A STABILIZATION ALGORITHM FOR LINEAR CONTROLLED SDE'S*

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Keywords : Stochastic differential equation, stabilization, stochastic gradient algorithm

Abstract

We consider a stochastic differential equation with linear feedback control :

$$dX_t = (A_0 + \rho\, B_0)\, X_t\, dt + \sum_{l=1}^{L} (A_l + \rho\, B_l)\, X_t \,\circ dW_l(t)$$

where $\rho \in \mathbb{R}$ is a one–dimensional feedback gain matrix. The problem is to find an online algorithm which adjust the parameter ρ in order to stabilize the system.

We propose a stochastic gradient method which minimize the Lyapunov exponent λ_ρ associated with the solution of the SDE with parameter ρ. We present some simulation tests.

Key words : stochastic system, stochastic gradient, stabilization.

1 Preliminaries

We consider the following linear stochastic differential equation in $\mathbb{R}^d$

$$dX_t = (A_0+\rho\, B_0)\, X_t\, dt+\sum_{l=1}^{L}(A_l+\rho\, B_l)\, X_t \circ dW_l(t)\ , \quad (1)$$

where $X_0 = x_0 \neq 0$, A_l and B_l are $d\times d$ matrices, $W_1,\dots,W_L$ are independent standard Wiener processes. Here "$\circ dW$" (resp. "dW") refer to the Stratonovich (resp. Itô) stochastic integral.

We denote

$$A_l^\rho \triangleq A_l + \rho\, B_l\ , \quad l = 0,\dots,L\ .$$

We define the Lyapunov exponent of the solution of (1) starting at x_0 with parameter ρ

$$\lambda_\rho(x_0) \triangleq \lim_{t\to\infty} \frac{1}{t} \log \|X_t\|\ . \quad (2)$$

Oseledec's multiplicative ergodic theorem [8] states that the limit (2) exists with probability one and that there are d fixed numbers $\lambda_1 \geq \cdots \geq \lambda_d$ – called the Lyapunov exponents of (1) – such that the random variable $\lambda_\rho(x_0)$ takes on only these values (see [1] for a review). Moreover, (1) is exponentially stable with probability one if and only if $\lambda_1 < 0$.

Let $S^{d-1} = \{x \in \mathbb{R}^d; \|x\| = 1\}$ denote the unit sphere of $\mathbb{R}^d$. We can define the projection of X_t onto the sphere by

$$U_t \triangleq \|X_t\|^{-1}\ X_t\ .$$

U_t is the solution of the following SDE on S^{d-1}

$$\begin{aligned} dU_t &= h(A_0^\rho, U_t)\, dt + \textstyle\sum_{l=1}^{L} h(A_l^\rho, U_t) \circ dW_l(t)\ , \\ U_0 &= u_0 \triangleq \|x_0\|^{-1}\, x_0 \end{aligned} \quad (3)$$

where

$$h(C,u) \triangleq C\, u - (C\, u, u)\, u\ . \quad (4)$$

Here (x, y) is the scalar product on $\mathbb{R}^d$ and $\|x\|^2 = (x,x)$.

Moreover,

$$\begin{aligned} \|X_t\| &= \|x_0\| \times \exp\Big\{ \int_0^t q(\rho, U_s)\, ds \\ &\qquad + \sum_{l=1}^{L} \int_0^t p_l(\rho, U_s)\, dW_l(s) \Big\} \end{aligned} \quad (5)$$

with

$$\begin{aligned} p_l(\rho,u) &\triangleq (A_l^\rho\, u, u)\ , \quad l = 0,\dots,L\ , \\ p_0'(\rho,u) &\triangleq \frac{1}{2} \sum_{l=1}^{L} [((A_l^\rho)^2\, u, u) \\ &\qquad + \|A_l^\rho\, u\|^2 - 2(A_l^\rho\, u, u)^2]\ , \\ q(\rho,u) &\triangleq p_0(\rho,u) + p_0'(\rho,u)\ . \end{aligned}$$

*ECC95, Roma.

For each matrix M, $h(M,-u) = -h(M,u)$, so that $h(M,\cdot)$ can be viewed as a vector field on the projective space P^{d-1} (obtained from S^{d-1} by identifying u and $-u$). Therefore (3) can be considered as a stochastic differential equation on P^{d-1} and (5) is still valid with this definition.

We make the following

Hypothesis 1.1 *For all $u \in P^{d-1}$ and $\rho \in \mathbb{R}$*

$$\dim \mathsf{Lie\ Algebra}\{h(A_l^\rho,\cdot)\,;\, l=0,\ldots,L\}(u) = 1\ .$$

Theorem 1.2 *Under Hypothesis 1.1, for all $\rho \in \mathbb{R}$*

(i) *The diffusion process U_t admits a unique invariant probability measure $\mu_U(\rho,du)$. Moreover $\mu_U(\rho,du)$ has a C^∞ density $p_U(\rho,u)$ with respect to the Lebesgue measure on P^{d-1} which solves the Fokker–Planck equation*

$$[\mathcal{L}_\rho^* p_U(\rho,\cdot)](u) = 0\ , \quad \forall u \in P^{d-1}\ ,$$

where $\mathcal{L}_\rho$ is the infinitesimal generator associated with Equation (3).

(ii) *The number*

$$\lambda_\rho \triangleq \int_{P^{d-1}} q(\rho,u)\,\mu_U(\rho,du)$$

is equal to the top Lyapunov exponent λ_1.

(iii) *For all $x_0 \in \mathbb{R}^d$, $x_0 \neq 0$, $\lambda_\rho(x_0) = \lambda_\rho$ with probability one. When $\lambda_\rho < 0$, the system (1) is exponentially stable with probability one.*

Proof See [3].

Proposition 1.3 *Under Hypothesis 1.1, the application $\mathbb{R} \ni \rho \mapsto \lambda_\rho \in \mathbb{R}$ is continuous.*

Proof We give here a sketch of the proof, for details see [6]. Let suppose, for sake of simplicity, that $B_l = 0$ for $l = 1,\ldots,L$ (i.e. the gain matrix parameter act only on the drift coefficient).

First, using the fact that the coefficient $h(M,\cdot)$ are globally Lipschitz on P^{d-1}, we have that for all $t \geq 0$, there exist $C_t < \infty$, such that for all ρ_1, $\rho_2 \in \mathbb{R}$

$$\sup_{u\in P^{d-1}} E\left\|U_t^{1,u} - U_t^{2,u}\right\|^2 \leq C_t\ \|\rho_1-\rho_2\|^2\ ,$$

where $U_t^{i,u}$ denote the solution of (3) with parameter ρ_i and starting at point u.

Then, let $\rho_n \to \rho$ as $n \to \infty$. We want to prove that $\lambda_{\rho_n} \to \lambda_\rho$ as $n \to \infty$. Let U_t^u (resp. $U_t^{n,u}$) denote the solution of Equation (3) with control matrix ρ (resp. ρ_n) and starting at point u. We have

$$\lim_{n\to\infty} \sup_{u\in P^{d-1}} E\,\|U_t^{n,u} - U_t^u\|^2 = 0\ . \qquad (6)$$

This last property allows us to prove that $\mu_U(\rho,du)$ is the weak limit of $\mu_U(\rho_n,du)$.

Finally,

$$\begin{aligned}
&|\lambda_{\rho_n} - \lambda_\rho| \\
&\quad\leq \left|\int_{P^{d-1}} q(\rho_n,u)\,\mu_U(\rho_n,du) - \int_{P^{d-1}} q(\rho,u)\,\mu_U(\rho,du)\right| \\
&\quad\leq \sup_{u\in P^{d-1}} |q(\rho_n,u) - q(\rho,u)| \\
&\qquad + \left|\int_{P^{d-1}} q(\rho,u)\,[\mu_U(\rho_n,du) - \mu_U(\rho,du)]\right|
\end{aligned}$$

which tends to 0. □

We deduce from Proposition 1.3 that $\mathcal{D} \triangleq \{\lambda_\rho; \rho \in \mathbb{R}\}$ is a connected interval of $\mathbb{R}$.

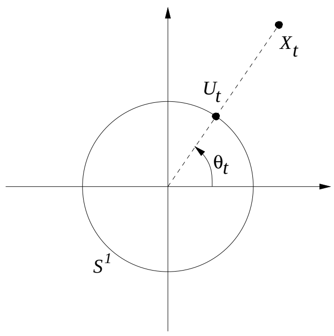
Definition 1.4 *System (1) is said to be* stabilizable *(resp.* non stabilizable*) if $\mathcal{D} \cap [-\infty,0[\neq \emptyset$ (resp. $\mathcal{D} \cap [-\infty,0[= \emptyset$).*

2 Special case : $d = 2$

We introduce the process θ_t defined by

$$U_t = \begin{pmatrix} \cos\theta_t \\ \sin\theta_t \end{pmatrix}\ . \qquad (7)$$

The process θ_t take is values in $[0,\pi]$ which can be identified to P^1.



We present two examples of stabilizable and non stabilizable systems (cf. [6] for more details).

Example 2.1 (Stabilizable system) *We consider the following SDE*

$$\begin{aligned}
dX_t &= \begin{pmatrix} -1 & 0 \\ 0 & -1+\rho \end{pmatrix} X_t\,dt \\
&\quad + \begin{pmatrix} 0 & \rho \\ -\rho & 0 \end{pmatrix} X_t \circ dW(t)\ .
\end{aligned}$$

In this case it is easy to prove that the Lyapunov exponent is

$$\lambda_\rho = -1 + \rho \int_0^\pi \sin^2(\theta)\,\nu(\rho,d\theta)\ , \qquad (8)$$

where $\nu(\rho, d\theta)$ denote the invariant measure associated with the process θ_t. Then, in [6], we prove that $\lambda_\rho \to -\infty$ as $\rho \to -\infty$. Which proves that the system is stabilizable. □

Example 2.2 (Unstabilizable system) *We consider the following SDE*

$$dX_t = \begin{pmatrix} 1 & 0 \\ 0 & 1+\rho \end{pmatrix} X_t\,dt + \begin{pmatrix} 0 & \rho \\ 0 & 0 \end{pmatrix} X_t \circ dW(t) .$$

In this case, in [6] we prove that $\lambda_\rho \geq 1$, for all ρ, so that the system is not stabilizable. □

3 Gradient of the Lyapunov exponent

In this section we restrict ourself to the two–dimensional case (i.e. $d = 2$) with one standard real Wiener input process (i.e. $L = 1$). We use the notation of the previous section. Throughout this section, ρ is fixed. Equation (1) reads

$$dX_t = A_0^\rho\, X_t\,dt + A_1^\rho\, X_t \circ dW_t , \tag{9}$$

where $X_0 = x_0 \neq 0 \in \mathbb{R}^2$ (here W is a standard real Brownian motion). We know that the system (1) is exponentially stable with probability one if and only if $\lambda_\rho < 0$. In order to find ρ which minimize λ_ρ we can use a gradient algorithm.

The next step is to derive the derivative of λ_ρ with respect to ρ. Let

$$G(\rho) \triangleq \frac{\partial}{\partial\rho}\lambda_\rho .$$

We know that under Hypothesis 1.1,

$$\lambda_\rho = \lim_{T\to\infty}\text{–a.s.}\ \frac{1}{T}\int_0^T q(\rho, U_t)\,dt .$$

Let

$$V_t \triangleq \frac{\partial U_t}{\partial\rho}$$

(which is well defined because all coefficients of Equation (3) are smooth enough) and the couple

$$Z_t \triangleq \begin{pmatrix} U_t \\ V_t \end{pmatrix} .$$

In this section we want show that, under certain conditions, the process Z_t admits a unique invariant measure $\mu_Z(\rho, du, dv)$.

Which will imply that

$$G(\rho) = \lim_{T\to\infty}\text{–a.s.}\ \frac{1}{T}\int_0^T Q(\rho, U_t, V_t)\,dt ,$$

where

$$Q(\rho, u, v) \triangleq \frac{\partial}{\partial\rho} q(\rho, u) + \frac{\partial}{\partial u} q(\rho, v) .$$

We define ξ_t the derivative with respect to ρ of the process θ_t defined by (7).

$$\xi_t \triangleq \frac{\partial}{\partial\rho}\theta_t . \tag{10}$$

Process θ_t is solution of the following SDE

$$d\theta_t = h(A_0^\rho, \theta_t)\,dt + h(A_1^\rho, \theta_t) \circ dW_t . \tag{11}$$

Let

$$F_i^\rho(\theta) \triangleq \frac{\partial}{\partial\theta} h(A_i^\rho, \theta) , \text{ and } f_i^\rho(\theta) \triangleq \frac{\partial}{\partial\rho} h(A_i^\rho, \theta) ,$$

for $i = 0, 1$. From (11), we get

$$d\xi_t = [F_0^\rho(\theta_t)\,\xi_t + f_0^\rho(\theta_t)]\,dt + [F_1^\rho(\theta_t)\,\xi_t + f_1^\rho(\theta_t)] \circ dW_t . \tag{12}$$

We define $\varphi_{t,s}$ as the fundamental solution associated with Equation (12), that is

$$d\varphi_t = F_0^\rho(\theta_t)\,\varphi_t\,dt + F_1^\rho(\theta_t)\,\varphi_t \circ dW_t , \quad \varphi_{s,s} = I . \tag{13}$$

We denote $\varphi_t \triangleq \varphi_{t,0}$. $\varphi_{s,t}$ is given by

$$\varphi_{t,s} = \exp\left(\int_s^t F_0^\rho(\theta_r)\,dr + \int_s^t F_1^\rho(\theta_r) \circ dW_r\right) .$$

Hypothesis 3.1 *There exist $\alpha < 0$ such that*

$$\lim_{t\to\infty}\text{–a.s.}\ \frac{1}{t}\log\|\varphi_t\| = \alpha .$$

Proposition 3.2 *Under hypotheses 1.1 and 3.1, Equation (12) admits a unique stationary solution.*

Proof The proof of this result is technical and will be presented elsewhere. □

This result imply that under Hypotheses 1.1 and 3.1, the process $Z_t = (U_t, V_t)$ (or the process (θ_t, ξ_t)) admits a unique invariant measure $\mu_Z(\rho, du, dv)$. So we can have the following representation for $G(\rho)$

$$\begin{aligned} G(\rho) &= \lim_{T\to\infty}\text{–a.s.}\ \frac{1}{T}\int_0^T Q(\rho, U_t, V_t)\,dt \\ &= \int Q(\rho, u, v)\,\mu_Z(\rho, du, dv) . \end{aligned} \tag{14}$$

4 Stabilization algorithm

Suppose that System (1) is stabilizable in the sense given in section 1. We can try to find a parameter ρ such that $\lambda_\rho < 0$ and the more this term is negative, the more System (1) will be stable. So a natural idea is to use a gradient algorithm which minimize λ_ρ.

$$\rho_{n+1} \leftarrow \rho_n - \gamma_n\,G(\rho_n) .$$

where $\gamma_n > 0$ is a decreasing gain parameter. Of course, this algorithm is not useful in practice because $G(\rho_n)$ cannot be computed explicitly. We present now a way to approximate such an expression.

5 Implementation

Let Δ be a time step. Given ρ_n, we simulate the processes U_t and V_t with fixed parameter ρ_n for $t \in [(n-1)\,\Delta, n\,\Delta)$, then we use the following update rule

$$\rho_{n+1} = \rho_n - \gamma_n\,\tilde{G}(\rho_n)\ , \quad n = 0, 2, \dots\ , \quad \rho_0 \text{ given}, \tag{15}$$

where

$$\tilde{G}(\rho_n) \triangleq \frac{1}{\Delta}\int_{n\Delta}^{(n+1)\Delta} Q(\rho_n, \tilde{U}_t^n, \tilde{V}_t^n)\,dt\ . \tag{16}$$

Where $(\tilde{U}^n, \tilde{V}^n)$ is given by "freezing" parameter ρ at the value ρ_n over the interval $t \in [n\,\Delta, (n+1)\,\Delta]$, i.e. it is the solution of (we explicitly denote the dependency with respect to parameter ρ)

$$\begin{aligned} dU_t^\rho &= h(A_0^\rho, U_t^\rho)\,dt + h(A_1^\rho, U_t^\rho) \circ dW_t \qquad (17)\\ dV_t^\rho &= \{\ell_0^\rho(U_t^\rho) + L_0^\rho(U_t^\rho, V_t^\rho)\}\,dt \\ &\quad + \{\ell_1^\rho(U_t^\rho) + L_1^\rho(U_t^\rho, V_t^\rho)\} \circ dW_t \qquad (18)\end{aligned}$$

for $(n-1)\,\Delta < t \le n\,\Delta$, $\rho = \rho_n$ and initial conditions

$$U^\rho_{(n-1)\Delta} = \tilde{U}^{n-1}_{(n-1)\Delta}\ , \quad V^\rho_{(n-1)\Delta} = \tilde{V}^{n-1}_{(n-1)\Delta}\ ,$$

where

$$\begin{aligned} L_i^\rho(U_t^\rho, V_t^\rho) &\triangleq \frac{\partial}{\partial u} h(A_i^\rho, U_t^\rho)\ V_t^\rho\ ,\\ \ell_i^\rho(U_t^\rho) &\triangleq \frac{\partial}{\partial \rho}\,h(A_i^\rho, U_t^\rho)\ ,\end{aligned}$$

for $i = 0, 1$.

For simulation, we discretize Equations (17,18) in time using a Milshtein scheme [10]. This simulation is achieved in two steps :

`1st step` We simulate (X_t^ρ, Y_t^ρ) for $t \in [n\Delta, (n+1)\Delta[$ with $\rho = \rho_n$. These processes are solution of the following Itô–sense system :

$$\begin{aligned} dX_t^\rho &= \left[A_0^\rho + \frac{1}{2}(A_1^\rho)^2\right]\,X_t^\rho\,dt + A_1^\rho\,X_t^\rho\,dW_t\\ dY_t^\rho &= S_t^\rho(X_t^\rho, Y_t^\rho)\,dt + [A_1^\rho\,Y_t^\rho + B_1\,X_t^\rho]\,dW_t\end{aligned}$$

where

$$\begin{aligned} S_t^\rho(X_t^\rho, Y_t^\rho) &\triangleq \left[A_0^\rho + \frac{1}{2}(A_1^\rho)^2\right]\,Y_t^\rho\\ &\quad + \frac{1}{2}(A_1\,B_1 + B_1\,A_1)\,X_t^\rho\\ &\quad + (B_0 + \rho\,B_1^2)\,X_t^\rho\ .\end{aligned}$$

Let $\bar{X}_n$ and $\bar{Y}_n$ be the approximations of $\tilde{X}^{\rho_n}_{n\Delta}$ and $\tilde{Y}^{\rho_n}_{n\Delta}$ respectively. We get :

$$\begin{aligned} \bar{X}_{n+1} &\leftarrow \bar{X}_n + A_1^{\rho_n}\,\bar{X}_n\,\Delta W_n + A_0^{\rho_n}\,\bar{X}_n\,\Delta\\ &\quad + \frac{1}{2}\,(A_1^{\rho_n})^2\,\bar{X}_n\,(\Delta W_n)^2\\ \bar{Y}_{n+1} &\leftarrow \bar{Y}_n + [A_1^{\rho_n}\,\bar{Y}_n^{\rho_n} + B_1\,\bar{X}_n^{\rho_n}]\,\Delta W_n\\ &\quad + \left[A_0^{\rho_n}\,\bar{Y}_n + \frac{1}{2}\,B_1\,A_1\,\bar{X}_n + B_0\,\bar{X}_n\right]\,\Delta\\ &\quad + \frac{1}{2}\,\left[(A_1^{\rho_n})^2\,\bar{Y}_n + A_1^{\rho_n}\,B_1\,\bar{X}_n\right]\,(\Delta W_n)^2\end{aligned}$$

where $\Delta W_n \triangleq W_{t_{n+1}} - W_{t_n} \sim \mathcal{N}(0, \Delta)$ is a white Gaussian noise.

`2nd step` We compute $(\bar{U}_n, \bar{V}_n)$, the approximation of $(\tilde{U}^{\rho_n}_{n\Delta}, \tilde{V}^{\rho_n}_{n\Delta})$, from $(X_n^{\rho_n}, Y_n^{\rho_n})$ using a projection technique :

$$\bar{U}_n \leftarrow \frac{\bar{X}_n}{\|\bar{X}_n\|}\ , \quad \bar{V}_n \leftarrow \frac{\bar{Y}_n - (\bar{U}_n, \bar{Y}_n)\,\bar{U}_n}{\|\bar{X}_n\|}\ .$$

We end up with the following algorithm :

$$\rho_{n+1} = \rho_n - \gamma_n\,Q(\rho_n, \bar{U}_n, \bar{V}_n)\ , \quad n = 0, 2, \dots\ ,$$

ρ_0 given. For practical implementation we use the standard following gain coefficient :

$$\gamma_n = c + \frac{b}{\max(1, n - n_0)}$$

where $c \ge 0$, $b > 0$, et $n_0 \ge 1$ are given.

6 Numerical simulation

We present three examples. In each case, for the initial value of ρ the system is unstable (i.e. $\lambda_\rho > 0$), but using the proposed algorithm we can tune parameter ρ up, in such way that process X stabilize.

6.1 Example 1

$$\begin{aligned} dX_t^1 &= -[1+\rho]\,X_t^1\,dt - [4+\rho]\,X_t^2 \circ dW(t)\ ,\\ dX_t^2 &= [2-\rho]\,X_t^2\,dt + [4+\rho]\,X_t^1 \circ dW(t)\ .\end{aligned}$$

Stochastic gradient algorithm parameters :

$$n_0 = 2.10^5,\ c = 10^{-5},\ b = 4.10^{-6},\ \rho_0 = 0.5$$

6.2 Example 2

$$\begin{aligned} dX_t^1 &= [2 + 0.1\,\rho]\,X_t^1\,dt\\ &\quad + [-2\,X_t^2 + 0.01\,\rho\,X_t^1] \circ dW(t)\ ,\\ dX_t^2 &= [X_t^1 - 5\,X_t^2]\,dt + [X_t^1 + 0.01\,\rho\,X_t^2] \circ dW(t)\ .\end{aligned}$$

Stochastic gradient algorithm parameters :

$$n_0 = 10^5,\ c = 9^{-4},\ b = 7.10^{-2},\ \rho_0 = 200\ .$$

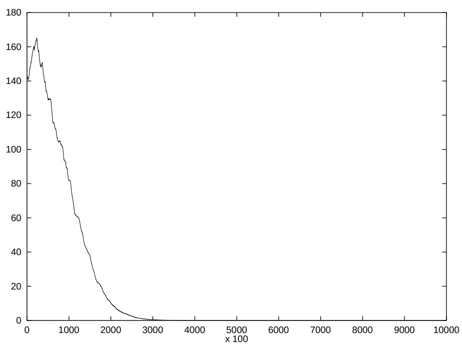


Figure 1: Example 1 : Graph $n \to \left\|\bar{X}_n\right\|$.

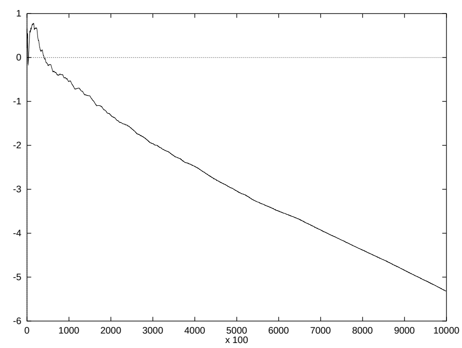


Figure 2: Example 1 : Graph $n \to \lambda_n$ (approximation of λ_{ρ_n}).

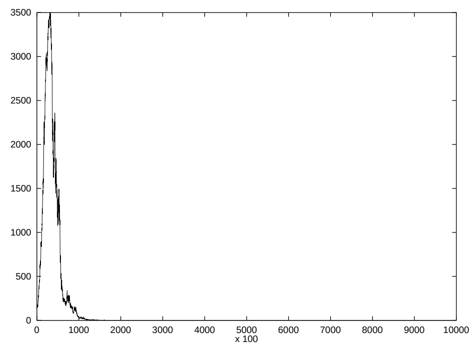


Figure 3: Example 2 : Graph $n \to \left\|\bar{X}_n\right\|$.

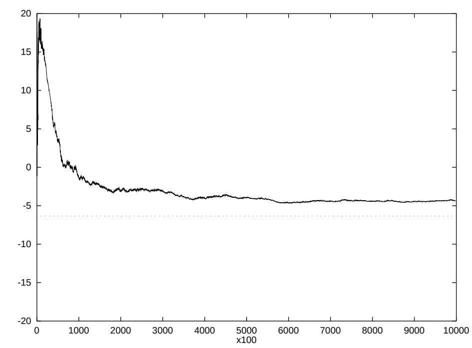


Figure 4: Example 2 : Graph $n \to \lambda_n$ (approximation of λ_{ρ_n}).

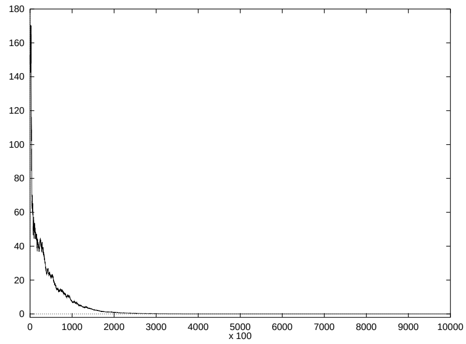


Figure 5: Example 3 : Graph $n \to \left\| \bar{X}_n \right\|$.

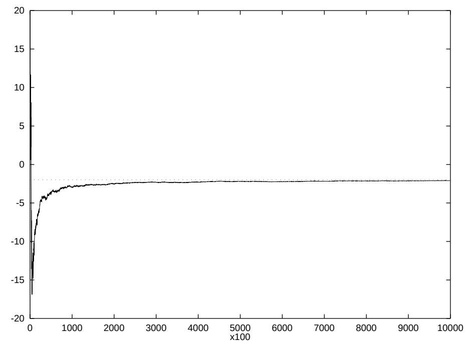


Figure 6: Example 3 : Graph $n \to \lambda_n$ (approximation of λ_{ρ_n}).

6.3 Exemple 3

$$
\begin{aligned}
dX_t^1 &= [1 + 0.1\,\rho]\,X_t^1\,dt \\
&\qquad +[0.1\,X_t^2 + 0.01\,\rho\,X_t^1] \circ dW(t)\ , \\
dX_t^2 &= -2\,X_t^2\,dt + [0.1\,X_t^1 + 0.01\,\rho\,X_t^2] \circ dW(t)\ .
\end{aligned}
$$

Stochastic gradient algorithm parameters :

$$n_0 = 2.10^5,\ c = 1,\ b = 3,\ \rho_0 = 100\ .$$

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