NUMERICAL METHODS IN ERGODIC STOCHASTIC CONTROL: APPLICATION TO SEMI–ACTIVE SHOCK–ABSORBER[1]

Fabien Campillo
INRIA/LATP
CMI
39, Rue F. Joliot–Curie
13453 Marseille cedex
France
e–mail: campillo@sophia.inria.fr

Abstract

In [2] we presented a numerical algorithm for the computation of the optimal feedback law in an ergodic stochastic optimal control problem. This method, based on the discretization of the associated Hamilton–Jacobi–Bellman equation, can be used only in low dimensions (2,4, or 6 in a parallel computer). For higher dimensional problems, we propose here to use a stochastic gradient algorithm in order to find the optimal feedback within a given subclass of parameterized controls. As is [2], we apply these techniques to the control of semi–active suspensions for road vehicles.

1 Introduction

In this paper we consider numerical procedures for stochastic control problems. Given a real case study (here we consider semi–active control of vehicle suspensions) we can use different methods. For low dimensional problems, we can use optimal methods : we discretize the Hamilton–Jacobi–Bellman equation (this approach is proposed in [2]). In higher dimensions this approach is cumbersome or even impossible to implement, in this case we can look for the optimal feedback in a given subclass of parameterized controls using a stochastic gradient algorithm. The aim of this paper is to compute the stochastic gradient in the simplest two–dimensional model relevant in our application, and to give some numerical results.

In section 3 we introduce the stochastic control problem of ergodic type in $\mathbb{R}^2$ motivated by the application to the control of suspension systems (see [2]). In section 4, we derive the stochastic gradient for the above problem.

2 An example : a semi–active suspension system

In this section we present a damping control method for a nonlinear suspension of a road vehicle (comprising a spring, a shock absorber, a mass, and taking into account the dry friction, cf. figure 1). The aim is to improve the ride comfort.

Among alternatives to classical suspension systems (passive systems) we distinguish between active and semi–active techniques. An active suspension system consists of force elements in addition to a spring and a damper assembly. Force elements continuously vary the force according to some control law. In general, an active system is expensive, complicated, and requires an external power source [8]. In contrast, a semi–active system requires no hydraulic power supply, and its hardware implementation is simpler and cheaper than a fully active system. A semi–active suspension system acts only on damping or spring laws, so it can only dissipate or store energy.

Here we consider a system with control on the damping law, the forces in the damper are generated by modulating its orifice for fluid flow [1, 10]. We use the simplest model which consists in a one degree–of–freedom model.

The equation of motion for a one degree–of–freedom model is (cf. figure 1 for the exact definition of the terms)

$$m\,\ddot y + c\,\dot y + k_s\,y + F_s\,\mathrm{sign}(\dot y) = -m\,\ddot e\ . \qquad (1)$$

$\ddot e$ denotes the input acceleration, i.e. it models the roughness of the road surface. The restoring force $k_s\,y + F_s\,\mathrm{sign}(\dot y)$, has a linear part $k_s\,y$, and a nonlinear part $F_s\,\mathrm{sign}(\dot y)$ which describes the dry friction force. The damping force is $c\,\dot y$ where $c > 0$

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a	absolute displacement of mass m
y	relative displacement ($y = a - e$)
e	stochastic input (surface road acceleration)
m	sprung mass
c	shock–absorber damping constant (controlled)
k_s	spring constant
F_s	dry friction constant

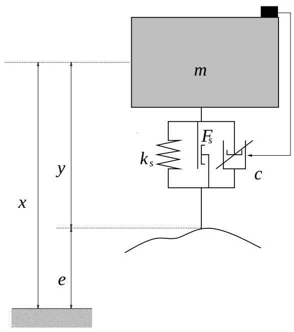

Figure 1: One degree–of–freedom model.

is the instantaneous damping coefficient (the control is acting on this term).

The problem is to compute a feedback law $c = c(y, \dot{y})$ such that the solution of the system (1) minimizes a criterion — related to the vibration comfort

$$J(u) \triangleq \lim_{T\to\infty} \frac{1}{T} E \int_0^T |\ddot{a}|^2 \, dt$$

(note that $\ddot{a} = \ddot{y} + \ddot{e}$). $\ddot{e}$ is supposed to be a white Gaussian noise process, $\ddot{e} = -\sigma \, dW/dt$ where W is a standard Wiener process.

Let

$$u = \frac{c}{m}\,, \; \alpha = \frac{k_s}{m}\,, \; \beta = \frac{F_s}{m}\,, \text{ and } X \triangleq \begin{pmatrix} y \\ \dot{y} \end{pmatrix}.$$

The constants α and β are strictly positive; u takes its values in $U = [u_1, u_2]$, $0 < u_1 < u_2 < \infty$.

3 A stochastic control problem

The process X_t is the solution of the following stochastic system in $\mathbb{R}^2$

$$dX_t^1 = X_t^2 \, dt\,, \tag{2}$$

$$dX_t^2 = -g(u(X_t), X_t)\, dt + \sigma\, dW_t \tag{3}$$

where $g(u,x) = u\,x^2 + \alpha\, x^1 + \beta \,\mathrm{sign}(x^2)$, and B is a standard Brownian motion; $x \mapsto u(x)$ is a feedback control which takes values in U.

We consider the following long time average cost functional

$$J(u) \triangleq \lim_{T\to\infty} \frac{1}{T} E \int_0^T f(u(X_t), X_t)\, dt \tag{4}$$

with $f = [g]^2$.

The non linear stochastic control problem is to find $\hat{u}$, in a suitable set $\mathcal{U}$ on feedback controls, which minimizes the cost function (4) under the constraint (2,3).

The "optimal" approach

This approach is presented in [2]. If the functional space $\mathcal{U}$ is *wide* enough, then this problem becomes, in a certain way, a minimization problem over the values taken by the control feedback instead over the functional space $\mathcal{U}$. Indeed, we come up with the Hamilton–Jacobi–Bellman (HJB) equation, that is we are looking for a function $v : \mathbb{R}^2 \to \mathbb{R}$, defined up to an additive constant, and a constant ρ such that:

$$\min_{u\in[u_1,u_2]} [\mathcal{L}^u \, v(x) + f(u,x)] = \rho\,, \; \forall x \in \mathbb{R}^2 \tag{5}$$

where $\mathcal{L}^u$ is the infinitesimal generator associated with the diffusion process (2), (3). Then,

$$\hat{u}(x) = \mathrm{Arg} \min_{u\in[u_1,u_2]} [\mathcal{L}^u \, v(x) + f(u,x)]\,.$$

One way to find (an approximation of) $\hat{u}$ is to solve numerically the HJB equation (5). Numerous methods exists, see [9].

Up to my knowledge, there is no result concerning existence and uniqueness of a solution for (5). This is du to the fact that, concerning the long–run non linear stochastic control problem, there is no result when the equations system is degenerated (the diffusion matrix is non uniformly elliptic) and the coefficients are discontinuous. But, in this context, the present case looks quite "academic". This is a good point in the sense that it is a nice textbook case.

The "parameterized" approach

The "optimal" solution proposed above could be useless ! Indeed, in high dimension, i.e. for real case studies, the approximation of the HJB equation is out of reach. Moreover, even if it was possible the result would be a "discretized" feedback control: for a given input you should find in a table the corresponding damping coefficient. Then, this approach is optimal *given* a set of quite restrictive hypothesis (the model for dry friction phenomena, the white Gaussian noise hypothesis etc.). So, for many reason, it is of interest to use a more restrictive class of feedback control function, in practice a class of parameterized control function.

Let $X_t(\theta)$ be the solution of the following stochastic system in $\mathbb{R}^2$

$$
\begin{aligned}
dX_t^1(\theta) &= X_t^2(\theta)\, dt\ , && (6)\\
dX_t^2(\theta) &= -g(\theta, X_t(\theta))\, dt + \sigma\, dB_t && (7)
\end{aligned}
$$

where $g(\theta,x) = u(\theta,x)\, x^2 + \alpha\, x^1 + \beta\, \mathrm{sign}(x^2)$, and $u(\theta,x)$ is a feedback control parameterized by $\theta \in \Theta$ (Θ is a compact subset of $\mathbb{R}^d$, for some $d \geq 1$). $u(\theta,x)$ takes values in $U = [u_1,u_2]\,,\ 0 < u_1 < u_2 < \infty$.

The components of $x \in \mathbb{R}^2$ (resp. of $X_t(\theta)$) are denoted by x^1 and x^2 (resp. by $X_t^1(\theta)$ and $X_t^2(\theta)$).

For each $\theta \in \Theta$, we consider the following long time average cost functional

$$
J(\theta) \triangleq \lim_{T\to\infty} \frac{1}{T} E \int_0^T f(\theta, X_t(\theta))\, dt \qquad (8)
$$

with $f = [g]^2$.

A class of feedback controls : Let $\mathcal{U}$ denote the set of feedback functions $u(\theta,x)$ such that

- $u : \Theta \times \mathbb{R}^2 \mapsto U = [u_1,u_2]$,
- for any $x \in \mathbb{R}^2$, $\theta \to u(\theta,x)$ is C^1,
- for any $\theta \in \Theta$, $x \to u(\theta,x)\, x^2$ is C^1 on $\mathbb{R}\times\mathbb{R}\setminus\{0\}$ with bounded derivatives.

This last condition allows u to be discontinuous at $x^2 = 0$; it is the case for the optimal control which we have computed in a previous work [2].

Note that in (7), the drift coefficient is C^1 on $\mathbb{R}\times\mathbb{R}\setminus\{0\}$, and it is discontinuous across $x^2 = 0$ with a jump of amplitude $-2\,\beta$.

In [7] we proved that, for any $\theta \in \Theta$, the system (6,7) admits a unique strong solution (this, in particular, means that in Equation (8), the expectation operator does not depend on θ).

4 Stochastic approximation algorithms

The problem is to find θ^* which minimizes the cost function (8). In [2] we have already proved that the cost function is of the form

$$
J(\theta) = \int_{\mathbb{R}^2} f(\theta,x)\, \mu_\theta^X(dx) \qquad (9)
$$

where μ_θ^X is the unique invariant measure of the process $X_t(\theta)$. We now compute the gradient of $J(\theta)$, which involves differentiating $X_t(\theta)$.

The gradient process

In [7] we investigate the regularity of the process $X_t(\theta)$ with respect to θ and we establish an equation for the gradient process

$$
Y_t(i,\theta) \triangleq \frac{\partial}{\partial\theta_i} X_t(\theta)\ .
$$

This derivation will give rise to the local time of the process $\{X_t^2(\theta)\}$ at 0.

The processes X_t^2 admits a local time at 0, which is the unique adapted, continuous, nondecreasing process L_t such that $L_0 = 0$, $\int_0^\infty \mathbf{1}_{\mathbb{R}\setminus\{0\}}(X_s^2)\, dL_s = 0$ a.s., and

$$
|X_t^2| = |X_0^2| + \int_0^t \mathrm{sign}(X_s^2)\, dX_s^2 + 2\, L_t\ . \qquad (10)
$$

Moreover,

$$
L_t = \lim_{\varepsilon\to 0}\text{–a.s.}\ \frac{\sigma^2}{2\varepsilon} \int_0^t \mathbf{1}_{[-\varepsilon,\varepsilon]}(X_s^2)\, ds\ . \qquad (11)
$$

Then, for each $t \geq 0$, $X_t(\theta)$ is mean–square differentiable w.r.t. the parameter θ_i ($i = 1,\ldots,d$), and $Y_t(i,\theta) \triangleq \partial X_t(\theta)/\partial\theta_i$ is the solution of the following system

$$
\begin{aligned}
dY_t(\theta) &= A(\theta, X_t(\theta))\, Y_t(\theta)\, dt \\
&\quad + B\, Y_t(\theta)\, dL_t + C(\theta, X_t(\theta))\, dt \qquad (12)
\end{aligned}
$$

with

$$
\begin{aligned}
A(\theta,x) &\triangleq \begin{pmatrix} 0 & 1 \\ -\bar{g}_{x^1}(\theta,x) & -\bar{g}_{x^2}(\theta,x) \end{pmatrix} \\
B &\triangleq \begin{pmatrix} 0 & 0 \\ 0 & -\frac{2\beta}{\sigma^2} \end{pmatrix} \\
C(\theta,x) &\triangleq \begin{pmatrix} 0 \\ -u_\theta(\theta,x) x^2 \end{pmatrix}\ .
\end{aligned}
$$

$Y_0 \equiv 0$, where L is the local time of X^2 at 0, and $\bar{g}$ is the regular part of g:

$$\bar{g}(\theta, x) \stackrel{\triangle}{=} u(\theta, x)\, x^2 + \alpha\, x^1 \,.$$

We denote $Y_t(\theta) = [Y_t(1,\theta)|\cdots|Y_t(d,\theta)]$.

For the sake of simplicity, from now on we suppose that θ is scalar, so we drop the subscript i. We have an explicit representation of $Y_t(\theta)$ in terms of $X_t(\theta)$. Let $\{\zeta_t(\theta)\,,\, t \geq 0\}$ be the solution of

$$d\zeta_t(\theta) \stackrel{\triangle}{=} A(\theta, X_t(\theta))\,{}_t(\theta)\, dt + B\,\zeta_t(\theta)\, dL_t$$

with $\zeta_0(\theta) = I$, then

$$Y_t(\theta) = \int_0^t \zeta_t(\theta)\,\zeta_s(\theta)^{-1}\, C(\theta, X_s(\theta))\, ds \,.$$

We note $\xi_t(\theta) = \begin{pmatrix} X_t(\theta) \\ Y_t(\theta) \end{pmatrix}$.

In [2] we showed that $X_t(\theta)$ admits a unique invariant measure μ_θ^X. In [7] we proved that the pair process $\xi_t(\theta)$ possesses a unique invariant measure μ_θ.

The gradient of the cost functional

Formally, if we take the derivative of the cost functional (8) with respect to θ, and if we interchange the derivation and the limit as $T \to \infty$, we get a formal result which appears to be rigorous:

$$\nabla_\theta\, J(\theta) = \lim_{T\to\infty} \tfrac{1}{T}\, E\,[\textstyle\int_0^T F(\theta, X_t(\theta), Y_t(\theta))\, dt + \int_0^T 4\,\alpha\,\beta\, X_t^1(\theta)\, Y_t^2(\theta)\, dL_t] \qquad (13)$$

where $F(\theta,x,y) = f_\theta(\theta,x) + \tilde{f}_x(\theta,x)\,.\,y$, and

$$\begin{aligned} \tilde{f}_{x^1}(\theta,x) &= f_{x^1}(\theta,x) \\ \tilde{f}_{x^2}(\theta,x) &= 2\,g(\theta,x)\,[u_{x^2}(\theta,x)\,x^2 + u(\theta,x)]\,. \end{aligned}$$

It is possible to prove that the process $(X_t(\theta), Y_t(\theta))$ admits a unique invariant measure μ_θ which is regular with respect to the parameter θ, from which one can conclude that the limit (13) is well defined.

Stochastic gradient algorithm

In order to minimize (9), we want to find $\theta^* \in \Theta$ such that

$$\nabla_\theta\, J(\theta)|_{\theta=\theta^*} = 0 \,. \qquad (14)$$

The associated stochastic gradient algorithm is the following : given $\Delta t > 0$ and $t_k \stackrel{\triangle}{=} k\,\Delta t$, we solve equations (6),(7),(12) with $\theta = \theta_k$ for $t_k \leq t < t_{k+1}$, and θ_k is given by

$$\begin{aligned} \theta_{k+1} = \theta_k - \gamma_k\,[F\,(\theta_k, X_{t_k}(\theta_k), Y_{t_k}(\theta_k))\;\Delta t \\ +4\,\beta\,\alpha\, X_{t_k}^1(\theta_k)\, Y_{t_k}^2(\theta_k)\,\Delta L_k]\,, \end{aligned}$$

where $\Delta L_k = L_{t_{k+1}} - L_{t_k}$, and where the sequence of positive gains $\{\gamma_k\}$ satisfies appropriate conditions.

5 Discretization

Time discretization

We approximate X_t by X_t^n given by the following Euler scheme

$$\begin{aligned} X_{k+1}^{n,1} &= X_k^{n,1} + X_k^{n,2}\,\Delta t\,, \\ X_{k+1}^{n,2} &= X_k^{n,2} - g(\theta, X_k^n)\,\Delta t + \sigma\,\Delta B_k^n \end{aligned} \qquad (15)$$

where $\Delta t \stackrel{\triangle}{=} t_{k+1} - t_k$ and $\Delta B_k^n \stackrel{\triangle}{=} B_{t_{k+1}} - B_{t_k}$ which is a i.i.d. $\mathcal{N}(0, t_{k+1} - t_k)$ Gaussian sequence.

For Y_t we also use an Euler scheme

$$\begin{aligned} Y_{k+1}^{n,1} &= Y_k^{n,1} + Y_k^{n,2}\,\Delta t\,, \\ Y_{k+1}^{n,2} &= Y_k^{n,2} \\ &\quad -[u_\theta(\theta, X_k^n) + u_x(\theta, X_k^n)\,Y_k^n]\,X_k^{n,2}\,\Delta t \\ &\quad -u(\theta, X_k^n)\,Y_k^{n,2}\,\Delta t \\ &\quad -\alpha\,Y_k^{n,1}\,\Delta t - \frac{2\beta}{\sigma^2} Y_k^{n,2}\,\Delta L_k^n \end{aligned}$$

where ΔL_k^n is an approximation of $L_{t_{k+1}} - L_{t_k}$ given by

$$\Delta L_k^n = \begin{cases} |X_{k+1}^{n,2}| & \text{if } X_k^{n,2}\,X_{k+1}^{n,2} < 0\,, \\ 0 & \text{otherwise,} \end{cases}$$

We get that, for any $t \geq 0$ and $\theta \in \Theta$, $X_t^n(\theta) \to X_t(\theta)$ in L^2 as $n \to \infty$.

Approximation of the local time

From (10)

$$L_t = \frac{1}{2}|X_t^2| - \frac{1}{2}|X_0^2| - \frac{1}{2}\int_0^t \operatorname{sign}(X_s^2)\, dX_s^2 \,. \quad (16)$$

We approximate L_t the following way (see [7]): in (16) we replace X_t^2 by the polygonal interpolation of the discrete process $X_k^{n,2}$ given in (15)

$$X_t^{n,2} \triangleq \sum_{k\geq 0} \frac{(t_{k+1}^n - t)\, X_{n_k^n}^{n,2} + (t - t_k^n)\, X_{n_{k+1}^n}^{n,2}}{t_{k+1}^n - t_k^n} \mathbf{1}_{[t_k^n, t_{k+1}^n)}(t)$$

with $t_k^n = k/n$, and we use:

$$L_t^n = \frac{1}{2}|X_t^{n,2}| - \frac{1}{2}|X_0^{n,2}| - \frac{1}{2}\int_0^t \mathrm{sign}(X_{\tau_n(s)}^{n,2})\, dX_s^{n,2}\,,$$

where $\tau_n(s) = t_k^n$ if $s \in [t_k^n, t_{k+1}^n)$. We have

$$L_{t_{k+1}^n}^n - L_{t_k^n}^n = \begin{cases} |X_{t_{k+1}^n}^{n,2}| & \text{if } X_{t_{k+1}^n}^{n,2} X_{t_k^n}^{n,2} < 0\,, \\ 0 & \text{otherwise.} \end{cases}$$

Then, for all t, $L_t^n \to L_t$ in L^2 as $n \to \infty$.

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