# Effective diffusion in vanishing viscosity*

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**Abstract**

We study the asymptotic behavior of effective diffusion for singular perturbed elliptic operators with potential first order terms. Assuming that the potential is a random perturbation of a fixed periodic function and that this perturbation does not affect essentially the structure of the potential, we prove the exponetial decay of the effective diffusion. Moreover, we establish its logarithmic asymptotics in terms of proper percolation level for the random potential.

**Keywords** homogenization, effective diffusion, singular perturbed operators, logarithmic asymptotics, random potential, percolation theory

In the present article we consider the asymptotics of the effective diffusion for elliptic operators with vanishing diffusion and with potential first order terms, the potential being a statistically homogeneous field.

The homogenization problems for singular perturbed operators have many important applications, among them fluid dynamics in porous media Bear [5] or groundwater pollution Fried [11]. In the recent years, various such questions were considered in detail for the operator with divergence–free vector fields. Many interesting asymptotics were constructed for the periodic case — Bensoussan–Lions–Papanicolaou [7], Fannjiang–Papanicolaou [9]) — and then in the random case — Avellaneda–Majda [1, 2, 4], Carmona–Xu [8].

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In contrast to divergence–free case, where the effective diffusion is usually much greater than the initial one (see, for instance, Fannjiang–Papanicolaou [9]), we typically have in case of potential vector field the exponential decay of the effective coefficients.

For operators with periodic coefficients this phenomenon was investigated in Kozlov [14] and Kozlov–Piatnitski [15] where the logarithmic asymptotics of effective coefficient was found in terms of "Morse properties" of the potential on the torus of periodicity.

The operator with first order terms which are not potential but show in a way similar behavior, were considered in Kozlov–Piatnitski [16], where the logarithmic asymptotics of the effective diffusion was established.

In the present work, we study a particular case of operators with random potential first order terms. Namely, we assume that the potential is a random perturbation of a given periodic function. Considering this random perturbation, we assume that it does not change essentially the topological structure of the initial potential. This allows us to use the results from the percolation theory and to find the required asymptotics in terms of the proper percolation levels.

All the exact assumptions are provided in Section 1. Then, in Section 2, we prove the general result on asymptotic behavior of homogenized coefficients. One of the key condition of this statement is non-explicit. In Section 3 we present a couple of sufficient conditions expressed in explicit terms.

# 1 The setup

Let us consider a potential on $\mathbb{R}^2$ (with orthonormal basis $\{e_1, e_2\}$) of the form $U = U_0 + U_1$ where $U_0$ is a deterministic smooth potential which is supposed to be periodic with with period 1 in each coordinate directions. We denote the cell of periodicity $[0,1]^2$ by $\square$ and identify it with the 2D taurus $\mathcal{T}$; $U_1$ is an isotropic random field, it represents a small random perturbation of $U_0$.

If $\mathsf{S}$ denotes the rotation matrix of angle $\pi/2$, we suppose that:

$(i)$ $U_0(\mathsf{S}\,x) = U_0(x)$ for all $x \in \mathcal{T}$,

$(ii)$ the distribution of $U_1$ is invariant with respect to any integer shift of $\mathbb{R}^2$ and to $S$: $\mathrm{law}(U_1(\mathsf{S}\,x)) = \mathrm{law}(U_1(x))$ for all $x \in \mathcal{T}$,

$(iii)$ there exists $\gamma_0 > 0$ such that $|U_1(x)| \le \gamma_0$, for all $x \in \mathbb{R}^2$, a.s.

The two first conditions ensure the isotropy of the effective media.

Under condition $(i)$, the potential $U_0$ has a specific structure: in the simplest case — other cases rely on the same arguments — the minimum number of degenerate points that $U_0$ could admit on $\mathcal{T}$, i.e. points $x$ such that $\nabla U_0(x) = 0$, is four: one minimum point $x_{\mathrm{min}}$, one maximum point $x_{\mathrm{max}}$, and two saddle points $x_{\mathrm{s}}$.

In $\mathbb{R}^2$ minimum points, maximum points, and saddle points will be denoted $x_{\mathrm{min}}$, $x_{\mathrm{max}}$, and $x_{\mathrm{s}}$ respectively.

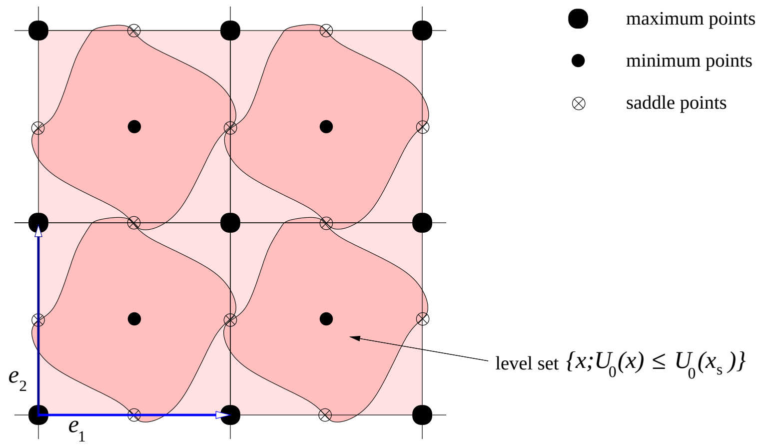


Figure 1: Example of structure for the potential $U_0$.

Without loss of generality we may assume that in $\mathbb{R}^2$, the set of maximum points is $\mathsf{X}_{\mathrm{max}} = \mathbb{Z}^2$, then the set of minimum points should coincide with $\mathsf{X}_{\mathrm{min}} = \mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$, and the set of saddle points with $\mathsf{X}_{\mathrm{s}} = \mathbb{Z}^2 + (0, \frac{1}{2}) \cup \mathbb{Z}^2 + (\frac{1}{2}, 0)$, (see Figure 1).

The case of a more general potential $U_0(x)$, having more singular points, including minimum points, could be treated with the same reasoning.

We make the following hypothesis:

**Hypothesis 1.1** *For all minimum point $x_{\mathrm{min}}$ (the set of minimum points is $\mathsf{X}_{\mathrm{min}}$), let us consider:*

$$\alpha(x_{\mathrm{min}}, e_i) = \inf_{X(\cdot) \in \mathcal{X}(x_{\mathrm{min}}, e_i)} \sup_{0 \leq t \leq 1} U(X(t)) \,, \; i = 1, 2 \,, \tag{1}$$

*where $\mathcal{X}(x_{\mathrm{min}}, e_i)$ is the set of functions $[0,1] \ni t \mapsto X(t) \in \mathbb{R}^2$ such that $X(0) = x_{\mathrm{min}}$, $X(1) = x_{\mathrm{min}} + e_i$, and which are homotetic to $[0,1] \ni t \mapsto X_0(t) \ni \mathbb{R}^2 \setminus \mathsf{X}_{\mathrm{max}}$ defined as follows: $X_0(t) = x_{\mathrm{min}} + t\, e_i$.*

*We suppose that $\{\alpha(x_{\min}, e_i)\,;\, x_{\min} \in \mathsf{X}_{\min},\ i = 1, 2\}$ is a family of independent random variables.*

This last assumption is rather non explicit, a couple of sufficient conditions that ensure the above independence are supplied in Section 3.

Consider the following homogenization problem:

$$\begin{cases} \mu\, \Delta u^\varepsilon(x) + \frac{1}{\varepsilon}\ \nabla_z[-U(z)]\big|_{z=\frac{x}{\varepsilon}} \cdot \nabla_x u^\varepsilon(x) = f(x)\,,\ x \in Q\,, \\ u^\varepsilon \in H_0^1(Q) \end{cases} \tag{2}$$

for some bounded domain $Q$ in $\mathbb{R}^2$.

We assume first that the viscosity parameter $\mu$ is small and fixed, and pass to the limit as $\varepsilon \downarrow 0$. Then we study the asymptotic behavior of the effective (scalar) diffusivity $\sigma(\mu)$ as $\mu \downarrow 0$.

## 2 Effective diffusion

In this section we show that, under Hypothesis 1.1, the logarithmic limit of the effective diffusivity $\sigma(\mu)$ can be found in terms of so called *fiber percolation level* (see xxxx) of the potential $U$.

After standard transformations, Equation (2) reads:

$$\mu\, e^{U(\frac{x}{\varepsilon})/\mu} \sum_{i=1,2} \frac{\partial}{\partial x_i}\left(e^{-U(\frac{x}{\varepsilon})/\mu}\,\frac{\partial}{\partial x_i} u^\varepsilon(x)\right) = f(x)$$

(from now on we will always suppose that $u^\varepsilon \in H_0^1(Q)$).

Multiply each term of this last equation by $e^{-U(\frac{x}{\varepsilon})/\mu}$ so that:

$$\mu \sum_{i=1,2} \frac{\partial}{\partial x_i}\left(e^{U(\frac{x}{\varepsilon})/\mu}\,\frac{\partial}{\partial x_i} u^\varepsilon(x)\right) = e^{-U(\frac{x}{\varepsilon})/\mu}\, f(x)\,. \tag{3}$$

Without loss of generality, we can assume that:

$$\operatorname{ess}\inf_{x\in\mathbb{R}^2} U(x) = 0\,. \tag{4}$$

Then, for any $\mu > 0$, $e^{-U(\frac{\cdot}{\varepsilon})/\mu} f \rightharpoonup \beta(\mu)\, f$ weakly in $L^2(Q)$ as $\varepsilon \downarrow 0$, where:

$$\beta(\mu) \triangleq \int_\square e^{-U(\frac{x}{\varepsilon})/\mu}\, dx\,.$$

Moreover $\mu \log \beta(\mu) \to 0$ as $\mu \downarrow 0$.

For each $\mu$, the family of operators appearing in (3) is coercive, uniformly in $\varepsilon$. Thus, it suffices to homogenize the following PDE:

$$\mu \sum_{i=1,2} \frac{\partial}{\partial x_i} \left( e^{U(\frac{x}{\varepsilon})/\mu} \frac{\partial}{\partial x_i} v^\varepsilon(x) \right) = \beta(\mu)\, f(x)\,. \tag{5}$$

Then, $v^\varepsilon \sim u^\varepsilon$ as $\varepsilon \downarrow 0$ (in the sense that these functions have the same limit in $H_0^1(Q)$ as $\varepsilon \downarrow 0$). Clearly, we can omit both factors $\mu$ and $\beta(\mu)$ for a while; we end up with the equation:

$$\sum_{i=1,2} \frac{\partial}{\partial x_i} \left( e^{U(\frac{x}{\varepsilon})/\mu} \frac{\partial}{\partial x_i} v^\varepsilon(x) \right) = f(x)\,. \tag{6}$$

According to Jikov et al [12], under above conditions, Equation (6) admits the effective diffusion matrix $\Sigma(\mu)$ which is isotropic: $\Sigma(\mu) = \sigma(\mu)\, I$ and the scalar effective diffusion coefficient is supplied by the following variational problem:

$$\sigma(\mu) = \liminf_{\varepsilon \downarrow 0} \inf_{\substack{v \,\in\, H^1(\square) \\ v(0,\cdot) \,\equiv\, 0 \\ v(1,\cdot) \,\equiv\, 1}} \int_\square e^{U(\frac{x}{\varepsilon})/\mu} \, |\nabla v(x)|^2 \, dx\,. \tag{7}$$

## *Lower bound*

Let $\square_1$ and $\square_2$ two neighbor cells, let say that $\square_2 = \square_1 + e_1$, and $x^1_{\min} \in \square_1$, $x^2_{\min} \in \square_2$ the corresponding minimum points of $U_0$. We introduce the following random open set:

$$G^-_\eta \triangleq \left\{ x \in \mathbb{R}^2 \ : \ U(x) < \eta \right\}$$

and the events:

- $A^0_\eta$ : the set of $\omega$ such that there is a path connecting $x^1_{\min}$ and $x^2_{\min}$ which belongs to $\mathcal{X}(x^1_{\min}, e_1)$ and which is included in $G^-_\eta(\omega)$.
- $A^n_\eta$ : the set of $\omega$ such that there is a smooth curve from $\mathcal{X}(x^1_{\min}, e_1)$ of length not greater than $n$ such that its $\frac{1}{n}$–neighborhood is included in $G^-_\eta(\omega)$.

In our case, $A^0_\eta$ could be also defined as the event: the set of $\omega$ such that $G^-_\eta(\omega) \cap (\square_1 \cup \square_2)$ contains $x^1_{\min}$ and $x^2_{\min}$.

Clearly $\cup_{n>0} A^n_\eta = A^0_\eta$ and, hence:

$$\lim_{n \uparrow \infty} P(A^n_\eta) = P(A^0_\eta) \,. \tag{8}$$

It is also obvious that under our assumptions on the structure of $U$, the above events are independent for different pairs $(x^1_{\min}, x^2_{\min})$ of neighbor minimum points.

Consider standard bond percolation model using minimum points of $U_0$ as sites, and let $p_c$ be the critical probability of the appearance of the infinite cluster: $p_c = \frac{1}{2}$. We define the *critical value* $\eta_c$ as follows:

$$P(A^0_{\eta_c}) = \frac{1}{2} \,,$$

or, if such a $\eta_c$ does not exist:

$$\eta_c = \inf\{\eta\,;\, P(A^0_\eta) < \tfrac{1}{2}\} = \sup\{\eta\,;\, P(A^0_\eta) > \tfrac{1}{2}\} \,. \tag{9}$$

This last equality is, in fact, an additional assumption which is supposed to be fulfilled later on.

For all $\gamma > 0$ small enough, $P(A^0_{\eta_c+\gamma}) > \frac{1}{2}$. Thus, using (8), $P(A^n_{\eta_c+\gamma}) > \frac{1}{2}$ for sufficiently large $n$. We fix such a $n$ and denote it by $n^0$; we also denote $p^0 = P(A^{n^0}_{\eta_c+\gamma})$.

We say that a bond $(x^1_{\min}, x^2_{\min})$ is open if the corresponding $\omega$ belongs to $A^{n^0}_{\eta_c+\gamma}(x^1_{\min}, x^2_{\min})$.

As proved in Kesten [13], for almost all realizations and for all sufficiently large $N$, the square $[0,N]^2$ contains at least $c(p^0)\,N$ mutually non intersecting channels connecting left and right sides of the square. Finally, we arrive at the following conclusions:

> *For sufficiently large $N$, $[0,N]^2$ contains at least $c(p^0)\,N$ mutually non intersecting smooth $\frac{1}{n^0}$–pipes connecting left and right sides of the square such that along each of these pipes:*
>
> $$U(y) < \eta_c + \gamma \,. \tag{10}$$

Denote the above pipes by $Q_1, \ldots, Q_{k(N)}$, $k(N) \geq c(p^0)\, N$. Without loss of generality we assume that for any function $x \mapsto u(x)$ such that $u(0,x_2) \equiv 0$, $u(N,x_2) \equiv 1$ we have:

$$\int_{Q_m} \frac{\partial u(y)}{\partial \ell}\, dy \geq \frac{1}{2}\,\frac{1}{n^0}$$

(here $\ell$ is a variable directed along the pipe after rescaling). Indeed, taking a smooth pipe included in $Q_m$ and choosing, if necessary, a larger value of $n_0$, one can achieve the above lower bound.

After rescaling $x = \varepsilon\, y$, $\varepsilon = 1/N$, we find:

$$\int_{Q_m^\varepsilon} \frac{\partial u(x)}{\partial \ell}\, dx \geq \frac{1}{2}\,\frac{1}{n^0}\,\varepsilon \text{ with } Q_m^\varepsilon = \varepsilon\, Q_m\,.$$

By the Shwartz inequality:

$$\varepsilon^2\,\frac{1}{(n^0)^2}\,\frac{1}{4} \leq \left[\int_{Q_m^\varepsilon} \frac{\partial u(x)}{\partial \ell}\, dx\right]^2 \leq |Q_m^\varepsilon| \int_{Q_m^\varepsilon} |\nabla u(x)|^2\, dx\,.$$

Thus,

$$\int_{Q_m^\varepsilon} |\nabla u(x)|^2\, dx \geq \frac{\varepsilon}{4}\,\frac{1}{(n^0)^2}\,\frac{1}{c_1(p^0)}\,.$$

Summing up over $m$ leads to:

$$\sum_{m=1}^{k(N)} \int_{Q_m^\varepsilon} |\nabla u(x)|^2\, dx \geq c(p^0)\,\varepsilon\,\frac{\varepsilon}{4}\,\frac{1}{(n^0)^2}\,\frac{1}{c_1(p^0)} = \frac{c(p^0)}{c_1(p^0)}\,\frac{1}{(2\,n^0)^2}\,.$$

From (10), we have:

$$\begin{aligned}
\int_{\Box} e^{-U(\frac{x}{\varepsilon})/\mu}\, |\nabla u(x)|^2\, dx &\geq \sum_{m=1}^{k(N)} \int_{Q_m^\varepsilon} e^{-U(\frac{x}{\varepsilon})/\mu}\, |\nabla u(x)|^2\, dx \\
&\geq e^{-(\eta_c+\gamma)/\mu} \sum_{m=1}^{k(N)} \int_{Q_m^\varepsilon} |\nabla u(x)|^2\, dx \\
&\geq e^{-(\eta_c+\gamma)/\mu}\,\frac{c(p^0)}{c_1(p^0)}\,\frac{1}{(2\,n^0)^2}\,.
\end{aligned}$$

Using Definition (7) of $\sigma(\mu)$, and taking into account the fact that $\gamma$ is an arbitrary positive number, we obtain:

$$\liminf_{\mu \downarrow 0}\, \mu\, \log \sigma(\mu) \geq -\eta_c\,.$$

*Upper bound*

Let $\square_1$ and $\square_2$ two neighbor cells, let say that $\square_2 = \square_1 + e_1$, and $x^1_{\max} \in \square_1$, $x^2_{\max} \in \square_2$ the corresponding maximum points of $U_0$.

We introduce the random set:

$$G^+_\eta \triangleq \{x \in \mathbb{R}^2 \,;\, U(x) > \eta\}$$

and the events:

- $B^0_\eta$: the set of $\omega$ such that there is a path connecting $x^1_{\max}$ and $x^2_{\max}$ which belongs to $\mathcal{X}(x^1_{\max}, e_1)$ and which is included in $G^+_\eta(\omega)$.
- $B^n_\eta$ : the set of $\omega$ such that there is a smooth curve of length not greater than $n$ such that its $\frac{1}{n}$–neighborhood is included in $G^+_\eta(\omega)$.

Comparing this setting with the one used for the proof of the lower bound, one can easily see that:

$$\eta_c = \max\{\eta\,;\, P(B^0_\eta) < \tfrac{1}{2}\} = \min\{\eta\,;\, P(B^0_\eta) > \tfrac{1}{2}\}\,.$$

Thus, for any small positive $\gamma$ we have:

$$P(B^0_{\eta_c-\gamma}) > \frac{1}{2}\,.$$

This implies the existence of $n_0 = n_0(\gamma) > 0$ such that:

$$P(B^{n_0}_{\eta_c-\gamma}) > \frac{1}{2}\,.$$

We use the notation $p^0 = P(B^{n_0}_{\eta_c-\gamma})$.

In the same way as above one can assert that for sufficiently large $N$, the square $[0,N]^2$ contains at least $c(p^0)\,N$ mutually non intersecting smooth $\frac{1}{n_0}$ pipes connecting bottom and top sides of the square such that along each of these pipes:

$$U(x) > \eta_c - \gamma\,.$$

We consider a specific test function $\bar{v}$ such that:

(*i*) $\bar{v}(0,x_2) \equiv 0$ and $\bar{v}(1,x_2) \equiv 1$, $\bar{v}$ is continuous, (*ii*) $\bar{v}$ is constant between any pair of channels (pipes), and also between $\{x\,;\,x_1=0\}$ and the first pipe from one side, and between the last pipe and $\{x\,;\,x_1=1\}$ from the other side,

(*iii*) crossing each channel (pipe), $\bar{v}$ makes a jump of amplitude $1/(c(p^0)\,N)$; inside a channel $\bar{v}$ is linear in the direction orthogonal to the curve that forms the channels.

Hence $|\nabla\bar{v}| \le n_0/c(p^0)$, and letting $\varepsilon = 1/N$, we get:

$$\begin{aligned}
\inf_{\substack{v\,\in\,H^1(\square)\\ v(0,\cdot)\,\equiv\,0\\ v(1,\cdot)\,\equiv\,1}} \int_\square e^{U(\frac{x}{\varepsilon})/\mu}\,|\nabla v(x)|^2\,dx \;&\le\; \int_\square e^{U(\frac{x}{\varepsilon})/\mu}\,|\nabla \bar{v}(x)|^2\,dx \\
&=\; \int_{\text{channels}} e^{-U(\frac{x}{\varepsilon})/\mu}\,|\nabla \bar{v}(x)|^2\,dx \\
&\le\; \frac{n_0^2}{c^2(p^0)}\,e^{-(\eta_c-\gamma)/\mu} \\
&=\; C(\gamma)\,e^{-(\eta_c-\gamma)/\mu}
\end{aligned}$$

here we also used the fact that $|\nabla\bar{v}| \equiv 0$ outside the channels. Back to Definition (7) of $\sigma(\mu)$, we get $\sigma(\mu) \le C(\gamma)\,e^{-(\eta_c-\gamma)/\mu}$. Taking the lim–sup as $\mu \downarrow 0$, we find:

$$\limsup_{\mu\downarrow 0}\,\mu\,\log\sigma(\mu) \le -\eta_c + \gamma\,.$$

Since $\gamma$ is an arbitrary positive number, this relation implies:

$$\limsup_{\mu\downarrow 0}\,\mu\,\log\sigma(\mu) \le -\eta_c\,.$$

## *Main result*

**Theorem 2.1** *Under above assumptions, in particular Hypothesis 1.1, the log–behavior of the effective diffusion $\sigma(\mu)$ in the small viscosity case is given by:*

$$\lim_{\mu\downarrow 0}\,\mu\,\log\sigma(\mu) = -\eta_c$$

*where $\eta_c$ is the critical value given by (9).*

## 3 Hypothesis 1.1: Sufficient conditions

In this section we provide two different sufficient conditions for validity of Hypothesis 1.1.

**Lemma 3.1** *Let the random field* $U_1$ *be equal to* 0 *everywhere in the vicinity of the level set* $\mathcal{L}_0 = \{x : U_0(x) = U_0(x_s)\}$ *except for some neighborhoods of the saddle points. Then for sufficiently small* $\gamma_0$ *the random variables* $\alpha(x_{\min}, e_i)$, $x_{\min} \in \mathsf{X}_{\min}$, $i = 1, 2$, *are independent.*

**Proof** Each periodic cell has two saddle points $x_s^1$ and $x_s^2$, for each one of these saddle points $x_s^i$ (see Figure 2) we denote by $x_{\min}^{i,-} = x_{\min}$ and $x_{\min}^{i,+} = x_{\min} + e_i$ the two neighbor minimum points, symbol + corresponds to the greater value of one of the coordinates. Similarly, by $x_{\max}^{i,+} = x_{\max}$ and $x_{\max}^{i,-} = x_{\max} - e_2$ (if $i = 1$), $= x_{\max} - e_1$ (if $i = 2$) we denote the neighbor maximum points.

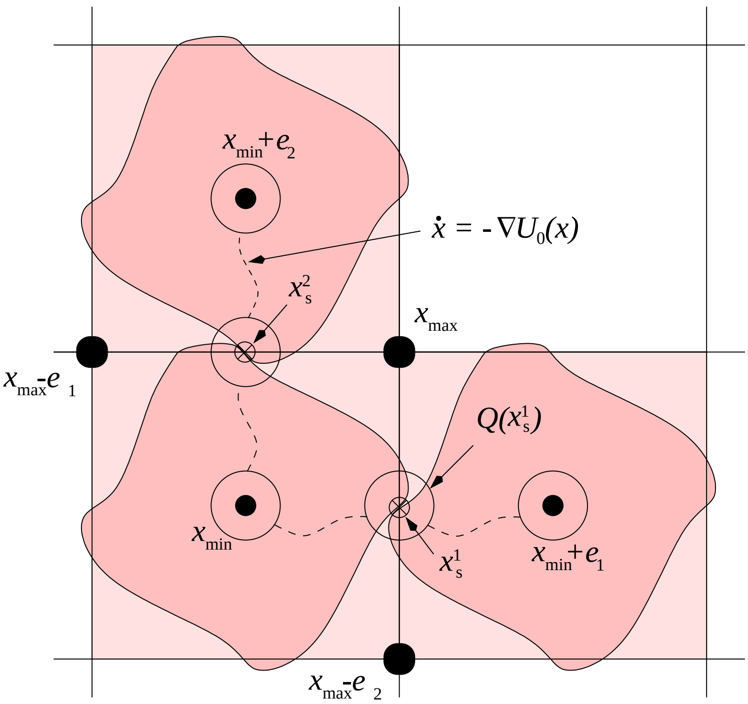


Figure 2: Sufficient condition, Lemma 3.1 (example of Figure 1).

We begin by constructing a periodic family of sufficiently small neighborhoods $Q(x_s)$ of saddle points $x_s \in \mathsf{X}_s$ that possesses the following properties (see Figure 2 and Figure 3):

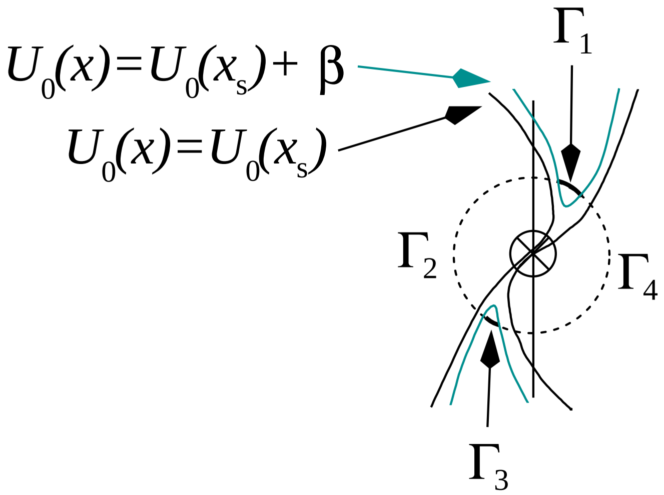


Figure 3: Zoom on point $x_s^1$ in Figure 2 with level lines $\{x\,;\,U_0(x) = U_0(x_s^1)\}$ and $\{x\,;\,U_0(x) = U_0(x_s^1) + \beta\}$, and the decomposition $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ of $\partial Q(x_s^1)$.

$(i)$ for all $x_s \in \mathsf{X}_s$, $Q(x_s)$ is a smooth domain that contains no singular points except $x_s$ ;

$(ii)$ the $\sigma$–algebras generated by $\{U_1(x),\ x \in Q(x_s)\}$, $x_s \in \mathsf{X}_s$, are independent ;

$(iii)$ there exists $\beta > 0$ such that, for all $x_s \in \mathsf{X}_s$, the following decomposition is valid: $\partial Q(x_s) = \cup_{j=1}^4 \Gamma_j$, where $\Gamma_j$ are connected components of $\partial Q(x_s)$ such that $U_0(x) > U_0(x_s) + \beta$ if $x \in \Gamma_1 \cup \Gamma_3$, and

$$\min_{x\in\Gamma_2} U_0(x) < U_0(x_s) - \beta, \quad \min_{x\in\Gamma_4} U_0(x) < U_0(x_s) - \beta,$$

$$\max_{x\in\Gamma_2\cup\Gamma_4} U_0(x) \le U_0(x_s) + \beta\,;$$

$(iv)$ for all $x_s \in \mathsf{X}_s$: if $x \in \partial Q(x_s)$ and $U_0(x_s) - \beta \le U_0(x) \le U_0(x_s) + \beta$ then $U_1(x) \equiv 0$ ;

($v$) all the trajectories of the equation $\dot{x} = -\nabla U_0(x)$ starting at $\Gamma_2$, are attracted with $x^{i,-}_{\rm min} = x_{\rm min}$ while the trajectories starting at $\Gamma_4$ are attracted with $x^{i,+}_{\rm min}$.

Under the above assumptions on $U_0$ and $U_1$ the said neighborhoods do evidently exist if $\beta$ is small enough.

We are going to show now that for $\gamma_0 < \beta/2$ the random variables $\alpha(x_{\rm min}, e_i)$ are independent. To this end we consider arbitrary two neighbor minimum points $x_{\rm min}$ and $x_{\rm min} + e_i$ and a minimizing sequence of curves $\{\varphi^\delta(\cdot)\}$ such that $\varphi^\delta(0) = x_{\rm min}$, $\varphi^\delta(1) = x_{\rm min} + e_i$, $\varphi^\delta \in \mathcal{X}(x_{\rm min}, e_i)$ and

$$\max_{0\le t\le 1} U(\varphi(t)) \le \alpha(x_{\rm min}, e_i) + \delta\,.$$

Due to the structure of $U_0$ and the choice of $Q(x^i_{\rm s})$, the intersection of $\varphi(\cdot)$ with $Q(x^i_{\rm s})$ is nontrivial for all sufficiently small $\delta$. It is also clear that $\varphi^\delta$ only intersects $\partial Q(x^i_{\rm s})$ at the points located at $\Gamma_2 \cup \Gamma_4$. Denote

$$\tau_1 = \max\{t\,;\,\varphi^\delta(t) \in \Gamma_2\}\,, \quad \tau_2 = \min\{t > \tau_1\,;\,\varphi^\delta(t) \in \Gamma_4\}\,.$$

Now one can replace the segments $\{\varphi(t)\,;\,0 \le t \le \tau_1\}$ and $\{\varphi(t)\,;\,\tau_2 \le t \le 1\}$ by the new ones in such a way that the curve $\tilde\varphi(\cdot)$ obtained is continuous, still belongs to $\mathcal{X}(x_{\rm min}, e_i)$ and satisfies the estimates:

$$U(\tilde\varphi(t)) < \alpha(x_{\rm min}, e_i)\,, \quad \text{for all } t < \tau_1 \text{ and } t > \tau_2\,.$$

Thus $\alpha(x_{\rm min}, e_i)$ only depends on $\{U_1(x)\,;\,x \in Q(x^i_{\rm s})\}$, and the statement of the lemma follows. □

The proof of the next assertion is similar to that of the preceding lemma and will be omitted.

**Lemma 3.2** *Let $U_1(x)$ be statistically homogeneous field (whose distributions are invariant w.r.t. any shifts) supported by Lipschitz functions, and suppose that*

$$|U_1(x)| \le \gamma_0\,,\ \ |U_1(x^1) - U_1(x^2)| \le \gamma_1 |x^1 - x^2|\,,\ \ x, x^1, x^2 \in \mathbb{R}^2,$$

*and that $\sigma\{U_1(x)\,;\,x \in S^1\}$ and $\sigma\{U_1(x)\,;\,x \in S^2\}$ are independent whenever $\operatorname{dist}(S^1, S^2) > \rho$. Then for sufficiently small $\gamma_0$, $\gamma_1$ and $\rho$ the random variables $\alpha(x_{\rm min}, e_i)$ are independent.*

## References


[1] M. AVELLANEDA and A.J. MAJDA. Mathematical models with exact renormalization for turbulent tranport. *Communications in Mathematical Physics*, 131:381–429, 1990.

[2] M. AVELLANEDA and A.J. MAJDA. Superdiffusion in nearly stratified flows. *J. Stat. Phys.*, 69(3–4):689–729, 1992.

[3] M. AVELLANEDA and A.J. MAJDA. Application of an approximate R–N–G theory, to a model for turbulent transport, with exact renormalization. In G.R. Sell et al., editor, *Turbulence in fluid flows. A dynamical systems approach*, The IMA volumes in Mathematics and its Applications, pages 1–31, New York, 1993. Springer Verlag.

[4] M. AVELLANEDA and A.J. MAJDA. Simple examples with features of renormalization for turbulent transport. *Phil. Trans. R. Soc. Lond. A*, 346:205–233, 1994.

[5] J.J. BEAR. *Dynamics of Fluids in Porous Media.* Elsevier, New York, 1972.

[6] A.Yu. BELYAEV and Ya.R. EFENDIEV. Homogenization of the stokes equations with a random potential. *Math. Notes*, 59(4):361–372, 1996.

[7] A. BENSOUSSAN, J.L. LIONS, and G. PAPANICOLAOU. *Asymptotic Analysis for Periodic Structures*, volume 5 of *Studies in Mathematics and its Applications.* North–Holland, 1978.

[8] R.A. CARMONA and L. XU. Homogenization for time dependent 2–D incompressible Gaussian flows. Preprint.

[9] A. FANNJIANG and G. PAPANICOLAOU. Convection enhanced diffusion for periodic flows. *SIAM Journal on Applied Mathematics*, 54:333–408, 1994.

[10] A. FANNJIANG and G. PAPANICOLAOU. Diffusion in turbulence. *Probability Theory and Related Fields*, 105:279–334, 1996.

[11] J. FRIED. *Groundwater Pollution.* Elsevier, Amsterdam, 1975.

[12] V. V. JIKOV, S. M. KOZLOV, and O. A. OLEINIK. *Homogenization of Differential Operators and Integral Functionals.* Springer Verlag, 1994.

[13] H. KESTEN. *Percolation Theory for Mathematicians*, volume 2 of *Progress in Probability.* Birkhäuser, Boston, 1982.

[14] S. KOZLOV. Geometric aspects of homogenization. *Russian Mathematical Surveys*, 44(2):91–144, 1989.

[15] S.M. KOZLOV and A.L. PYATNITSKIĬ. Averaging on a background of vanishing viscosity. *Math. USSR Sbornik*, 70(1):241–261, 1991.

[16] A. L. PYATNITSKY and S.M. KOZLOV. Homogenization and vanishing viscosity. In B. Grigelionis, editor, *Probability Theory and Mathematical Statistics. Proc. Fifth Vilnius Conference, 1989*, pages 330–339. VSP/Mokslas, 1990.