Homogenization of Random Parabolic Operator with Large Potential in locally periodic media

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Abstract. We study nonstationary linearized reaction-diffusion problem in a medium with locally periodic microstructure. Under the assumption that the characteristics of the medium are random stationary rapidly oscillating functions of time, we construct and justify a homogenized problem.
Keywords. Random operators, homogenization, averaging.

1 Introduction

This work is aimed at homogenization of nonstationary linear parabolic operators with a large potential, that describe nonstationary linearized reaction-diffusion problems in a medium with locally periodic microstructure. We suppose the characteristics of the medium are random stationary rapidly changing in time functions and, moreover, they show the diffusive behaviour. Under these assumptions we construct a homogenized model and justify the onvergence.

Previously, similar problems for equations with purely periodic coefficients were considered in Campillo et al. (2001). However, it turns out that the decomposition technique developed there relies essentially on periodicity and does not apply in the locally periodic case under consideration. For the problem considered here, we propose another approach based on the technique of correctors and on the results on convergence of martingales.

The homogenization problems for elliptic and parabolic equation were widely discussed in the existing literature, see, for instance, Jikov et al. (1994), Bensoussan et al. (1978), and the bibliography there. The equations with large low order terms usually require special consideration, there is a number of particular cases that were explored carefully, among them the operators with divergence free convection terms (Fannjiang and Papanicolaou (1996, 1997), Avellaneda and Majda (1994)) and with potential convection terms (Kozlov and Piatnitski (1991)), elliptic operators with periodic coefficients (Kozlov (1984), Kozlov and Piatnitski (1996), Pardoux (1999)).

As was already pointed out in Campillo et al. (2001), for the equations studied here the limit behaviour of solutions depends crucially on whether the oscillation in time 'faster" than that in spatial variables or not. To clarify this assertion let us denote by ε the microscopic length scale (the period) of the space microstructure and by ε^α the typical size of inhomogeneity in time, ε being a small positive parameter. Then for $\alpha > 2$ the usual homogenization result holds, i.e. individual solutions converge, as $\varepsilon \to 0$, to a solution of limit deterministic parabolic equation.

On the contrary, for $\alpha \leq 2$ the standard homogenization results may fail to hold; in particular, in this case the individual solutions do not converge (neither a.s. nor in probability). In order to describe the limit behaviour of our model we introduce the distributions of solutions in a suitable functional space and then show that these distributions converge weakly to a solution of a limit martingale problem.

In this paper we also exploit some ideas originally developed in Bouc and Pardoux (1984) and Viot (1976).

2 Setting of the problem and the notation.

We study the asymptotic behavior, as $\varepsilon \downarrow 0$ of the solution to the following Cauchy problem:

$$\partial_t u^\varepsilon(t,x) = \operatorname{div}\left[a\left(x, \tfrac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}}\right) \nabla u^\varepsilon(t,x)\right] + \frac{1}{\varepsilon^{1\wedge\frac{\alpha}{2}}}\, c\left(x, \tfrac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}}\right) u^\varepsilon(t,x)\,, \tag{1}$$

$$u^\varepsilon(0,x) = u_0(x) \qquad u_0 \in L^2(\mathbb{R}^n), \tag{2}$$

where $x \in \mathbb{R}^n$, $t \in [0,T]$, α is a positive parameter and $T > 0$ is fixed.

The coefficients $a(x,z,y)$ and $c(x,z,y)$ are one-periodic in z (we will identify one-periodic functions with functions on the unit torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$), and $\{\xi_t\}_{t\geq 0}$ is a stationary ergodic diffusion process, with values in $\mathbb{R}^d$, given by $d\xi_t = B(\xi_t)\,dt + \sigma(\xi_t)\,dW_t$. We suppose this process has unique invariant measure that admits a density $\rho(\cdot)$.

Let us introduce the following operators: the infinitesimal generator of the diffusion process $\{\xi_t\}_{t\geq 0}$

$$\mathcal{L}g(y) = \sum_{1\leq k,l\leq d} q^{kl}(y)\, g_{y_k y_l}(y) + \sum_{1\leq k\leq d} B_k(y)\, g_{y_k}(y) \tag{3}$$

with $q = \frac{1}{2}\sigma\,\sigma^*$, and

$$\mathcal{A}h(z) = \operatorname{div}_z(a(x,z,y)\,\nabla_z h(z))\,. \tag{4}$$

Throughout this paper we use the following notation. In $\mathbb{R}^n$, $x \cdot x'$ denotes the scalar product and $|\cdot|$ is the corresponding norm. In $L^2(\mathbb{R}^n)$, $(\cdot,\cdot)$ stands for the inner product, and $\|\cdot\|$ for the norm. Other norms are indicated explicitly.

For a function $(z,y) \mapsto f(z,y)$, we define (partial) mean values as follows:

$$\overline{\langle f(\cdot,\cdot)\rangle} = \int_{\mathbb{T}_n}\int_{\mathbb{R}^d} f(z,y)\,\rho(y)\,dy\,dz\,,$$
$$\langle f(\cdot,y)\rangle = \int_{\mathbb{T}_n} f(z,y)\,dz\,, \qquad \overline{f(z,\cdot)} = \int_{\mathbb{R}^d} f(z,y)\,\rho(y)\,dy\,.$$

Our assumptions on the coefficients of (1) and on the process ξ_t are as follows.

H1 There exist $C_1 > 0$ and $p_1,\ p_2 \in \mathbb{N}$ such that for all $(x,z,y) \in \mathbb{R}^n \times \mathbb{T}^n \times \mathbb{R}^d$ and $1 \le i,j \le n$, $1 \le k,l \le d$ the estimates hold

$$\begin{aligned} |\nabla_z a^{ij}(x,z,y)| + |\nabla_y a^{ij}(x,z,y)| &\le C_1\,(1+|y|^{p_1})\,, \\ |c(x,z,y)| + |\nabla_z c(x,z,y)| + |\nabla_y c(x,z,y)| &\le C_1\,(1+|y|^{p_2})\,, \\ |a^{ij}(x,z,y)| + |q^{kl}(y)| + |\nabla_y q^{ij}(y)| &\le C_1\,, \end{aligned}$$

where ∇_z and ∇_y stand for the space gradient with respect to z and y respectively.
H2 Operators $\mathcal{L}$ and $\mathcal{A}$ are uniformly elliptic : there is a constant $C_2 > 0$ such that for all $(z,y) \in \mathbb{R}^n \times \mathbb{T}^n \times \mathbb{R}^d$

$$C_2\,|z'|^2 \le (a(x,z,y)\,z')\cdot z'\,,\ \forall z' \in \mathbb{R}^n\,, \qquad C_2\,|y'|^2 \le (q(y)\,y')\cdot y'\,,\ \forall y' \in \mathbb{R}^d\,.$$

H3 The vector function B is uniformly bounded as well as its derivatives and there exist constants $\mu > -1$, $r_1 > 0$ and $C_3 > 0$ such that $B(y)\cdot \frac{y}{|y|} \le -C_3\,|y|^{\mu}$ for all y with $|y| > r_1$.
H4 For all $x \in \mathbb{R}^n$, the relation holds $\overline{\langle c(x,\cdot,\cdot)\rangle} = 0$.

Under Hypotheses **H1–H3**, the process $\{\xi_t\}$ admits the unique invariant measure with smooth density $\rho(y)$ given by :

$$\mathcal{L}^*\rho = 0 \text{ on } \mathbb{R}^d\,, \text{ and } \int_{\mathbb{R}^d} \rho(y)\,dy = 1\,. \tag{5}$$

Moreover, the density $\rho(\cdot)$ decays faster than any negative power of $|y|$, as $|y| \uparrow \infty$, see Pardoux and Veretennikov (2001).

Let (ζ_t, ξ_t) be the diffusion process associated to the infinitesimal generator $\mathcal{A}+\mathcal{L}$ with values in $\mathbb{T}^n \times \mathbb{R}^d$, and denote by $L^2_\rho(\mathbb{T}^n \times \mathbb{R}^d)$ the weighted space with the norm :

$$\|f\|^2_\rho = \int_{\mathbb{T}^n}\int_{\mathbb{R}^d} f(z,y)^2\,\rho(y)\,dy\,dz\,.$$

Also, we introduce the spaces :

$$\begin{aligned} \bar{L}^2_\rho(\mathbb{T}^n \times \mathbb{R}^d) &= \left\{ f \in L^2_\rho(\mathbb{T}^n \times \mathbb{R}^d)\,;\ \overline{\langle f(\cdot,\cdot)\rangle} = 0 \right\}, \\ \bar{H}^1_\rho(\mathbb{T}^n \times \mathbb{R}^d) &= \left\{ f \in \bar{L}^2_\rho(\mathbb{T}^n \times \mathbb{R}^d)\,;\ |\nabla_z f| + |\nabla_y f| \in L^2_\rho(\mathbb{T}^n \times \mathbb{R}^d) \right\}\,. \end{aligned}$$

In order to formulate our main results we need the following statement.

Lemma 19. *Let $f \in \bar{L}^2_\rho(\mathbb{T}^n \times \mathbb{R}^d)$ and suppose that :*

$$|f(z,y)| \le C_5\,(1+|y|^p)\,,\ \forall (z,y) \in \mathbb{T}^n \times \mathbb{R}^d$$

for some constants $C_5 > 0$ and $p \in \mathbb{N}$. Then the equation :

$$(\mathcal{A}+\mathcal{L})\,u(z,y) = f(z,y)$$

has a solution $\bar{H}^1_\rho(\mathbb{T}^n \times \mathbb{R}^d)$, the solution is unique and the estimate

$$|u(z,y)| \le C_6\,(1+|y|^{p_1})\,,\ \forall (z,y) \in \mathbb{T}^n \times \mathbb{R}^d$$

holds uniformly in $x \in \mathbb{R}^n$; moreover, p_1 only depends on p and μ while the constant C_6 depends on C_5, p and μ (we assume that the dimensions are fixed).

If, in addition, there exists $N > 0$ such that for all $n_1,\ n_2 \in \mathbb{N}$ with $n_1 + n_2 \le N$ we have : $\left|\frac{\partial^{n_1+n_2}}{\partial z^{n_1}\,\partial y^{n_2}}\,f(z,y)\right| \le C_5\,(1+|y|^p)\,,\ \forall (z,y) \in \mathbb{T}^n \times \mathbb{R}^d$, then $\left|\frac{\partial^{n_1+n_2}}{\partial z^{n_1}\,\partial y^{n_2}}\,u(z,y)\right| \le C_6\,(1+|y|^{p_1})\,,\ \forall (z,y) \in \mathbb{T}^n \times \mathbb{R}^d$.

3 The results

Let $L^2_w(\mathbb{R}^n)$ denote the space $L^2(\mathbb{R}^n)$ endowed with the weak topology. We define:

$$\Omega_T = L^2_w((0,T);H^1(\mathbb{R}^n)) \cap \mathcal{C}([0,T];L^2_w(\mathbb{R}^n)) \tag{6}$$

endowed with supremum of the topology of uniform convergence over the space $\mathcal{C}([0,T];L^2_w(\mathbb{R}^n))$ and weak topology over the space $L^2((0,T);H^1(\mathbb{R}^n))$. Ω_T is a Lusin and regular space, denote by $\mathcal{F}$ its Borel σ–field and by $\{\mathcal{F}_t\}_{0\leq t\leq T}$ the associated filtration.

For any $\varepsilon > 0$, let Q^ε be the law of $\{u^\varepsilon(t)\}_{0\leq t\leq T}$ in Ω_T. Then, Q^ε is a Radon probability measure on $(\Omega_T, \mathcal{F})$.

The asymptotic behavior of u^ε, as $\varepsilon \downarrow 0$, depends essentially on whether $\alpha = 2$, or $\alpha < 2$, or $\alpha > 2$.

Case $\alpha = 2$**.** First, we introduce the following functions:

$$\begin{aligned}
\hat{a}^{ij}(x) &= \overline{\langle a^{ik}(x,\cdot,\cdot)\,(\delta_{kj} + \partial_{z_k}\Psi^j(x,\cdot,\cdot))\rangle}\,,\\
\hat{b}(x) &= \overline{\langle \Psi(x,\cdot,\cdot)\,c(x,\cdot,\cdot) + a^{ij}(x,\cdot,\cdot)\,\partial_{z_j} g(x,\cdot,\cdot)\rangle}\,,\\
\hat{c}(x) &= \overline{\langle g(x,\cdot,\cdot)\,c(x,\cdot,\cdot)\rangle} + \mathrm{div}_x \overline{\langle a(x,\cdot,\cdot)\,\nabla_z g(x,\cdot,\cdot)\rangle}
\end{aligned}$$

where $g(x,z,y)$ and $\Psi(x,z,y)$ are solutions of:

$$(\mathcal{A}+\mathcal{L})g(x,z,y) = -c(x,z,y)\,, \qquad (z,y) \in \mathbb{T}^n \times \mathbb{R}^d, \tag{7}$$

$$(\mathcal{A}+\mathcal{L})\Psi(x,z,y) = -\sum_{j=1}^{n} a^{ij}_{z_j}(x,z,y)\,, \qquad (z,y) \in \mathbb{T}^n \times \mathbb{R}^d; \tag{8}$$

here $x \in \mathbb{R}^n$ appears as a parameter.

Theorem 1. *Let $\alpha = 2$, then under Hypotheses* **H1–H4***, the law Q^ε of the solution u^ε of Equations (1)–(2) converges weakly, as $\varepsilon \downarrow 0$, in the space Ω_T to $\hat{Q}$, a solution of the martingale problem with drift operator*

$$\hat{A}u = \mathrm{div}\big(\hat{a}\,\nabla u - \hat{b}\,u\big) + \hat{c}\,u\,,$$

covariance operator

$$(\hat{R}(u)\phi,\phi) = \Big(\Big(\overline{\langle q\,\langle \nabla_y g(x,\cdot,\cdot)\rangle \cdot \nabla_y \langle g(x,\cdot,\cdot)\rangle\rangle}\,u,\phi\Big)\ ,\ u\,\phi\Big)\ .$$

and initial condition $u_0(x)$.

Case $\alpha < 2$. Let:

$$\hat{a}^{ij}(x) = \overline{\langle a^{ik}(x,\cdot,\cdot)\,(\delta_{kj} + \partial_{z_k}\Psi^j(x,\cdot,\cdot))\rangle}\,, \qquad \hat{c}(x) = \overline{\langle g(x,\cdot,\cdot)\,c(x,\cdot,\cdot)\rangle}$$

where $g(x,y)$ and $\Psi(x,z,y)$ are solutions of:

$$\mathcal{L}g(x,y) = -\langle c(x,\cdot,y)\rangle\,, \tag{9}$$

$$\mathcal{A}\Psi(x,z,y) = -\sum_{j=1}^{n} a^{ij}_{z_j}(x,z,y)\,, \tag{10}$$

for any $(z,y) \in \mathbb{T}^n \times \mathbb{R}^d$ and $j = 1,\ldots,n$, here $x \in \mathbb{R}^n$ (resp. $(x,y) \in \mathbb{R}^n \times \mathbb{R}^d$) appears as a parameter in (9) (resp. (10)).

Theorem 2. *Let $\alpha < 2$, then under Hypotheses* **H1–H4** *the law Q^ε of the solution u^ε of Equations (1)–(2) converges weakly, as $\varepsilon \downarrow 0$, in space Ω_T to $\hat{Q}$, the solution of the martingale problem with drift operator*

$$\hat{A}u = \operatorname{div}\big(\hat{a}\,\nabla u\big) + \hat{c}\,u\,,$$

covariance operator

$$(\hat{R}(u)\phi, \phi) = \Big(\Big(\overline{(q\,\nabla_y g(x,\cdot)\cdot\nabla_y g(x,\cdot))}\,u, \phi\Big)\ ,\ u\,\phi\Big)\ .$$

and initial condition $u_0(x)$.

Case $\alpha > 2$. Let $\hat{u}$ be the solution of the following Cauchy problem:

$$\hat{u}_t(t,x) = \operatorname{div}(\hat{a}(x)\,\nabla\hat{u}(t,x)) + \hat{c}(x)\,\hat{u}(t,x)\,, \quad \hat{u}(0,x) = u_0(x) \tag{11}$$

with $(t,x) \in [0,T]\times\mathbb{R}^n$ and:

$$\begin{aligned}\hat{a}(x) &= \langle \bar{a}(x,\cdot,\cdot)\,(I + \nabla_z\Psi(x,\cdot))\rangle\,,\\ \hat{c}(x) &= \langle g(x,\cdot)\,\bar{c}(x,\cdot,\cdot)\rangle + \operatorname{div}_x\langle\bar{a}\,\nabla_z g\rangle\,,\end{aligned} \tag{12}$$

where the functions g, Ψ are solutions of equations:

$$\bar{A}g(x,z) = -\overline{c(x,z,\cdot)}\,, \qquad \bar{A}\Psi(x,z) = -\sum_{j=1}^{n}\overline{a^{ij}_{z_j}(x,z,\cdot)} \tag{13}$$

for $(x,z)\in\mathbb{R}^n\times\mathbb{T}^n$, and the operator $\bar{A}$ is defined by:

$$\bar{A}\phi(z) = \operatorname{div}_z\Big(\overline{a(x,z,\cdot)}\,\nabla_z\phi(z)\Big)\ .$$

In (13) x appears as a parameter.

Theorem 3. *For $\alpha > 2$, under Hypotheses* **H1–H4**, *the solution u^ε of Equations (1)–(2) converges in probability in the space Ω_T to the solution $\hat{u}$ of the limit problem (11).*

Corollary 4. *For $\alpha > 2$, under Hypotheses* **H1–H4**, *we have*

$$P\text{-}\lim_{\varepsilon\downarrow 0}\|u^\varepsilon - \hat{u}\|_{L^2((0,T)\times\mathbb{R}^n)} = 0 \tag{14}$$

where u^ε (resp. $\hat{u}$) is the solution of Equations (1)–(2) (resp. (11)).

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