Proceedings of the 34th
Conference on Decision & Control
New Orleans, LA - December 1995

A STABILIZATION ALGORITHM FOR LINEAR CONTROLLED SDE'S

Fabien Campillo and Abdoulaye Traoré

INRIA Sophia–Antipolis, BP 93
06902 Sophia–Antipolis Cédex, France
{campillo, traore}@sophia.inria.fr

Abstract

We consider a stochastic differential equation with linear feedback control with a one–dimensional feedback gain parameter. The problem is to find an online algorithm which adjust this gain in order to stabilize the system. We propose a stochastic gradient method which minimize the Lyapunov exponent associated with the solution of this stochastic differential equation. We present some simulation tests.

1. Preliminaries

We consider the following linear stochastic differential equation in $\mathbb{R}^d$

$$dX_t = A_0^\rho X_t\,dt + A_1^\rho X_t \circ dW_t\,,\; X_0 = x_0 \neq 0 \quad (1)$$

where

$$A_i^\rho = A_i + \rho B_i\,, \quad i = 0, 1\,,$$

A_i and B_i are $d \times d$ matrices, $\rho \in \mathbb{R}$, W is a standard Wiener process. Here "$\circ dW$" (resp. "dW") refer to the Stratonovich (resp. Itô) stochastic integral.

We define the Lyapunov exponent of the solution of (1) starting at x_0 with parameter ρ

$$\lambda_\rho \triangleq \lim_{t\to\infty} \frac{1}{t} \log \|X_t\|\;.$$

Equation (1) is exponentially stable with probability one if and only if, under suitable conditions, $\lambda_\rho < 0$.

Let $S^{d-1} = \{x \in \mathbb{R}^d; \|x\| = 1\}$. We can define the projection of X_t onto S^{d-1} by $U_t = \|X_t\|^{-1} X_t$, it is the solution of the following SDE on S^{d-1}

$$dU_t = h_0^\rho(U_t)\,dt + h_1^\rho(U_t) \circ dW_t\,,\; U_0 = \frac{x_0}{\|x_0\|} \quad (2)$$

$$h_i^\rho(u) \triangleq A_i^\rho u - (A_i^\rho u, u)\,u\,, \quad i = 0, 1\,.$$

Here (x, y) is the scalar product and $\|x\|^2 = (x, x)$.

For each matrix M, $h(M, -u) = -h(M, u)$, so that $h(M, \cdot)$ can be viewed as a vector field on the projective space P^{d-1} (obtained from S^{d-1} by identifying u and $-u$). Therefore (2) can be considered as a stochastic differential equation on P^{d-1}.

Hypothesis 1.1 *For all $u \in P^{d-1}$ and $\rho \in \mathbb{R}$*
$$\dim \text{Lie Algebra}\{h_i^\rho(\cdot)\,, i = 0, 1\}(u) = 1.$$

Proposition 1.2 (See [1]) *Under Hypothesis 1.1, process U_t admits a unique invariant probability measure μ_U^ρ, and*

$$\lambda_\rho = \int_{P^{d-1}} q^\rho(u)\,\mu_U^\rho(du)$$

with

$$q^\rho(u) \triangleq (A_0^\rho u, u) + \frac{((A_1^\rho)^2 u, u) + \|A_1^\rho u\|^2}{2} - (A_1^\rho u, u)^2\;.$$

In [3], we proved that, under Hypothesis 1.1, application $\rho \mapsto \lambda_\rho$ is continuous. We deduce that $\mathcal{D} = \{\lambda_\rho; \rho \in \mathbb{R}\}$ is a connected interval of $\mathbb{R}$, then System (1) is said to be *stabilizable* (resp. *non stabilizable*) if $\mathcal{D} \cap [-\infty, 0[\neq \emptyset$ (resp. $\mathcal{D} \cap [-\infty, 0[= \emptyset$).

2. Gradient of the Lyapunov exponent

Here, we suppose $d = 2$. We introduce the process θ_t defined by $U_t = (\cos\theta_t, \sin\theta_t)^*$. The process θ_t take is values in $[0, \pi]$ which can be identified to P^1.

Throughout this section, ρ is fixed. We know that the system (1) is exponentially stable with probability one if and only if $\lambda_\rho < 0$. In order to find ρ which minimize λ_ρ we can use a gradient algorithm.

Under Hypothesis 1.1

$$\lambda_\rho = \operatorname*{lim\text{–}a.s.}_{T\to\infty} \frac{1}{T} \int_0^T q^\rho(U_t)\,dt\;.$$

Let $\xi_t = \partial\theta_t/\partial\rho$, we have

$$\begin{aligned} d\theta_t &= h_0^\rho(\theta_t)\,dt + h_1^\rho(\theta_t) \circ dW_t\,, \\ d\xi_t &= F_0^\rho(\theta_t, \xi_t)\,dt + F_1^\rho(\theta_t, \xi_t) \circ dW_t\,. \end{aligned} \quad (3)$$

where

$$F_i^\rho(\theta, \xi) \triangleq \frac{\partial}{\partial\theta} h_i^\rho(\theta)\,\xi + \frac{\partial}{\partial\rho} h_i^\rho(\theta)\,, \quad i = 0, 1\,.$$

Let ϕ_t be the fundamental solution of (3).

Proposition 2.1 *Suppose that there exist $\alpha < 0$ such that $\frac{1}{t}\log\|\phi_t\| \to \alpha$ p.s. as $t \to \infty$, and that Hypothesis 1.1 is fulfilled, then Equation (3) admits a unique stationary solution.*

The proof of this result is technical and will be presented elsewhere. We deduce that the process Z_t :

$$Z_t = \begin{pmatrix} U_t \\ V_t \end{pmatrix} \quad \text{with} \quad V_t = \frac{\partial U_t}{\partial \rho} ,$$

admits a unique invariant measure, and

$$\begin{aligned} G(\rho) &\triangleq \frac{\partial \lambda_\rho}{\partial \rho} = \lim_{T\to\infty}\text{-a.s.}\, \frac{1}{T}\int_0^T Q^\rho(U_t, V_t)\, dt , \\ Q^\rho(u,v) &\triangleq \frac{\partial}{\partial \rho} q^\rho(u) + \frac{\partial}{\partial u} q^\rho(u).v . \end{aligned}$$

3. Stabilization algorithm

We want to find a parameter ρ such that $\lambda_\rho < 0$. A natural idea is to use a gradient algorithm which minimize λ_ρ.

$$\rho_{n+1} \leftarrow \rho_n - \gamma_n\, G(\rho_n) , \quad n \geq 0 .$$

where $\gamma_n > 0$ is a decreasing gain parameter.

Let Δ be a time step and

$$I_n^\Delta \triangleq [n\,\Delta, (n+1)\,\Delta) , \quad n \geq 0 .$$

We use an (approximated) gradient algorithm given in two two steps :

Simulation step We simulate (X_t, Y_t) solution of (1) over I_n^Δ with $\rho = \rho_n$ fixed. Let $\bar{X}_{n+1}$ and $\bar{Y}_{n+1}$ be the value of the Milshtein scheme approximations of these processes at the end of the interval I_n^Δ (i.e. at $t = (n+1)\,\Delta$).

We compute $(\bar{U}_{n+1}, \bar{V}_{n+1})$ using a projection technique :

$$\bar{U}_{n+1} \leftarrow \frac{\bar{X}_{n+1}}{\|\bar{X}_{n+1}\|} , \; \bar{V}_{n+1} \leftarrow \frac{\bar{Y}_{n+1} - (\bar{U}_{n+1}, \bar{Y}_{n+1})\, \bar{U}_{n+1}}{\|\bar{X}_{n+1}\|} .$$

Update step We end up with the following algorithm :

$$\rho_{n+1} \leftarrow \rho_n - \gamma_n\, Q^{\rho_n}(\bar{U}_{n+1}, \bar{V}_{n+1}) , \quad n \geq 0$$

ρ_0 given. For practical implementation we use the standard following gain coefficient

$$\gamma_n = c + \frac{b}{\max(1, n - n_0)} , \quad n \geq 0 ,$$

$c \geq 0$, $b > 0$, and $n_0 \geq 1$ given.

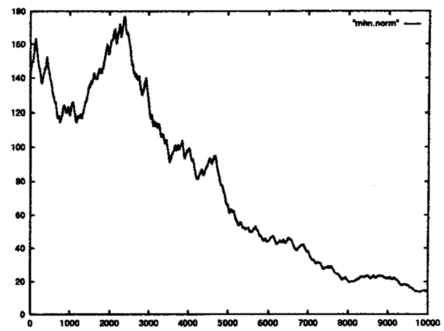

Fig. 1. Example : Graph $n \to \|\bar{X}_n\|$.

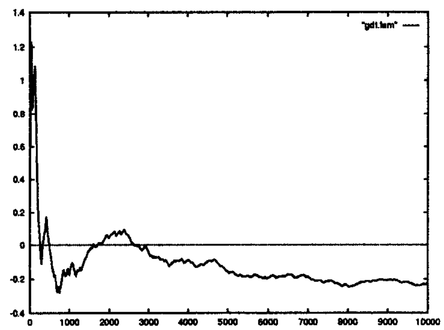

Fig. 2. Example : Graph $n \to \lambda_n$ (approximation of λ_{ρ_n}).

4. Numerical simulation

We take : $X_0 = (100, 100)^*$, $Y_0 = (0.01, 0.01)^*$, $\Delta = 10^{-4}$, we make 10^5 iterations, and

$$\begin{aligned} A_0 &= \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} & B_0 &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\ A_1 &= \begin{pmatrix} 0 & -4 \\ 4 & 0 \end{pmatrix} & B_1 &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

$n_0 = 2\,10^4$, $c = 1\,10^{-1}$, $b = 4\,10^{-6}$, $\rho_0 = 0.5$.

References

[1] L. Arnold, E. Oeljeklaus, and E. Pardoux. Almost sure and moment stability for linear itô equations. In L. Arnold and V. Wihstutz, editors, *Lyapunov Exponents, Bremen–1984*, volume 1186 of *Lecture Notes in Mathematics*, pages 129–159, Berlin, 1986. Springer Verlag.

[2] P.H. Baxendale and E.M. Henning. Stabilization of a linear system via rotational control. *Random & Computational Dynamics*, 1(4):395–421, 1992–93.

[3] F. Campillo and A. Traoré. Lyapunov exponents of controlled SDE's and stabilizability property : Some examples. Rapport de Recherche 2397, INRIA, November 1994.