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Analysis and Approximation of a Stochastic Growth Model with Extinction

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Abstract We consider a stochastic growth model for which extinction eventually occurs almost surely. The associated complete Fokker–Planck equation describing the law of the process is established and studied. This equation combines a PDE and an ODE, connected one to each other. We then design a finite differences numerical scheme under a probabilistic viewpoint. The model and its approximation are evaluated through numerical simulations.

Keywords Logistic model · Markov processes · Diffusion processes · Extinction · Fokker–Planck equation · PDE

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1 Introduction

Most of the growth models in population dynamics and ecology are based on ordinary differential equations (ODE). Among these models, the logistic population growth model was

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first introduced by Verhulst (1838) to take into account crowding effect, by damping the *per capita growth rate* in the Malthusian growth model; this model reads:

$$\dot{x}(t) = r\,x(t)\left(1 - \frac{x(t)}{K}\right), \qquad x(0) = x_0 > 0\,,\ t \geq 0 \tag{1}$$

where $x(t) \geq 0$ is the density of some population, $r > 0$ the growth rate and $K > x_0$ the carrying capacity of the environment. This formulation leaves aside the intrinsic stochasticity resulting from diversity in the population or from random fluctuations of the environment. Even so, these demographic and environmental noises are known to be important in ecology, biology and environmental sciences. The need for stochastic model is therefore grounded, see e.g. Kendall (1949) .

Beside, this description obscures two issues. On the one hand the model (1) cannot account for a possible extinction of the population. Consequently, this model cannot be used to carry inference on the unknown parameters based on an observed data set where extinction occurs. On the other hand, it does not involve the birth rate $\lambda > 0$ and the death rate $\mu > 0$, but only their balance $r = \lambda - \mu$. Again, the rates λ and μ cannot be identified from observations.

Stochastic counterparts of Eq. 1 may overcome these two issues. Stochastic logistic models can be obtained by adding a random ad hoc perturbation term in Eq. 1. In Nåsell (2001), a stochastic approach using pure jump Markov processes that feature a logistic mechanism, is proposed. Diffusion approximations of such processes lead to SDE's, see Section 2. For these approaches, there will obviously be many stochastic models derived from or leading to the same deterministic model (1), but with different qualitative behaviors, see Schurz (2007). In Skiadas (2010), a stochastic logistic model without extinction is presented.

In this paper, we will consider the stochastic logistic model given by the following SDE:

$$\mathrm{d}X_t = (\lambda - \mu - \alpha\,X_t)\,X_t\,\mathrm{d}t + \rho\,\sqrt{(\lambda + \mu + \alpha\,X_t)\,X_t}\,\mathrm{d}B_t \tag{2}$$

where $\alpha > 0$ the logistic coefficient and $\rho > 0$ the noise intensity which relates to the order of magnitude of the underlying population (see Section 2); B_t is a standard Brownian motion; the law of the initial condition X_0 is supported by $\mathbb{R}_+$; B_t and X_0 are supposed independents. This stochastic model explicitly handles the question of extinction, see Section 3. It also features the birth and death rates through their difference $\lambda - \mu$ and their sum $\lambda + \mu$, allowing statistical inference on the two parameters separately. Hereafter we will rewrite Eq. 2 as

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t\,, \tag{3}$$

where $b(x) = (\lambda - \mu - \alpha\,x)\,x$ and $\sigma(x) = \rho\,\sqrt{(\lambda + \mu + \alpha\,x)\,x}$ are the drift and diffusion coefficients respectively. We also define $a(x) \stackrel{\text{def}}{=} [\sigma(x)]^2$.

Due to the Markov nature of the process (3), the distribution law of X_t is given in a weak form by the Kolmogorov Forward Equation. For diffusion process that never becomes extinct, this equation reduces to the Fokker-Planck equation for the transition density. An originality of this work lies in the fact that the solution of this Kolmogorov equation fails to have a density with respect to the Lebesgue measure on $\mathbb{R}_+$. This problem was already noticed in Campillo et al. (2014), where the authors derived a closed result for a two–dimensional model, from a numerical perspective. It is expected that a more thorough study of the one–dimensional case will be of valuable help in higher dimension.

The model is presented and studied in Sections 2 to 5. We prove existence and uniqueness in Section 3 and we prove that $x = 0$ is an exit boundary point according to Feller terminology in Section 4. We investigate the *complete* form of the Fokker-Planck equation (Feller 1952) that gives the evolution of the transition kernel of the diffusion process $\{X_t\}_{0 \leq t \leq T}$ in

Section 5. The finite difference numerical scheme is introduced in Section 6. Section 7 is devoted to numerical experiments.

2 Diffusion Approximation

Although our objective is to study the diffusion process (3), we give hereafter a possible derivation of this equation. Let us consider at a microscopic scale, a population size N_t subject to a birth and death process that features a logistic mechanism, e.g.:

$$\mathbb{P}(N_{t+h} = n' | N_t = n) \underset{h\to 0}{\simeq} \begin{cases} h\,\lambda\,n + o(h)\,, & \text{if } n' = n+1\,, \\ h\,\left(\mu + \frac{\alpha}{\kappa}\,n\right)\,n + o(h)\,, & \text{if } n' = n-1\,, \\ 1 - h\,\left(\lambda + \mu + \frac{\alpha}{\kappa}\,n\right)\,n + o(h)\,, & \text{if } n' = n\,, \\ o(h)\,, & \text{otherwise} \end{cases} \tag{4}$$

with $\lambda > 0$, $\mu > 0$, $\alpha > 0$, $\kappa > 0$; here the birth per capita rate λ is constant and the death per capita rate $\mu + \frac{\alpha}{\kappa}\,n$ increases linearly with the population size n. The rescaled process:

$$X_t^\kappa \overset{\text{def}}{=} \frac{1}{\kappa}\,N_t$$

is a pure jump Markov process with values in $\frac{1}{\kappa}\,\mathbb{N}$ where κ denotes the order of magnitude of the population size N_t of interest. Its distribution law is characterized by the infinitesimal generator:

$$\lim_{t\to 0} \frac{\mathbb{E}(\phi(X_t^\kappa)|X_0^\kappa = x) - \phi(x)}{t} = \lambda\,x\,\left[\phi\left(x + \tfrac{1}{\kappa}\right) - \phi(x)\right] + (\mu + \alpha\,x)\,x\,\left[\phi\left(x - \tfrac{1}{\kappa}\right) - \phi(x)\right]$$

defined for any bounded function ϕ. In large population scale, that is for κ large, a *diffusion approximation* of X_t^κ is obtained by performing a second order Taylor expansion of regular functions ϕ. This yields the operator $\mathscr{A}$ defined by

$$\begin{aligned} \mathscr{A}\phi(x) &\overset{\text{def}}{=} b(x)\,\phi'(x) + \tfrac{1}{2}\,a(x)\,\phi''(x) \\ &= (\lambda - \mu - \alpha\,x)\;x\,\phi'(x) + \tfrac{1}{2}\,\rho^2\;(\lambda + \mu + \alpha\,x)\;x\,\phi''(x) \end{aligned} \tag{5}$$

which is the infinitesimal generator of the diffusion process solution of Eq. 3.

Note that Eq. 3 can be rewritten as

$$\mathrm{d}X_t = r\,\left(1 - \frac{X_t}{K}\right)\,X_t\,\mathrm{d}t + \rho\,\sqrt{r'\,\left(1 + \frac{X_t}{K'}\right)\,X_t}\,\mathrm{d}B_t$$

with $r = \lambda - \mu$, $K = \frac{r}{\alpha}$, $r' = \lambda + \mu$, $K' = \frac{r'}{\alpha}$ and $\rho = \frac{1}{\sqrt{\kappa}}$, so that its instantaneous mean has the same form as in Eq. 1. However, the deterministic model (1) does not describe the evolution of $\mathbb{E}[X_t]$ but rather the large population asymptotics of X_t. Diffusion approximation technique is frequently encountered in life sciences. A typical example of a two–dimensional bioreactor is described in Campillo et al. (2011) or Joannides and Larramendy-Valverde (2013). The relationship between the distribution laws of the process X_t^κ and its *diffusion approximation* X_t for large κ is made precise in Ethier and Kurtz (1986, Chap. 7 and 11).

3 Existence and Uniqueness for SDE (3)

The drift function b is locally Lipschitz on $\mathbb{R}$ but fails to satisfy the usual linear growth condition. On the other hand, the diffusion function σ is not locally Lipschitz on $\mathbb{R}$. Nevertheless, we have

Lemma 1 *For any non-negative initial condition $X_0 \in L^2$, there exists a unique non-negative solution to* Eq. 3.

Proof We first deal with the diffusion term. Consider $h : \mathbb{R} \mapsto \mathbb{R}$ globally Lipschitz and of linear growth, suppose also that $h(0) = 0$. Then introduce for $\ell \geq 1$

$$\sigma_\ell(x) \stackrel{\text{def}}{=} \begin{cases} \sigma(x)\,, & \text{if } \frac{1}{\ell} \leq x\,, \\ \rho\sqrt{\frac{1}{\ell}\left(\lambda+\mu+\frac{\alpha}{\ell}\right)(2\,\ell\,x-1)}\,, & \text{if } \frac{1}{2\ell} < x < \frac{1}{\ell}\,, \\ 0\,, & \text{if } x \leq \frac{1}{2\ell}\,. \end{cases}$$

This function is globally Lipschitz so that SDE

$$\mathrm{d}Y_t^\ell = h\left(Y_t^\ell\right)\mathrm{d}t + \sigma_\ell\left(Y_t^\ell\right)\mathrm{d}B_t$$

has a unique solution with a.s. continuous path, for any initial condition $Y_0 \in L^2$. If Y_0 is non-negative, then Y_t^ℓ remains non-negative by almost sure continuity of the paths and by the condition $h(0) = 0$. Define $T_\ell = \inf\left\{t \geq 0\,;\ Y_t^\ell \leq \frac{1}{\ell}\right\}$ and note that Y_t^ℓ and $Y_t^{\ell'}$ coincide up to time $T_{\ell'}$ when $\ell' \leq \ell$ (Durrett (1996), Lemma 2.8, Chap.5). The process $Y_t = Y_t^\ell$ is then well defined up to time $T_\infty = \lim_{\ell\uparrow\infty} T_\ell$. Now since σ_ℓ and σ coincide on $\left[\frac{1}{\ell}, +\infty\right)$, Y_t is a solution to the following SDE

$$\mathrm{d}Y_t = h(Y_t)\,\mathrm{d}t + \sigma(Y_t)\,\mathrm{d}B_t \tag{6}$$

on time interval $[0, T_\infty)$. As above, Y_t remains non-negative for any non-negative initial condition Y_0, on $[0, T_\infty)$. On the event $\{T_\infty < \infty\}$, we define $Y_t = 0$ for $t \geq T_\infty$, which is an obvious solution of Eq. 6 with initial condition $\tilde{Y}_0 = 0$. We conclude that Eq. 6 has a unique non-negative solution for all $t \geq 0$ and initial condition Y_0, square integrable non-negative random variable.

Likewise, let for $\ell \geq 1$

$$b_\ell(x) \stackrel{\text{def}}{=} \begin{cases} b(x)\,, & \text{if } x \leq \ell\,, \\ (\lambda-\mu-\alpha\,\ell)\,(2\,\ell-x)\,, & \text{if } \ell < x < 2\,\ell\,, \\ 0\,, & \text{if } 2\,\ell \leq x\,. \end{cases}$$

Since b_ℓ is globally Lipschitz and bounded, the preceding result applies so that

$$\mathrm{d}X_t^\ell = b_\ell\left(X_t^\ell\right)\mathrm{d}t + \sigma\left(X_t^\ell\right)\mathrm{d}B_t$$

has a unique non-negative solution for each ℓ. Consider the stopping times $S_\ell = \inf\{t \geq 0\,;\ X_t^\ell \geq \ell\}$ and $S_\infty = \lim_{\ell\uparrow\infty} S_\ell$, and define $X_t = X_t^\ell$ for $t \leq S_\ell$. Since b_ℓ and b coincide on $[0, \ell]$, it holds

$$X_{t\wedge S_\ell} = X_0 + \int_0^{t\wedge S_\ell} b(X_s)\,\mathrm{d}s + \int_0^{t\wedge S_\ell} \sigma(X_s)\,\mathrm{d}B_s\,.$$

The last term is a martingale, so by the optional stopping theorem

$$\begin{aligned}\mathbb{E}(X_{t\wedge S_\ell}) &= \mathbb{E}(X_0) + \mathbb{E}\left(\int_0^{t\wedge S_\ell} b(X_s)\,\mathrm{d}s\right)\\ &= \mathbb{E}(X_0) + \int_0^t \mathbb{E}[1_{(s\le S_\ell)}\, b(X_{s\wedge S_\ell})]\,\mathrm{d}s\\ &\le \mathbb{E}(X_0) + \int_0^t \mathbb{E}[b(X_{s\wedge S_\ell})]\,\mathrm{d}s\\ &\le \mathbb{E}(X_0) + \int_0^t \mathbb{E}[(\lambda-\mu)\, X_{s\wedge S_\ell}]\,\mathrm{d}s \le \mathbb{E}(X_0) + (\lambda-\mu)\int_0^t \mathbb{E}[X_{s\wedge S_\ell}]\,\mathrm{d}s\,.\end{aligned}$$

The Gronwall lemma yields $\mathbb{E}(X_{T\wedge S_\ell}) \le \mathbb{E}(X_0)\ \exp\{(\lambda-\mu)\,T\}$, for all $T > 0$, and from Fatou's lemma:

$$\forall\, T > 0, \qquad \mathbb{E}(X_{T\wedge S_\infty}) \le \liminf_{\ell\to\infty} \mathbb{E}(X_{T\wedge S_\ell}) \le \mathbb{E}(X_0)\ \exp\{(\lambda-\mu)\,T\}$$

so that $\mathbb{P}(S_\infty \le T) = 0$. Hence $S_\infty = \infty$ p.s. and the lemma is proved. □

Remark 1 The Lipschitz condition on the diffusion coefficient can be weakened, see Yamada and Watanabe (1971). If we do so, we obtain existence and uniqueness of a weak solution. The existence of a strong solution would require a little more effort.

4 Extinction Time

For $y \ge 0$, let:

$$\tau_y \stackrel{\text{def}}{=} \inf\{t \ge 0\,;\ X_t = y\}$$

As a by–product of the proof of the existence and uniqueness of solution of Eq. 3 given in Section 2, we find that the process remains in the interval $[0, +\infty)$. We also show that $X_t = 0$ for $t \ge \tau_0$, but whether the boundary 0 could be reach in finite time or not is still to be determined. A complete description of the possible behavior at the boundary points has been established by Feller (1952). A detailed review of these results can be found in Chapter 15 of Karlin and Taylor (1981). The following lemma states that 0 is an *exit* boundary point according to Feller terminology: it is reached in an almost surely finite time and no interior point in $(0, +\infty)$ can be reached starting from 0.

Lemma 2 *Extinction occurs almost surely in finite time, that is for all* $x \ge 0$, $\mathbb{P}_x(\tau_0 < \infty) = 1$ *where* $\mathbb{P}_x$ *is the probability measure such that* $X_0 = x$.

Proof For $0 < x_l < x < x_r$, we have

$$\mathbb{P}_x(\tau_{x_l} < \tau_{x_r}) = \frac{S(x_r) - S(x)}{S(x_r) - S(x_l)}$$

where S is the *scale function*, see e.g. Klebaner (2005) Theorem 6.17, or Kallenberg (1997) Chapter 20, defined by

$$S(x) \stackrel{\text{def}}{=} \int_\eta^x \exp\left\{-\int_\eta^y \frac{2\,b(z)}{a(z)}\mathrm{d}z\right\}\mathrm{d}y\,.$$

The choice of the lower bound η in the integrals will appear to be unimportant and could be chosen arbitrarily within (x_l, x_r) since the relevant expressions involve only differences of the function S. A straightforward computation gives for this particular case

$$S(x) = C_\eta \int_\eta^x s(y)\,\mathrm{d}y$$

where

$$s(y) \stackrel{\text{def}}{=} \left(\mathrm{e}^y\,(\lambda + \mu + \alpha\, y)^{-\frac{2\lambda}{\alpha}}\right)^{\frac{2}{\rho^2}}$$

and C_η is a constant depending on η only, so that

$$\mathbb{P}_x(\tau_{x_l} < \tau_{x_r}) = \frac{\int_\eta^{x_r} s(y)\,\mathrm{d}y - \int_\eta^x s(y)\,\mathrm{d}y}{\int_\eta^{x_r} s(y)\,\mathrm{d}y - \int_\eta^{x_l} s(y)\,\mathrm{d}y} = 1 - \frac{\int_{x_l}^x s(y)\,\mathrm{d}y}{\int_{x_l}^{x_r} s(y)\,\mathrm{d}y}\,.$$

Taking the limit as $x_l \downarrow 0$ yields

$$\mathbb{P}_x(\tau_0 < \tau_{x_r}) = 1 - \frac{\int_0^x s(y)\,\mathrm{d}y}{\int_0^{x_r} s(y)\,\mathrm{d}y}$$

where both integrals are finite since s is continuous on the compact $[0, x_r]$. For the same reason, we have $\lim_{x_r\uparrow\infty} \int_0^{x_r} s(y)\,\mathrm{d}y = \infty$. Note also that we already have from Section 3, $\lim_{x_r\uparrow\infty} \tau_{x_r} = \infty$, a.s. since explosion does not occur. The probability of ultimate extinction is then

$$\mathbb{P}_x(\tau_0 < \infty) = \lim_{x_r\uparrow\infty} \mathbb{P}_x(\tau_0 < \tau_{x_r}) = 1\,.$$

□

5 Complete Fokker-Planck Equation

We denote by:[1]

$$Q_t(\mathrm{d}y \,|\, x) \stackrel{\text{def}}{=} \mathbb{P}(X_{s+t} \in \mathrm{d}y | X_s = x)$$

the transition kernel of the Markov process $\{X_t\}_{t\geq 0}$ and by $\pi_t(\mathrm{d}y) = (\pi_0 Q_t)(\mathrm{d}y)$ the distribution of X_t. We note that $Q_t(\mathrm{d}y \,|\, x)$ is not absolutely continuous with respect to $\mathrm{d}y$, the Lebesgue measure on $\mathbb{R}_+$, because it gives positive probability to the boundary point 0. The Lebesgue decomposition of $Q_t(\cdot \,|\, x)$ into absolutely continuous and singular parts reads

$$Q_t(\mathrm{d}y \,|\, x) = E_t(x)\,\delta_0(\mathrm{d}y) + p_t(y \,|\, x)\,\mathrm{d}y\,. \tag{7}$$

The transition kernel $Q_t(\mathrm{d}y \,|\, x)$ is a probability measure for any $x \geq 0$, so that the *extinction probability* starting from x is

$$E_t(x) = 1 - \int_0^\infty p_t(y \,|\, x)\,\mathrm{d}y\,.$$

[1] Let K and K' be two transition kernels on $\mathbb{R}_+$. Throughout this paper, we use the following notations:

– left action on test function: $Kf(x) \stackrel{\text{def}}{=} \int_{\mathbb{R}_+} f(y)\,K(\mathrm{d}y \,|\, x)$,

– right action on measure: $(\nu K)(\mathrm{d}y) \stackrel{\text{def}}{=} \int_{\mathbb{R}_+} \nu(\mathrm{d}x)\,K(\mathrm{d}y \,|\, x)$,

The transition kernel $Q_t(\mathrm{d}y \mid x)$ is absolutely continuous with respect to the reference measure on $\mathbb{R}_+$

$$m(\mathrm{d}y) \stackrel{\text{def}}{=} \delta_0(\mathrm{d}y) + \mathrm{d}y$$

with density

$$q_t(y \mid x) \stackrel{\text{def}}{=} \begin{array}{ll} E_t(x)\,, & \text{if } y = 0\,, \\ p_t(y \mid x)\,, & \text{otherwise.} \end{array} \tag{8a}$$

We suppose also that the initial distribution π_0 is absolutely continuous with respect to the reference measure $m(\mathrm{d}y)$, and we let:

$$q_0(y) \stackrel{\text{def}}{=} \frac{\pi_0(\mathrm{d}y)}{m(\mathrm{d}y)} = \begin{cases} E_0\,, & \text{if } y = 0\,, \\ p_0(y)\,, & \text{otherwise.} \end{cases} \tag{8b}$$

We now establish the evolution equations for $E_t(x)$ and $p_t(y \mid x)$, for any $x > 0$ fixed. Note that for $x = 0$, $p_t(y, \mid x) = 0$ and $E_t(x) = 1$ for all $t \geq 0$ and all $y \in \mathbb{R}_+$. The Kolmogorov forward equation describes the evolution of Q_t in a weak sense:

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_t f = Q_t(\mathscr{A} f) = Q_t\left(b\, f' + \frac{1}{2} a\, f''\right), \quad \forall f \in \mathscr{C}_K^\infty(\mathbb{R}_+) \tag{9}$$

where $\mathscr{A}$ is the infinitesimal generator previously defined by Eq. 5 and $\mathscr{C}_K^\infty(\mathbb{R}_+)$ is the set of functions differentiable for all degrees of differentiation and with compact support included in $[0, +\infty)$. Using decomposition (7),

$$\begin{aligned} Q_t(\mathscr{A} f)(x) &= \mathscr{A} f(0) + \int_0^\infty b(y)\, p_t(y \mid x)\, f'(y)\, \mathrm{d}y + \frac{1}{2} \int_0^\infty a(y)\, p_t(y \mid x)\, f''(y)\, \mathrm{d}y \\ &= \int_0^\infty b(y)\, p_t(y \mid x)\, f'(y)\, \mathrm{d}y + \frac{1}{2} \int_0^\infty a(y)\, p_t(y \mid x)\, f''(y)\, \mathrm{d}y\,. \end{aligned}$$

Note that $\mathscr{A} f(0) = 0$ since both the drift and diffusion terms vanish at 0. A first integration by parts gives

$$\begin{aligned} \int_0^\infty b(y)\, p_t(y \mid x)\, f'(y)\, \mathrm{d}y &= [b(y)\, p_t(y \mid x)\, f(y)]_0^\infty - \int_0^\infty \frac{\partial (b\, p_t(\cdot \mid x))}{\partial y}(y)\, f(y)\, \mathrm{d}y \\ &= -\int_0^\infty \frac{\partial (b(y)\, p_t(y \mid x))}{\partial y}\, f(y)\, \mathrm{d}y\,, \end{aligned}$$

and similarly

$$\int_0^\infty a(y)\, p_t(y \mid x)\, f''(y)\, \mathrm{d}y = -\int_0^\infty \frac{\partial (a(y)\, p_t(y \mid x))}{\partial y}\, f'(y)\, \mathrm{d}y$$

by the same property. In the above integrals, the non-integral terms vanish at ∞ because $f \in \mathscr{C}_K^\infty(\mathbb{R}_+)$, but they vanish at 0 because $b(0) = a(0) = 0$. A second integration by parts gives

$$\begin{aligned} &-\int_0^\infty \frac{\partial (a(y)\, p_t(y \mid x))}{\partial y}\, f'(y)\, \mathrm{d}y \\ &\quad = -\left[\frac{\partial (a(y)\, p_t(y \mid x))}{\partial y}\, f(y)\right]_0^\infty + \int_0^\infty \frac{\partial^2 (a(y)\, p_t(y \mid x))}{\partial y^2}\, f(y)\, \mathrm{d}y \\ &\quad = \left.\frac{\partial (a(y)\, p_t(y \mid x))}{\partial y}\right|_{y=0} f(0) + \int_0^\infty \frac{\partial^2 (a(y)\, p_t(y \mid x))}{\partial y^2}\, f(y)\, \mathrm{d}y\,. \end{aligned}$$

We define $\mathscr{A}^*$ the formal adjoint operator of $\mathscr{A}$ acting on the "forward" space variable y only by

$$\mathscr{A}^* p_t(y \mid x) = -\frac{\partial[b(y)\, p_t(y \mid x)]}{\partial y} + \frac{1}{2}\frac{\partial^2[a(y)\, p_t(y \mid x)]}{\partial y^2},$$

and we finally have the decomposition

$$Q_t(\mathscr{A} f)(x) = \frac{1}{2}\left.\frac{\partial(a(y)\, p_t(y \mid x))}{\partial y}\right|_{y=0} f(0) + \int_0^\infty \mathscr{A}^* p_t(y \mid x)\, f(y)\, \mathrm{d}y\,. \tag{10}$$

In view of Eq. 9, the first term of this decomposition has a nice interpretation: it is the rate of increase of the extinction probability at time t, expressed as a probability flux through the boundary 0 (up to a minus sign). Indeed, considering test functions f_ϵ, such that $f_\epsilon(0) = 1$, $f_\epsilon(y) = 0$ for $y \geq \epsilon$ and with first two derivatives vanishing at 0, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[E_t(x) + \int_0^\infty p_t(y|x) f_\epsilon(y)\, \mathrm{d}y\right] = \frac{1}{2}\left.\frac{\partial(a(y)\, p_t(y \mid x))}{\partial y}\right|_{y=0} + \int_0^\infty \mathscr{A}^* p_t(y|x) f_\epsilon(y)\, \mathrm{d}y\,.$$

The integrals vanish as $\epsilon \downarrow 0$ so that we obtain the differential equation satisfied by $E_t(x)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}E_t(x) = \frac{1}{2}\left.\frac{\partial\big(a(y)\, p_t(y \mid x)\big)}{\partial y}\right|_{y=0} p_t(0 \mid x)\,, \quad E_0(x) = 0\,. \tag{11a}$$

On the other hand, the Fokker–Planck equation for the absolutely continuous part $p_t(\cdot \mid x)$ is obtained by considering test functions vanishing at 0 in Eq. 10:

$$\frac{\partial p_t(y \mid x)}{\partial t} = \mathscr{A}^* p_t(y \mid x)\,, \quad \lim_{t \downarrow 0} p_t(y \mid x)\, \mathrm{d}y = \delta_x(\mathrm{d}y) \tag{11b}$$

which is a PDE in a classical sense describing the evolution of the process *before extinction*. It follows that $y \mapsto p_t(y \mid x)$ is the density of a *defective distribution*. This equation has been extensively studied by Feller (1952). A notable result of the latter work is that no boundary condition at 0 is required for Eq. 11b to have a unique solution in $L^1(0, \infty)$, the space of equivalence classes of Lebesgue integrable functions from $(0, \infty)$ to $(0, \infty)$. The regularity of the density is inherited from that of the coefficients b and a, see Friedman (1964). In Chapters 5 and 6 of Schuss (2010), the multidimensional case is investigated.

Remark 2 According to Lemma 2, $E_t(x)$ increases to 1, so that $Q_t(\cdot \mid x)$ will eventually degenerate to the Dirac mass at 0. We note that this convergence may be slow, i.e. that the contribution of the Dirac mass in Eq. 7 may not be significant for the time scale at which the system is observed. This feature is investigated in Grasman and van Herwaarden (1999). Related to this point, we can express the density of the process conditioned to non-extinction from both parts of $Q_t(\cdot \mid x)$ by

$$y \longmapsto \bar{p}_t(y \mid x) = \frac{p_t(y \mid x)}{1 - E_t(x)}\,.$$

Even if p_t decreases to 0 as t grows, its normalized version $\bar{p}_t$ might have a non-trivial limit that do not depend on x any more, called a Yaglom limit of the process. Yaglom limits are quasi-stationary distributions, that is stationary distributions for the process conditioned to non-extinction. An extensive bibliography on this subject can be found in Pollett (2014).

6 Finite Difference Approximation of the Fokker-Planck Equation

Notice that the density searched for consists of two distinctive parts. The continuous component p_t can be approximated independently whereas the discrete component E_t strongly depends on p_t. This suggests that we must first design an approximation to p_t from which the approximation of E_t can be deduced. In all cases, any acceptable approximation should be a probability density.

Numerical approximations of Eq. 11b can be obtained by classical methods of numerical analysis of PDEs, paying attention to the specific features of our model. Indeed, any appropriate discretization scheme should correctly handle the degeneracy (vanishing diffusion) at 0. Also the approximated solution should remain non negative and integrate to at most 1, since it approaches a defective probability distribution. Finally, the mass default must be a consistent approximation of Eq. 11a. The approach presented in Kushner and Dupuis (1992) seems natural in this context, because it allows a straightforward interpretation of the discretized operator in terms of generator of a Markov process. See Campillo et al. (2014) for such a discretization method applied to a two–dimensional model.

We discretize the space as a regular grid:

$$x_\ell = \ell\, h\,, \quad \ell = 0, \ldots, L$$

for h and L given. Note that this grid is finite so that it does not cover the whole support of $p_t(\cdot \mid x)$. In numerical experiments, the range of the grid will have to be large enough so that any artificial boundary condition imposed at x_L will cause limited harm. In practice, any computation leading to a density that differs significantly from 0 should be discarded and repeated with larger L.

More importantly, the boundary point 0 has a twofold status; as the node x_0 of the grid, it enters the computation of the continuous component $p_t(0 \mid x)$ and as an absorbing state, it carries the extinction probability E_t. It is thus legitimate to introduce an additional *cemetery point* Υ at location 0. Indeed, such a decomposition of the point 0 gives the expected smoothness of p at the boundary observed on Fig. 3.

We now derive the finite difference approximation of the continuous part p_t, returning to the weak formulation. For suitable test function ϕ,

$$\begin{aligned}\int_0^\infty p_t(y \mid x)\, \mathscr{A}\phi(y)\, \mathrm{d}y \;\simeq\; & \frac{h}{2}\, p_t(0 \mid x)\, \mathscr{A}\phi(0) + h \sum_{\ell=1}^{L-1} p_t(x_\ell \mid x)\, \mathscr{A}\phi(x_\ell) \\ & + \frac{h}{2}\, p_t(x_L \mid x)\, \mathscr{A}\phi(x_L) \simeq \sum_{\ell=0}^{L} P_t(l)\, \mathscr{A}\phi(x_\ell)\end{aligned}$$

with

$$P_t(0) \simeq \frac{h}{2}\, p_t(0 \mid x), \quad P_t(L) \simeq \frac{h}{2}\, p_t(x_L \mid x), \quad P_t(\ell) \simeq h\, p_t(x_\ell \mid x), \;\; \text{for } 0 < \ell < L\,. \tag{12}$$

We also need to define $P_t(\Upsilon) \simeq E_t(x)$. When designing our approximation, we expect $P_t(\cdot)$ to be a discrete probability distribution on $\{\Upsilon, x_0, \ldots, x_L\}$. The differential operator $\mathscr{A}$ is

now replaced by its finite difference approximation, denoted A, using an up–wind scheme, which reads for an interior point x_ℓ with $1 \le \ell \le L-1$:

$$\phi'(x_\ell) \simeq \begin{cases} \dfrac{\phi(x_{\ell+1}) - \phi(x_\ell)}{h}, & \text{if } b(x_\ell) \ge 0\,, \\ \dfrac{\phi(x_\ell) - \phi(x_{\ell-1})}{h}, & \text{if } b(x_\ell) < 0\,, \end{cases}$$
$$\phi''(x_\ell) \simeq \frac{\phi(x_{\ell+1}) - 2\,\phi(x_\ell) + \phi(x_{\ell-1})}{h^2}\,.$$

The resulting approximation can be written as

$$\mathscr{A}\,\phi(x_\ell) \simeq A_{\ell,\ell-1}\,\phi(x_{\ell-1}) + A_{\ell,\ell}\,\phi(x_\ell) + A_{\ell,\ell+1}\,\phi(x_{\ell+1}), \quad \ell = 1, \ldots, (L-1)\,,$$

with

$$\forall\, \ell = 1, \ldots, L-1, \quad A_{\ell,\ell-1} = \frac{b^-(x_\ell)}{h} + \frac{a(x_\ell)}{2\,h^2}\,,$$
$$A_{\ell,\ell} = -\frac{|b(x_\ell)|}{h} - \frac{a(x_\ell)}{h^2}\,,$$
$$A_{\ell,\ell+1} = \frac{b^+(x_\ell)}{h} + \frac{a(x_\ell)}{2\,h^2}\,.$$

Appropriate boundary conditions at x_0 and x_L will be given later on. It is enlightening to interpret this operator A as the infinitesimal generator of a pure jump Markov process on the grid $(\Upsilon, x_0, \ldots, x_L)$. Indeed, the extra–diagonal terms of A, considered as a matrix, are non–negative and the sum on each row is 0. $P_t(\ell)$ is then the probability that this process occupies site x_ℓ at time t. From an interior point x_ℓ, this process jumps to one of its neighbors with a bias directed according to the drift. This interpretation suggests how to complete the three lines of A not yet defined. The right boundary is simple: in order for the jump process to remain on the grid, its behavior at boundary x_L has to be prescribed artificially. There is no canonical choice between absorbing or reflecting boundary condition, since both corrupt the theoretical behavior.

We choose the reflecting boundary condition at x_L that reads:

$$A_{L,L-1} = \frac{|b(x_L)|}{h} + \frac{a(x_L)}{h^2}, \quad A_{L,L} = -\frac{|b(x_L)|}{h} - \frac{a(x_L)}{h^2}\,. \tag{13}$$

The sum on the last row is 0, so that there is no probability leak at boundary x_L. The boundary condition at 0 requires more care. We set all coefficient of the first line to 0, since it corresponds to the absorbing state Υ. We introduce the notation $\mathbf{P}_t = (P_t(\ell))_{\ell=\Upsilon,0,\ldots,L}$ for the law of the jump process at time t starting from x. This probability distribution solves the Fokker–Planck equation for jump processes that reads

$$\dot{\mathbf{P}}_t = A^*\mathbf{P}_t\,. \tag{14}$$

Observe that the first ODE of system (14) is

$$\dot{P}_t(0) = A_{0,0}\,P_t(0) + A_{1,0}\,P_t(1)$$

where

$$A_{1,0} = \frac{b^-(h)}{h} + \frac{a(h)}{2\,h^2}\,.$$

Using Eq. 12, this gives an approximation

$$\frac{\partial p_t(0\,|\,x)}{\partial t} \simeq \left(2\,\frac{b^-(h)}{h} + \frac{a(h)}{h^2}\right)\, p_t(h\,|\,x) + A_{0,0}\, p_t(0\,|\,x)\,. \tag{15}$$

An analogy with interior points suggests to set $A_{0,0}$ such that Eq. 15 is a finite difference approximation for $\lim_{y\downarrow 0} \mathscr{A}^* p_t(y\,|\,x)$:

$$\lim_{y\downarrow 0} \mathscr{A}^* p_t(y\,|\,x) \;=\; -b'(0)\, p_t(0\,|\,x) + \frac{1}{2}\, a''(0)\, p_t(0\,|\,x) + \frac{\partial p_t(y\,|\,x)}{\partial y}\Big|_{y=0} a'(0)\,.$$

This limit involves only the first derivative of $p_t(y\,|\,x)$ due to the vanishing diffusion. With

$$\frac{\partial p_t(y\,|\,x)}{\partial y}\Big|_{y=0} \simeq \frac{p_t(h\,|\,x) - p_t(0\,|\,x)}{h}$$

we obtain the approximation

$$\frac{\partial p_t(0\,|\,x)}{\partial t} \simeq p_t(0\,|\,x)\left[-b'(0) + \tfrac{1}{2}\, a''(0) - \tfrac{1}{h}\, a'(0)\right] + p_t(h\,|\,x)\tfrac{1}{h}\, a'(0)\,. \tag{16}$$

Using

$$a(h) = h\, a'(0) + h^2\, \frac{1}{2}\, a''(0)$$

in Eq. 15 we get

$$\frac{\partial p_t(0\,|\,x)}{\partial t} \simeq 2\,\frac{b^-(h)}{h}\, p_t(h\,|\,x) + p_t(h\,|\,x)\frac{1}{h}\, a'(0) + \frac{1}{2}\, a''(0)\, p_t(h\,|\,x) + A_{0,0}\, p_t(0\,|\,x)\,.$$

Also, $p_t(h\,|\,x) \simeq p_t(0\,|\,x)$ gives

$$\frac{\partial p_t(0\,|\,x)}{\partial t} \simeq p_t(0\,|\,x)\left[2\,\frac{b^-(h)}{h} + \frac{1}{2}\, a''(0) + A_{0,0}\right] + p_t(h\,|\,x)\frac{1}{h}\, a'(0)\,.$$

Now since

$$b^-(h) = \frac{|b(h)| - b(h)}{2} \quad \text{and} \quad b(h) \simeq h\, b'(0)$$

we finally have

$$\frac{\partial p_t(0\,|\,x)}{\partial t} \simeq p_t(0\,|\,x)\left[|b'(0)| - b'(0) + \frac{1}{2}\, a''(0) + A_{0,0}\right] + p_t(h\,|\,x)\frac{1}{h}\, a'(0)\,.$$

In order to have an approximation of Eq. 16, we must set

$$A_{0,0} = -|b'(0)| - \frac{1}{h}\, a'(0)\,. \tag{17}$$

This diagonal term of A is non–negative as expected. We see that the state 0 of the jump process is not absorbing since $A_{0,0} \neq 0$, but act as a transition state towards extinction Υ. Since there is no reason to allow a jump to an interior point, we also set

$$\forall\, \ell = 1, \ldots L \quad A_{0,\,\ell} = 0, \quad \text{and} \quad A_{0,\Upsilon} = -A_{0,0}\,. \tag{18}$$

Observe that, from Eq. 14 the probability of extinction $P_t(\Upsilon)$ satisfies the evolution equation

$$\dot{P}_t(\Upsilon) = -A_{0,0}\, P_t(0) = \left(\frac{1}{2}\, a'(0) + \frac{h}{2}\, |b'(0)|\right)\, p_t(0\,|\,x)\,.$$

When $h \downarrow 0$, this equation is consistent with Eq. 11a which gives the rate of extinction. Notice that $A_{0,0}$ could have been chosen so that the above equation exactly matches Eq. 11a, but then Eqs. 15 and 16 would not match so closely. Finally, the complete set of boundary conditions is given by Eqs. 13, 17 and 18.

Remark 3 It could be argued that distinguishing the cemetery point Υ from 0 is not necessary, so that boundary condition (17) is useless. Indeed, the first interior point x_1 receives no probability flux from x_0. Therefore, imposing an absorbing boundary condition at x_0 would give the same values for the other points of the grid. Doing so would reduce to considering Fokker–Planck equation with absorbing boundary condition instead of complete Fokker–Planck equation. However the value $P_t(0)$ would now be the extinction probability, which is completely distinct from the value of the density at x_0 that we still need, e.g. for parameter inference.

For a final instant $T > 0$, Eq. 14 is discretized in time using the Euler implicit scheme on the interval $[0, T]$ by

$$[I - \delta A]^* \tilde{\mathbf{P}}_{t_{k+1}} = \tilde{\mathbf{P}}_{t_k}, \quad k = 0, \ldots, n-1$$

where $t_k \stackrel{\text{def}}{=} k\,\delta$ with $\delta = T/n$, n given. The initial condition is approximated by

$$\tilde{P}_{t_0}(l) = \begin{cases} 1, & \text{if } \ell = \ell_0, \\ 0, & \text{otherwise,} \end{cases}$$

where x_{ℓ_0} is the nearest neighbor in the grid of the initial condition x. According to Eq. 12, the numerical solution $\tilde{\mathbf{P}}_T$ yields a numerical approximation $\tilde{p}_T(x_\ell \,|\, x)$ for the density at a grid point, that can be linearly interpolated to obtain an approximation $\tilde{p}_T(y \,|\, x)$ for $0 \leq y \leq x_L$. The likelihood function is then approximated by

$$q_T(x, y) \simeq \begin{cases} P_T(\Upsilon), & \text{if } y = 0, \\ \tilde{p}_T(y \,|\, x), & \text{if } y \in]0, x_L]. \end{cases}$$

Remark 4 This discretization scheme is unconditionally stable, but h and δ have to be chosen in a coherent way. Indeed, $-A(\ell, \ell)$ gives the expectation of the holding time of the pure jump Markov process. We see that the order of magnitude of the holding time is $\frac{1}{h^2}$. The time step δ should then be chosen small enough to ensure that not too many jumps occur within an interval of length δ.

The numerical treatment of the Fokker–Planck equation in the degenerate case has already been considered in the numerical analysis, see for example Cacio et al. (2011). The approach adopted in this work retains the probabilistic meaning of the objects involved, at the cost of a possible loss of accuracy.

7 Numerical Experiments

We first explore empirically SDE Eq. 3 by simulating N trajectories. The simplest algorithm is the Euler–Maruyama scheme, restricted to non–negative values, that is

$$\bar{X}_{t_{k+1}} = \max\left(0, \bar{X}_{t_k} + \delta\, b(\bar{X}_{t_k}) + \sqrt{\delta}\,\sigma(\bar{X}_{t_k})\, w_k\right), \quad k = 0, \ldots, n-1 \tag{19}$$

with $\bar{X}_0 = x$ and where $(w_k)_{0,\ldots,n-1}$ are i.i.d. $\mathcal{N}(0, 1)$. Figure 1 shows the results for the parameter sets given by Table 1. These values are chosen so as to observe a short transient phase where the population grows quickly, starting from a small initial value, and a stationary phase with a high noise intensity. Within the time interval of the simulation, we observe

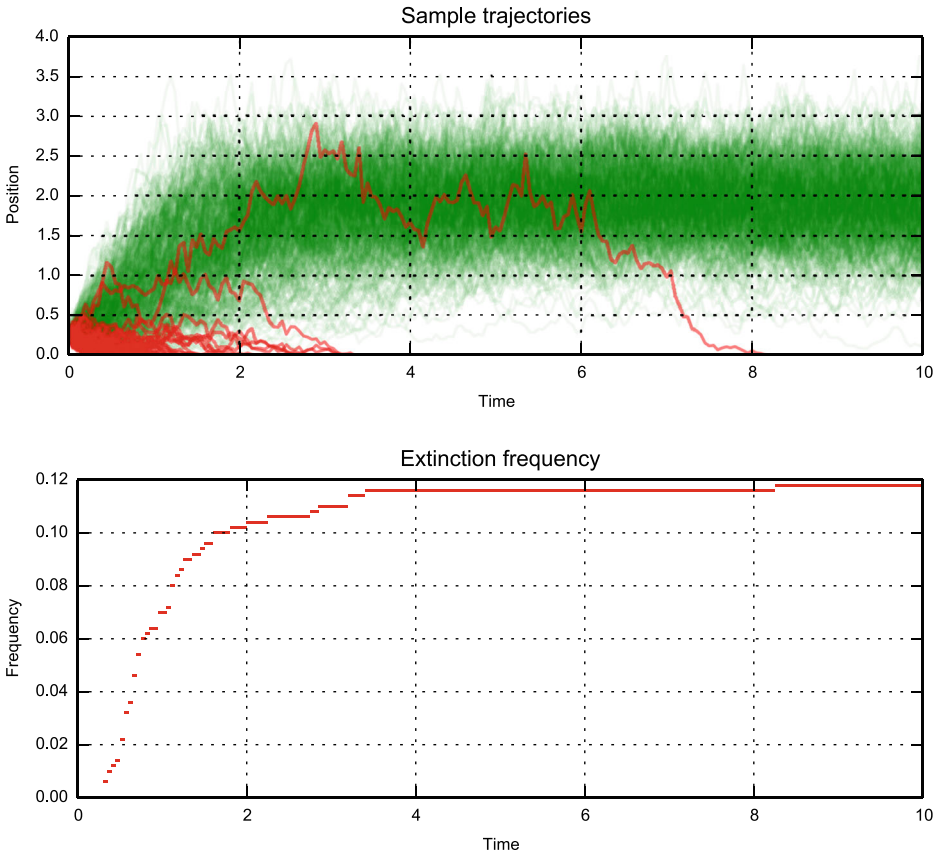

Fig. 1 On the *top plot*: simulation of $n = 500$ trajectories according to dynamics (3). Trajectories in red become extinct before final time. One of them completes the transient phase. On the *bottom plot*, the corresponding extinction frequency

the three possible behaviors:

- Early extinction during the transient phase. This stems from the fact that the initial level of the population is small and the noise intensity is high.
- Completed transient phase followed by noisy fluctuations around a natural carrying capacity. Even if it is known that all trajectories will eventually reach 0, all of them but one survive on the short term.
- Late extinction after reaching the stationary phase. Only one trajectory is concerned.

Table 1 Parameters

Model						Simulation		
λ	μ	α	ρ	x_0	T	h	δ	N
20	18	1	10^{-1}	0.25	10	10^{-3}	10^{-3}	500

Due to early extinction, the estimated extinction probability grows quickly in the first part, and seems to approach an asymptotic value. Running the simulation on a much larger period of time will result in an extinction frequency reaching 1 slowly.

Remark 5 The Euler-Maruyama scheme, although widely used for its simplicity suffers some well known drawbacks. Higher order schemes such as the Milstein scheme, see e.g. Kloeden and Platen (1992) can be considered if one is concerned with numerical accuracy. The recent *Exact Algorithm* (EA) (Beskos and Roberts 2005) is also of interest. In any case, these alternative algorithms should also be modified to handle correctly the extinction.

We now evaluate the numerical performance of the finite difference method on the scenario defined by the same model parameters. Figure 2 shows a contour plot of the numerical solution of Fokker–Planck (11) obtained by implementing the finite difference scheme of Section 6. Simulation parameters are also given by Table 1. This Figure is to be compared with Fig. 1. As expected, the first two moments quickly move to stationary values, after a

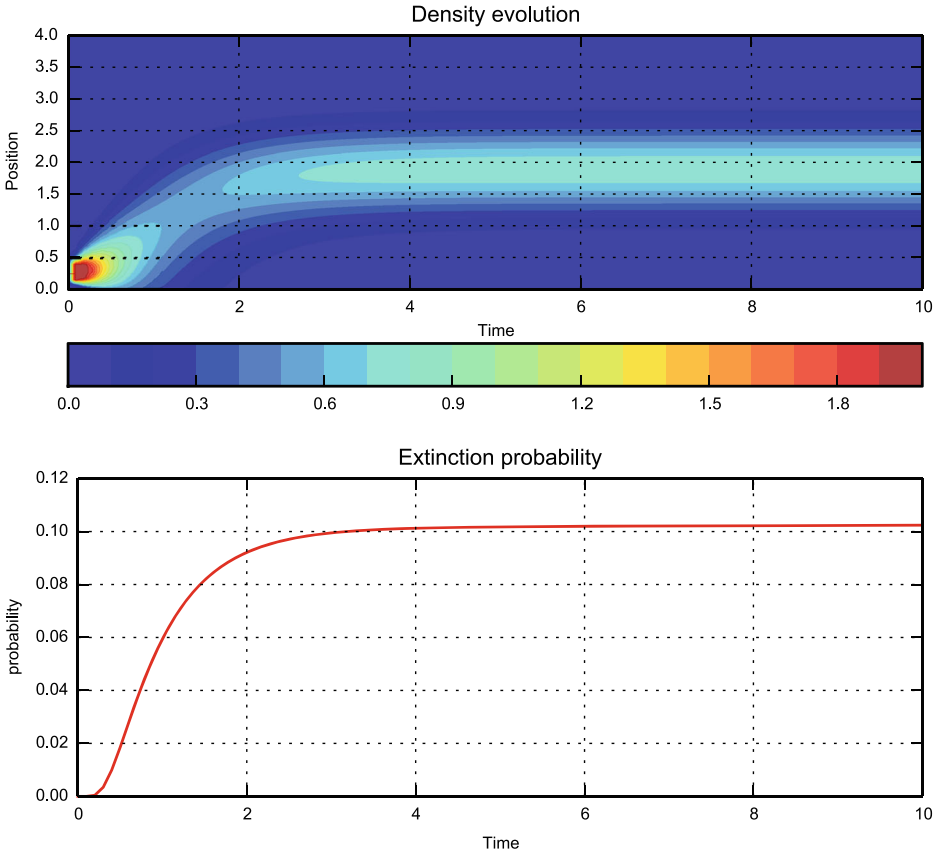

Fig. 2 Finite difference approximation of Fokker–Planck (11). Diffusion induces a quick loss of mass in the transient phase. The loss of mass is then much slower

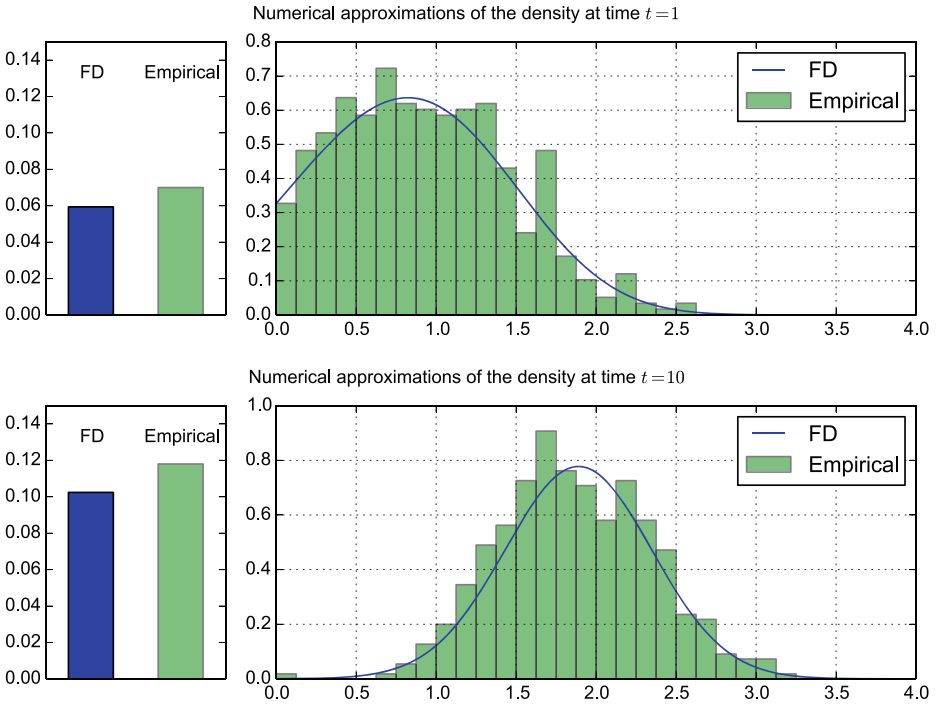

Fig. 3 Numerical solution of the complete Fokker–Planck equation and empirical distribution at time $t = 1$ (*top row*) and $t = 10$ (*bottom row*). First plot is within the transient phase. The extinction probability has already reached a high level, but keeps on growing. The high values of the density near the absorbing point 0 induces a heavy flow of probability mass through this point, increasing the extinction probability. *Bottom row plot* is within the stationary phase, with a probability mass loss much slower. For both pictures, the empirical density closely match the finite difference approximation

transient phase where a large amount of mass is lost. On the long term, the mass keeps on decreasing slowly, although the latter fact is not clearly readable.

This solution is then plotted at particular instants $t = 1$ and $t = 10$ on Fig. 3. As long as the stationary regime has not yet been reached (top row), the defective density takes high values near frontier point 0. This induced a probability flow towards 0 and increases the extinction probability as expected. The extinction probability grows much more slowly within the stationary regime (bottom row), since the probability flow through boundary point 0 is low.

8 Concluding Remarks

Growth models featuring a logistic mechanism are numerous in the literature for population that are subject to limitations imposed by the environment. Choosing the most appropriate depends strongly on the application one is interested in. We now summarize the main features of the model proposed in this paper. Firstly, it is obtained from an individual jump model by a diffusion approximation principle. This may be viewed as a mesoscopic scale description, between microscopic (individual based) scale and macroscopic scale, the latter being described mainly by ODEs. The diffusion coefficient follows naturally from the

original jump process and is not chosen in an ad hoc way. Consequently, the birth and death rates appear, under a different form, in both drift and diffusion coefficient. This is a key point allowing identification of these rates separately, and not only their difference. Secondly, extinction occurs almost surely in final time. Even if this event happens in a late future, this feature may be essential for some applications. For example, considering this model is appropriate for data sets in which extinction does occur for some trajectories.

The present work actually originates from a parameter estimation problem, when the processes is observed in discrete time. Most likelihood based methods for parameter inference for diffusion processes generally require that the law of the diffusion at time $t > 0$ has a density w.r.t. the Lebesgue measure. This is obviously not the case for processes being absorbed at 0 in final time. This law is nevertheless dominated by the sum of the Lebesgue measure and the Dirac mass at 0. In this paper, we found that its density is given by the solution of the complete Fokker–Planck equation. From this solution, a likelihood function can be built. Beside, since a further work is needed to conduct parameter inference, the study of the discretization methods presented here was limited to numerical experiments.

Finally, the approach presented here is not specific to the logistic case but can also be considered for models for which one or both boundaries are exit points. A corresponding complete Fokker–Planck equation can be derived and numerically solved with the appropriate boundary conditions.

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