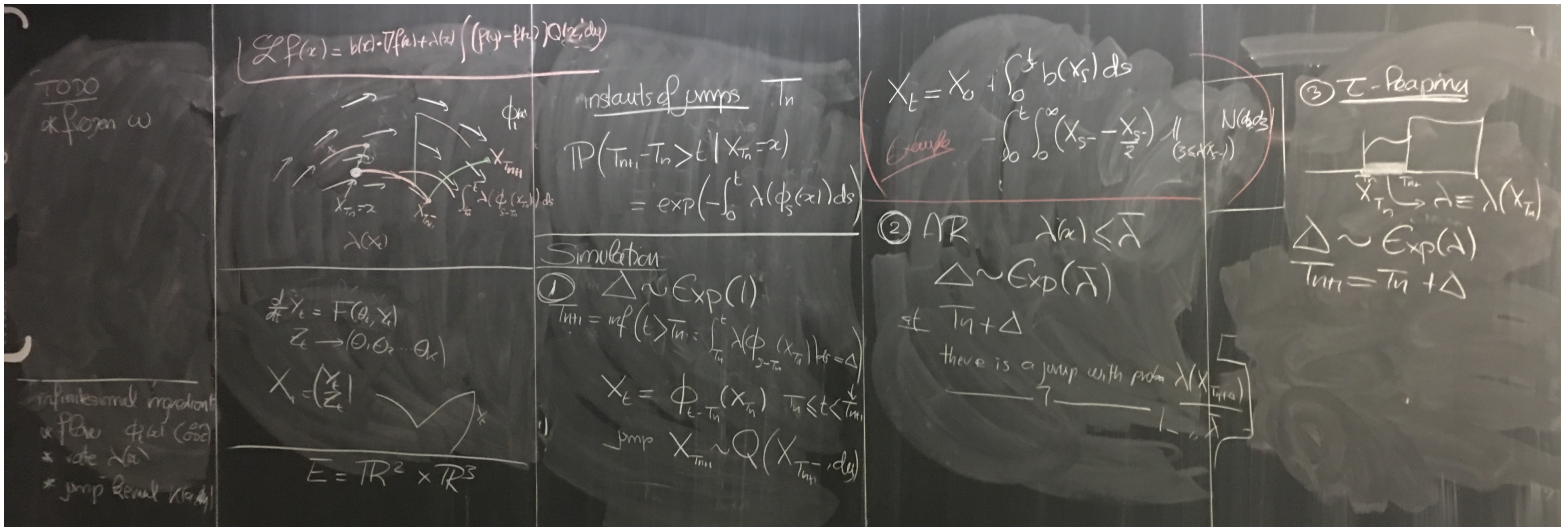
- examples
  - linear model
    $\mathrm{d}X_t = (\lambda-\mu)\,X_t\,\mathrm{d}t+\sqrt{(\lambda+\mu)\,X_t}\,\mathrm{d}B_t$
  - logistic model
    $\mathrm{d}X_t = (\lambda-\mu-\varepsilon\,X_t)\,X_t\,\mathrm{d}t+\sqrt{(\lambda+\mu+\varepsilon\,X_t)\,X_t}\,\mathrm{d}B_t$

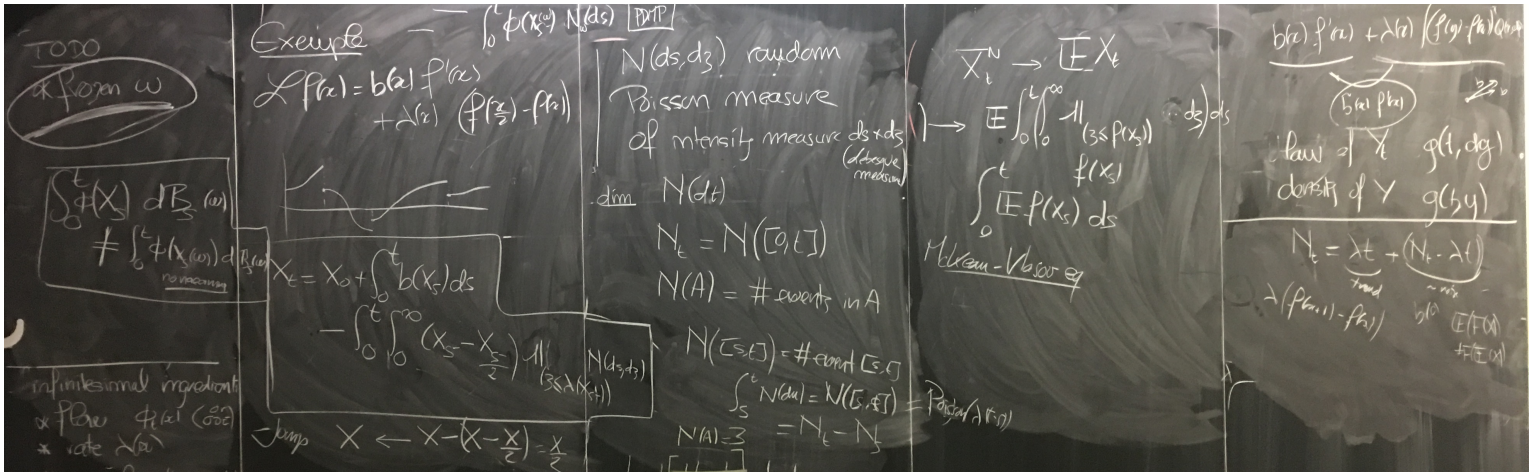
pics I


Simulation
in $T_n+\Delta$
there is a jump (+1)
with proba
(Acceptance Rejection method)
Monte Carlo
-thinning method
-Gillespie
jump
4* (Approximation)


PDMP in $\mathbb{R}^d$
instants of jumps T_n
Simulation
4* (Approximation)

pics II


TODO
frozen ω
instants of jumps
Simulation
AR
τ-leaping
E = ℝ² × ℝ³


TODO
frozen ω
Example
Poisson measure
of intensity measure
dim
Mckean-Vlasov eq

plan

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plan

section plan

lois de probabilités[W] I

- une variable aléatoire est une quantité issue d'une expérience aléatoire, son comportement est déterminée par sa loi de probabilité
  - variable finie: $X \to \{1, \ldots, N\}$, la loi de X est:

$$p_i \stackrel{\text{def}}{=} \mathbb{P}(X = i) \qquad i = 1, \ldots, N$$

avec $p_i \geq 0$, $\sum_1^N p_i = 1$ et

$$\mathbb{P}(X \in A) = \sum_{i \in A} p_i \qquad \mathbb{E}(\varphi(X)) = \sum_{i=1}^N p_i \, \varphi(i)$$

  - variable discrète: e.g. $X \to \mathbb{N}$, la loi de X est:

$$p_n \stackrel{\text{def}}{=} \mathbb{P}(X = n) \qquad n \in \mathbb{N}$$

avec $p_i \geq 0$, $\sum_0^\infty p_i = 1$ et

$$\mathbb{P}(X \in A) = \sum_{i \in A} p_i \qquad \mathbb{E}(\varphi(X)) = \sum_{n=1}^\infty p_i \, \varphi(i)$$

lois de probabilités[W] II

- variable continue: e.g. $X \to \mathbb{R}$ ou $\mathbb{R}^d$, sa loi se présente généralement sous la forme d'une densité:

$$p(x) \geq 0 \qquad x \in \mathbb{R}$$

avec $p(x) \geq 0$ et $\int p(x)\,\mathrm{d}x = 1$ et

$$\mathbb{P}(X \in A) = \int_A p(x)\,\mathrm{d}x \qquad \mathbb{E}(\varphi(X)) = \int p(x)\,\varphi(x)\,\mathrm{d}x$$

retour

loi de Bernoulli[w] & lois discrètes finies[w] I

- loi de Bernoulli ou épreuve de Bernoulli, $X \sim \mathcal{B}(p)$

$$\mathbb{P}(X = 1) = 1 - \mathbb{P}(X = 0) = p \qquad \text{(2 issues)}$$

$p \in [0, 1]$, la plus simple des lois de probabilité

- $\mathbb{E}(X) = 1\,p + 0\,(1 - p) = p$
- $\mathrm{var}(X) = \mathbb{E}(X^2) - p^2 = p\,(1 - p)$

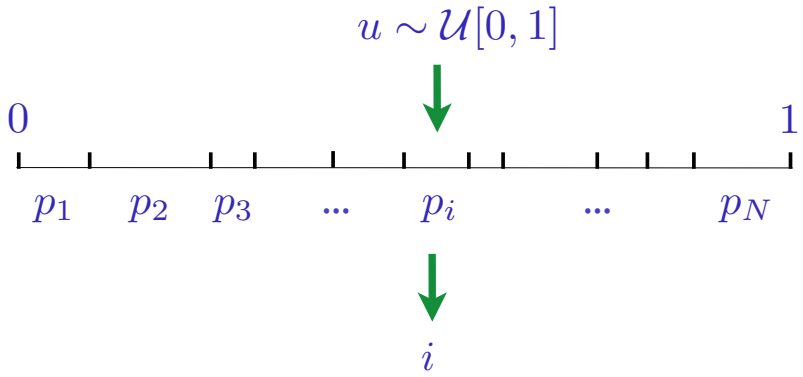
- loi de probabilité discrète et finie, N issues:

$$p_i \qquad i = 1, \ldots, N \qquad (p_i \geq 0\,,\ \textstyle\sum_{i=1}^{N} p_i = 1)$$

comment échantillonner selon cette loi?

loi de Bernoulli[W] & lois discrètes finies[W] II

- méthode d'inversion (de la fonction de répartition)



- un premier code matlab basique

```
function e = simu_loi_discrete1(probas,n)
e      = zeros(1,n);
probas = probas/sum(probas); % normaliser par la somme
probas = cumsum(probas);     % calculer la fonction de repartition
for i=1:n
    k=1;
    r=rand(1,1);
    while r>probas(k) % simulation par inversion
        k = k+1;
    end
    e(i)=k;
end
```

loi de Bernoulli[W] & lois discrètes finies[W] III

- une version vectorisée (moins lisible)

```
function x = simu_loi_discrete2(probas,n)
k      = length(probas);
probas = reshape(probas,k,1);
probas = cumsum(probas)/sum(probas);
x      = sum(repmat(rand(1,n),k,1)> repmat(probas,1,n),1)+1;
```

- c'est mieux

```
>> t = cputime; simu_loi_discrete1([1 1],10000000); cputime-t
ans =
   26.2400
>> t = cputime; simu_loi_discrete2([1 1],10000000); cputime-t
ans =
    1.4600
```

- mais il y a encore mieux

```
>> t = cputime; randsample([0 1],10000000,true,[1 1]); cputime-t
ans =
    0.5300
```

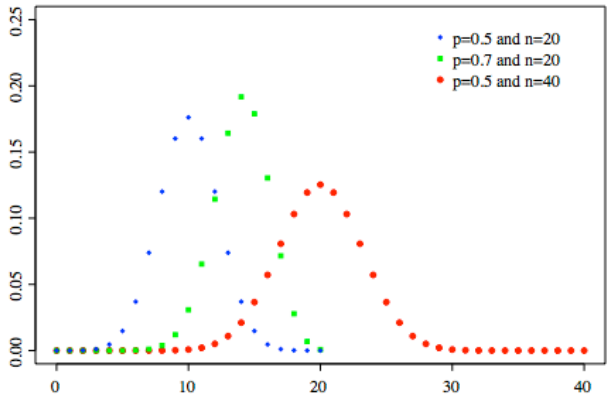
loi binomiale[W]

- On renouvelle n fois une épreuve de Bernoulli de paramètre p (succès avec probabilité p, échec avec probabilité $1-p$) et on note X le nombre de succès, X suit une loi binomiale de paramètres n et p, $\mathcal{B}in(n,p)$

$$\mathbb{P}(X=k) = \binom{n}{k} p^k \, (1-p)^{n-k}$$

pour $k = 0, \ldots, n$, où $\binom{k}{n} = \frac{n!}{k!\,(n-k)!}$

$$\mathbb{E}(X) = n\,p \quad \mathrm{var}(X) = n\,p\,(1-p)$$



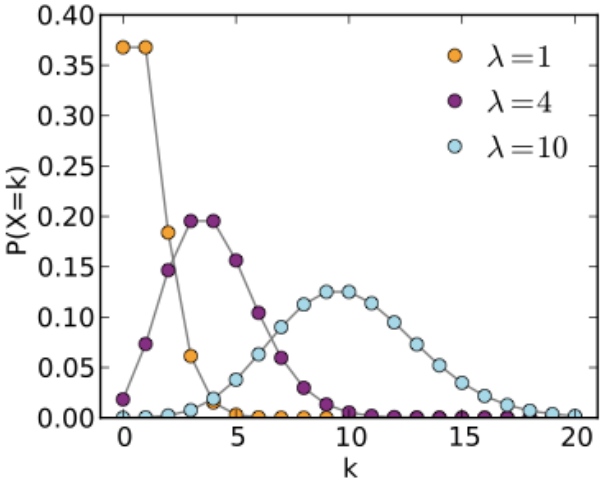
- lorsque $n\,p_n \to 0$ alors $\mathcal{B}in(n,p) \simeq \mathcal{P}(a)$, (ex: $n > 30$ et $np < 5$, ou $n > 50$ et $p < 0.1$)
- pour n grand et p pas trop proche de 0 ni de 1: $\mathcal{B}in(n,p) \simeq \mathcal{N}(n\,p, n\,p\,(1-p))$

loi de Poisson[W]

- $X \sim \mathcal{P}(\lambda)$

$$\mathbb{P}(X = k) = \frac{\lambda^k}{k!}\, e^{-\lambda}$$
$$\mathbb{E}(X) = \mathrm{var}(X) = \lambda$$

- La somme S_n d'un grand nombre de variables de Bernoulli indépendantes de petit paramètre p est $\simeq \mathcal{P}(n\,p)$

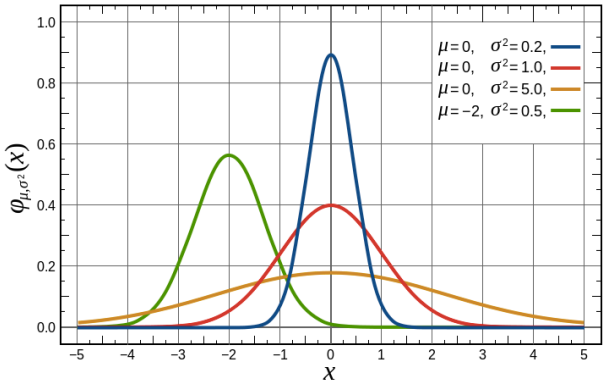


retour

loi normale[W]

- $X \sim \mathcal{N}(\mu, \sigma^2)$

$$p(x) = \frac{1}{\sqrt{2\,\pi\sigma^2}}\,\exp\Big(-\frac{|x-\mu|^2}{2\,\sigma^2}\Big)$$
$$\mathbb{E}(X) = \mu \;\; \mathrm{var}(X) = \sigma^2$$



- si $X \sim \mathcal{N}(\mu_1, \sigma_1^2)$, $Y \sim \mathcal{N}(\mu_2, \sigma_2^2)$ sont indépendants alors $X + Y \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$ ou encore il existe $Z \sim \mathcal{N}(0, 1)$ tel que $X + Y = \mu_1 + \mu_2 + \sqrt{\sigma_1^2 + \sigma_2^2}\, Z$

section plan

preuve de l'équation de Fokker-Planck I

- on pose

$$q(t,x) = p_n(t) \qquad x = \frac{n}{V}$$

d'après l'équation maitresse:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} p_n(t) &= -\Big[\lambda(n)\, p_n(t) - \lambda(n-1)\, p_{n-1}(t)\Big] \\
&\qquad + \Big[\mu(n+1)\, p_{n+1}(t) - \mu(n)\, p_n(t)\Big] \\
\frac{\mathrm{d}}{\mathrm{d}t} q(t,x) &= -\Big[\lambda(V\,x)\, q(t,x) - \lambda(V\,(x-1/V))\, q(t,x-1/V)\Big] \\
&\qquad + \Big[\mu(V\,(x+1/V))\, q(t,x+1/V) - \mu(V\,x)\, q(t,V\,x)\Big] \\
&= -V\,\Big[\bar\lambda(x)\, q(t,x) - \bar\lambda(x-1/V)\, q(t,x-1/V)\Big] \\
&\qquad + V\,\Big[\bar\mu(x+1/V)\, q(t,x+1/V) - \bar\mu(x)\, q(t,V\,x)\Big]
\end{aligned}$$

car $\lambda(V\,x)/V \simeq \bar\lambda(x)$ et $\mu(V\,x)/V \simeq \bar\mu(x)$

preuve de l'équation de Fokker-Planck II

mais

$$\bar{\lambda}(x)\, q(t,x) \simeq \bar{\lambda}(x-1/V)\, q(t,x-1/V) - \tfrac{1}{V}\left[\bar{\lambda}(x)\, q(t,x)\right]' + \tfrac{1}{2\,V^2}\left[\bar{\lambda}(x)\, q(t,x)\right]''$$

$$\bar{\mu}(x)\, q(t,x) \simeq \bar{\mu}(x+1/V)\, q(t,x+1/V) + \tfrac{1}{V}\left[\bar{\mu}(x)\, q(t,x)\right]' + \tfrac{1}{2\,V^2}\left[\bar{\mu}(x)\, q(t,x)\right]''$$

donc

$$\frac{\partial}{\partial t} q(t,x) = -\left[(\bar{\lambda}(x) - \bar{\mu}(x))\, q(t,x)\right]' + \frac{1}{2\,V}\left[(\bar{\lambda}(x) + \bar{\mu}(x))\, q(t,x)\right]''$$

section plan

vers la RRE

$$
\begin{aligned}
\frac{\mathrm{d}\mathbb{E}(X(t))}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\sum_x x\, p_x(t) \\
&= \sum_j \sum_x x\left[\lambda_j(x-\nu_j)\, p_{x-\nu_j}(t) - \lambda_j(x)\, p_x(t)\right] \\
&= \sum_j \sum_x x\, \lambda_j(x-\nu_j)\, p_{x-\nu_j}(t) - \sum_j \sum_x x\, \lambda_j(x)\, p_x(t) \\
&= \sum_j \sum_{x'} (x'+\nu_j)\, \lambda_j(x')\, p_{x'}(t) - \sum_j \sum_x x\, \lambda_j(x)\, p_x(t) \\
&= \sum_j \sum_x \nu_j\, \lambda_j(x)\, p_x(t) = \sum_j \nu_j\, \mathbb{E}\big[\lambda_j(X(t))\big]
\end{aligned}
$$

retour

references

- par analogie avec les réactions chimiques élémentaires:

$\emptyset \to \bullet\!\!\!\!\bullet \to$	*ordre 0*	$\emptyset \xrightarrow{c_j}$ produit	processus de Poisson
$\bullet \to \bullet\!\!\!\!\bullet \to$	*ordre 1*	$\mathrm{S}_i \xrightarrow{c_j}$ produit	naissance et mort linéaire
$\bullet, \bullet \to \bullet\!\!\!\!\bullet \to$	*ordre 2*	$2\mathrm{S}_i \xrightarrow{c_j}$ produit	logistique
$\bullet, \bullet \to \bullet\!\!\!\!\bullet \to$	*ordre 2*	$\mathrm{S}_i + \mathrm{S}_{i'} \xrightarrow{c_j}$ produit	Lotka-Volterra

- au-delà
  - modèle de naissance et mort non-linéaire
  - modèle du chemostat
  - modèles individu-centrés