Homogenization of Random Parabolic Operator with Large Potential

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Section F1. Probability
Stochastic analysis

We study the averaging problem for a divergence form random parabolic operators with a large potential and with coefficients rapidly oscillating both in space and time variables. The homogenization problems for various random structures are widely discussed in the physical and mathematical literature, see, for example, Jikov et al [1] and its bibliography.

Usually, in models for homogenization in random media, the randomness in spatial variables and the presence of a group of transformation preserving some probability measure, are supposed. In our model we assume that the medium has a periodic microscopic structure while the dynamics of the system is random and, moreover, diffusive (the equations without potential were previously considered in Kleptsyna–Piatnitski [2]); we consider the asymptotic behavior, $\varepsilon \downarrow 0$, of the solution of the following Cauchy problem:

$$u^\varepsilon_t(t,x) = \operatorname{div}\left[a\left(\tfrac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}}\right) \nabla u^\varepsilon(t,x)\right] + \tfrac{1}{\varepsilon^{1\wedge\frac{\alpha}{2}}}\, c\left(\tfrac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}}\right) u^\varepsilon(t,x) \tag{1}$$

($u^\varepsilon(0,x) = u_0(x)$, $x \in \mathbb{R}^n$, $t \in [0,T]$), where $\alpha > 0$ is a parameter.

The coefficients $a(z,y)$ and $c(z,y)$ are periodic in z (i.e. $z \in \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$), and $\{\xi_t\}_{t\geq 0}$ is stationary ergodic diffusion process, with values in $\mathbb{R}^d$, given by

$$d\xi_t = B(\xi_t)\,dt + \sigma(\xi_t)\,dW_t\,. \tag{2}$$

We suppose good mixing and localization properties. Our approach requires an exponential decay of the density $\rho(\cdot)$ of the invariant measure of ξ_s; a condition of the Khasminski type $B(y)\cdot y/|y| < -c,\ c > 0$, is sufficient and seems to be close to the minimal one in the case of a general bounded potential $c(z,y)$.

Let $\mathcal{L}$ be the infinitesimal generator of the diffusion process $\{\xi_t\}$, so that $\mathcal{L}^*\rho = 0$ and $\int \rho\,dy = 1$, and $\mathcal{A}^\varepsilon = \operatorname{div}\left(a(\frac{x}{\varepsilon},y)\,\nabla h(x)\right)$ ($\mathcal{A}$ will denote $\mathcal{A}^\varepsilon$ for $\varepsilon = 1$). We suppose that the operators $\mathcal{L}$ and $\mathcal{A}$ are uniformly elliptic, and that 0 is an isolated point of the spectrum of operator $\mathcal{L}$ (in fact, this condition can be omitted, but in this case much more delicate considerations are required).

We suppose that the coefficients a, c, and q are uniformly bounded as well as their derivatives. We also will suppose that $\overline{\langle c(\cdot,\cdot)\rangle} = 0$ which is not a restriction.

Here we use the notation:

$$\overline{\langle f(\cdot,\cdot)\rangle} = \int\int f(z,y)\,\rho(y)\,dy\,dz\,,\ \langle f(\cdot,y)\rangle = \int f(z,y)\,dz\,,\ \overline{f(z,\cdot)} = \int f(z,y)\,\rho(y)\,dy\,.$$

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Let $L^2_\rho(\mathbb{T}^n \times \mathbb{R}^d)$ denote the weighted space with the norm $\|f\|^2_\rho = \overline{\langle f(\cdot,\cdot)^2\rangle}$; $\bar{L}^2_\rho(\mathbb{T}^n \times \mathbb{R}^d)$ the subset of functions $f \in L^2_\rho(\mathbb{T}^n \times \mathbb{R}^d)$ such that $\overline{\langle f(\cdot,\cdot)\rangle} = 0$; and $\bar{H}^1_\rho(\mathbb{T}^n \times \mathbb{R}^d)$ the subset of functions $f \in \bar{L}^2_\rho(\mathbb{T}^n \times \mathbb{R}^d)$ such that $|\nabla_x f| + |\nabla_z f| \in L^2_\rho(\mathbb{T}^n \times \mathbb{R}^d)$.

Let $L^2_w(\mathbb{R}^n)$ denote the space $L^2(\mathbb{R}^n)$ endowed with the weak topology. Define $\Omega = L^2_w((0,T); H^1(\mathbb{R}^n)) \cap \mathcal{C}([0,T]; L^2_w(\mathbb{R}^n))$, and $\mathcal{F}$ the corresponding Borel σ–field.

Case $\alpha < 2$ **:** The law Q^ε of $\{u^\varepsilon(t)\,;\, 0 \le t \le T\}$ converge to $\hat{Q}$, the law of the solution of the equation:

$$d\hat{u}(t,x) = \Big[\mathrm{div}(\hat{a}\,\nabla\hat{u}(t,x)) + \hat{c}\,\hat{u}(t,x)\Big]\,dt + \lambda\,\hat{u}(t,x)\,d\hat{W}_t \tag{3}$$

where $\hat{W}$ is standard Brownian motion in $\mathbb{R}$ and: $\hat{a} = \overline{\langle a\,(I + \nabla_z\Psi)\rangle}$, $\hat{c} = \overline{\langle G\,c\rangle}$, $\lambda^2 = 2\,\overline{(q\,\nabla G\,,\,\nabla G)}$. The functions $G \in H^1_\rho(\mathbb{R}^d)$ and $\Psi^i \in \bar{H}^1_\rho(\mathbb{T}^n \times \mathbb{R}^d)$ are the solutions of the equations:

$$\mathcal{L}G(y) = -\langle c(\cdot,y)\rangle\,, \qquad \mathcal{A}\Psi^i(z,y) = -\textstyle\sum_{j=1}^n a^{ij}_{z_j}(z,y)\,.$$

Case $\alpha = 2$ **:** The law Q^ε of $\{u^\varepsilon(t)\,;\, 0 \le t \le T\}$ converge to $\hat{Q}$, the law of the solution of the equation:

$$d\hat{u}(t,x) = \Big[\mathrm{div}(\hat{a}\,\nabla\hat{u}(t,x)) - \hat{b}\cdot\nabla\hat{u}(t,x) + \hat{c}\,\hat{u}(t,x)\Big]\,dt + \lambda\,\hat{u}(t,x)\,d\hat{W}_t \tag{4}$$

where $\hat{W}$ is a standard Brownian motion in $\mathbb{R}$, and: $\hat{a} = \overline{\langle a\,(I + \nabla_z\Psi)\rangle}$, $\hat{b} = \overline{\langle \Psi\,c + a\,\nabla_z G\rangle}$, $\hat{c} = \overline{\langle G\,c\rangle}$, $\lambda^2 = 2\,\overline{\langle (q\,\nabla_y G\,,\,\nabla_y G)\rangle}$. The functions G, $\Psi^j \in \bar{H}^1_\rho(\mathbb{T}^n \times \mathbb{R}^d)$ are the solutions of the equations:

$$(\mathcal{A} + \mathcal{L})G(z,y) = -c(z,y)\,, \qquad (\mathcal{A} + \mathcal{L})\Psi^j(z,y) = -\textstyle\sum_{i=1}^n a^{ij}_{z_i}(z,y)\,.$$

Case $\alpha > 2$ **:** The limit $\|u^\varepsilon - \hat{u}\|_{L^2((0,T)\times\mathbb{R}^n)} \to 0$ holds in probability, where $\hat{u}$ is the solution of the equation:

$$\hat{u}_t(t,x) = \mathrm{div}(\hat{a}\,\nabla\hat{u}(t,x)) + \hat{c}\,\hat{u}(t,x) \tag{5}$$

with $\hat{a} = \langle \bar{a}\,(I + \nabla_z\Psi)\rangle$ and $\hat{c} = \langle G\,\bar{c}\rangle$, where the functions G, $\Psi^i \in \bar{H}^1(\mathbb{T}^n)$ are solutions of equations:

$$\mathrm{div}(\overline{a(z,\cdot)}\,\nabla G(z)) = -\overline{c(z,\cdot)}\,, \qquad \mathrm{div}(\overline{a(z,\cdot)}\,\nabla\Psi^i(z)) = -\textstyle\sum_{j=1}^n \overline{a^{ij}_{z_j}(z,\cdot)}\,.$$

Comments The proof of the three previous theorems relies on the same scheme: first we establish the tightness of the family $\{Q^\varepsilon\,;\,\varepsilon > 0\}$ and then we identify the limit.

According to [2], the absence of the potential always leads to a deterministic form of homogenized problem whose operator involves neither stochastic nor lower order terms.

The exact statements and proofs will be given together with Some works in progress.

References

[1] V. V. JIKOV, S. M. KOZLOV, and O. A. OLEINIK. *Homogenization of Differential Operators and Integral Functionals.* Springer Verlag, 1994.

[2] M.L. KLEPTSYNA and A.L. PIATNITSKI. Homogenization of random parabolic operators. *GAKUTO International Series, Mathemathical Sciences and Applications*, 9:241–255, 1997.