Optimal Ergodic Control of Nonlinear Stochastic Systems

Fabien Campillo

Abstract

We study a class of ergodic stochastic control problems for diffusion processes. We describe the basic ideas concerning the Hamilton–Jacobi–Bellman equation. For a given class of control problems we establish an existence and uniqueness property of the invariant measure. Then we present a numerical approximation to the optimal feedback control based on the discretization of the infinitesimal generator using finite difference schemes. Finally, we apply these techniques to the control of semi–active suspensions for road vehicle.

1 Introduction

This paper deals with a numerical procedure for optimal stochastic control problems and its application to a non trivial example. This procedure consists in approximating the non linear Hamilton–Jacobi–Bellman partial differential equation which is formally satisfied by the minimal cost function. We use finite difference techniques and with a suitable choice of the schemes, the resulting discrete equation can be viewed as the dynamic programming equation for the minimal cost function for the optimal control of a certain Markov process with finite state space [26].

In section 2, we present the different Hamilton–Jacobi–Bellman (HJB in short) equations which arise in the optimal control of diffusion processes in $\mathbb{R}^n$ (for different cases: finite horizon, infinite horizon with discounted or undiscounted cost functions). We stress the intuitive setup of the HJB equations rather than the mathematical aspects.

In section 3, we introduce a particular class — denoted by $\mathcal{C}$ — of ergodic control problems. Some characteristics of this problem are non classical (the diffusion is degenerate, the coefficients are non linear and discontinuous) and there is no available result concerning the HJB equation. Even the first step — giving a meaning to the cost function — is non trivial; this point is treated here (we prove the existence and the uniqueness of the invariant measure associated with the system, for any given admissible control). This class of problems derives from a particular application in control of suspension systems [4].

In section 4, the approximation procedure is detailed in a more general context than the class $\mathcal{C}$. For the special case of the class $\mathcal{C}$ we have already stated two types of results [4]: existence and uniqueness property for the discrete HJB

equation (with convergence of the algorithm used for solving it) and a convergence property of the approximation as the discretization step tends to 0. Finally, we apply these techniques to the suspension problem [4, 3] and perform some numerical tests; related suboptimal and adaptive techniques may be found in [3].

2 The HJB Equation

In this section we give an *intuitive* presentation of the HJB equations. For mathematical treatments of this problem one can consult: for the deterministic control [7, 17, 30], for the stochastic control of diffusion processes on finite horizon [7, 5, 17], for the control of diffusion processes on infinite horizon [7, 31, 36], for the control of Markov chains [10], for the control of Markov processes on infinite horizon [15], for the probabilistic aspects [24] and for numerical aspects [1, 6, 14, 21, 25, 26, 28, 32, 34, 35]. For other aspects of the stochastic control theory of diffusion processes one can consult [9] for the impulse control problem, [8] for the optimal stopping time problem and [20] for the stochastic maximum principle.

2.1 Finite–horizon problem

The problem We consider a diffusion process on $\mathbb{R}^n$

$$dX_t = b(X_t, u(X_t, t))\, dt + \sigma(X_t)\, dW_t \ , \quad 0 \le t \le T \ , \quad X_0 = x_0 \in \mathbb{R}^n \ , \quad (1)$$

where W is a n–dimensional standard Wiener process. We suppose that $u \in \mathcal{U}$, $\mathcal{U}$ is a given class of admissible controls which take values in $U \subset \mathbb{R}^k$. Suppose that for any (x, t), $\{u(x, t); u \in \mathcal{U}\} = U$. We will give later an example of such a class.

We fix the instant $T > 0$ and we define the following cost functional

$$J(u) \stackrel{\triangle}{=} E\left[\int_0^T f(X_t, u(X_t, t))\, dt + h(X_T)\right] \ . \quad (2)$$

The stochastic control problem is to find $\hat{u} \in \mathcal{U}$ which minimizes the cost functional J among all the admissible controls.

We introduce the infinitesimal generator associated with the system (1)

$$\mathcal{L}^u \phi(x) \stackrel{\triangle}{=} \sum_{i=1}^n b_i(x, u) \frac{\partial \phi(x)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 \phi(x)}{\partial x_i\, \partial x_j} \ . \quad (3)$$

with $a(x) \stackrel{\triangle}{=} \sigma(x)\, \sigma(x)^*$.

The HJB equation

Definition 1. The *value function* is defined by

$$v(x,t) \triangleq \inf_{u \in \mathcal{U}} J_{x,t}(u) , \tag{4}$$

with[4]

$$J_{x,t}(u) \triangleq E_{x,t}^u \left[\int_t^T f(X_s, u(X_s, s)) \, ds + h(X_T) \right] . \tag{5}$$

Optimality Principle For any $h > 0$,

$$v(x,t) = \inf_{u \in \mathcal{U}} E_{x,t}^u \left[\int_t^{t+h} f(X_s, u(X_s, s)) \, ds + v(X_{t+h}, t+h) \right] . \tag{6}$$

Proof. First we introduce (a version of) the *Dynamic Programming Principle*: suppose that $(x,s) \to \hat{u}^t(x,s)$ is the optimal feedback for the control problem over the time interval $[t,T]$ starting at point x, i.e.

$$J_{x,t}(\hat{u}^t) = v(x,t)$$

then, given $h > 0$, $\hat{u}^t|_{[t+h,T]}$ is the optimal feedback for the control problem over the time interval $[t,T]$ starting at point X_{t+h}, i.e.

$$J_{X_{t+h}, t+h}(\hat{u}^t) = v(X_{t+h}, t+h) \quad \text{a.s.} . \tag{7}$$

Now we consider

$$\overline{u}(x,s) = \begin{cases} \tilde{u}(x,s) , & s \in [t, t+h] , \\ \hat{u}^t(x,s) , & s \in]t+h, T] , \end{cases}$$

for some control $\tilde{u}$. By definition of $v(x,t)$ and using (7), we have

$$v(x,t) \leq J_{x,t}(\overline{u})$$

$$= E_{x,t}^u \left[\int_t^{t+h} f(X, \tilde{u}) \, ds + \int_{t+h}^T f(X, \hat{u}^t) \, ds + h(X_T) \right]$$

$$= E_{x,t}^u \left[\int_t^{t+h} f(X, \tilde{u}) \, ds + E_{X_{t+h}, t+h}^{\hat{u}^t} \left(\int_{t+h}^T f(X, \hat{u}^t) \, ds + h(X_T) \right) \right]$$

$$= E_{x,t}^u \left[\int_t^{t+h} f(X, \tilde{u}) \, ds + v(X_{t+h}, t+h) \right] .$$

---

[4] *Notations:* The term $E_{x,t}^u$ in expressions of the form

$$E_{x,t}^u \int_{t_1}^{t_2} \Psi(X_t, t) \, dt \text{ or } dW_t ,$$

means that we consider the diffusion process X_t solution of (1) starting from point x at time t (i.e. $X_t = x$) and using the feedback control u. When $t = 0$, $E_{x,t}^u$ is denoted by E.

· So we get

$$v(x,t) \le E_{x,t}^u \left[\int_t^{t+h} f(X_s, \tilde{u}(X_s, s))\, ds + v(X_{t+h}, t+h) \right] ,$$

and the equality holds if $\bar{u} = \hat{u}^t$, which proves (6).

Hamilton–Jacobi–Bellman equation The value function (4) satisfies the following equation

$$\frac{\partial}{\partial t} v(x,t) + \inf_{u \in U} \left[\mathcal{L}^u v(x,t) + f(x,u) \right] = 0 , \quad v(x,T) = h(x) . \tag{8}$$

Proof. For notational convenience we consider the 1-dimensional case. $E_{x,t}^u$ is denoted by E. Using (6)

$$\inf_{u \in \mathcal{U}} E \left[\frac{v(X_{t+h}, t+h) - v(x,t)}{h} + \frac{1}{h} \int_t^{t+h} f(X_s, u(X_s, s))\, ds \right] = 0 . \tag{9}$$

Moreover

$$v(X_{t+h}, t+h) = v(x,t) + v'(x,t)\,(X_{t+h} - x)$$
$$+ \frac{1}{2} v''(x,t)\,(X_{t+h} - x)^2 + h\,\frac{\partial}{\partial t} v(x,t) + o(h) ,$$

so

$$E \left[\frac{v(X_{t+h}, t+h) - v(x,t)}{h} \right] = v'(x,t)\, E \left[\frac{X_{t+h} - x}{h} \right] \tag{10}$$
$$+ \frac{1}{2} v''(x,t)\, E \left[\frac{(X_{t+h} - x)^2}{h} \right]$$
$$+ \frac{\partial}{\partial t} v(x,t) + o(1) .$$

Equation (1) with initial condition $X_t = x$, leads to

$$E \left[\frac{X_{t+h} - x}{h} \right] = E\,\frac{1}{h} \int_t^{t+h} b(X_s, u(X_s, s))\, ds + E\,\frac{1}{h} \int_t^r \sigma(X_s)\, dW_s$$
$$= E\,\frac{1}{h} \int_t^{t+h} b(X_s, u(X_s, s))\, ds$$
$$\underset{h \to 0}{\longrightarrow} b(x, u(x,t)) . \tag{11}$$

Similarly

$$E\left[\frac{(X_{t+h}-x)^2}{h}\right] = \frac{1}{h}E\left[\int_t^{t+h} b(X_s,u(X_s,s))\,ds\right]^2$$

$$+\frac{2}{h}E\left[\int_t^{t+h} b(X_s,u(X_s,s))\,ds\int_t^{t+h}\sigma(X_s)\,dW_s\right]$$

$$+\frac{1}{h}E\left[\int_t^{t+h}\sigma(X_s)\,dW_s\right]^2$$

$$\underset{h\to0}{\simeq}\frac{1}{h}E\left[\int_t^{t+h}\sigma(X_s)\,dW_s\right]^2$$

$$=\frac{1}{h}E\int_t^{t+h}\sigma^2(X_s)\,ds$$

$$\underset{h\to0}{\longrightarrow}\sigma^2(x)\ . \tag{12}$$

¿From (11,12) and (10) we get

$$E\left[\frac{v(X_{t+h},t+h)-v(x,t)}{h}\right]$$

$$\underset{h\to0}{\simeq}v'(x,t)\,b(x,u(x,t))+\frac{1}{2}v''(x,t)\,\sigma^2(x)+\frac{\partial}{\partial t}v(x,t)\ .$$

Let $h\to0$ in (9), so we get

$$\inf_{u\in\mathcal{U}}\left[\frac{\partial}{\partial t}v(x,t)+v'(x,t)\,b(x,u(x,t))+\frac{1}{2}v''(x,t)\,\sigma^2(x)+f(x,u(x,t))\right]=0\ .$$

This shows (8). The condition at final time is an easy consequence of (5,4). The preceding argument is made rigorous if we know *a priori* that

$$v\in C^{2,1}(\mathbb{R}^n\times]0,\infty[)\ ,$$

but the main question lies in the fact that v is not in general $C^{2,1}$ (cf. [31]).

Verification result Let v be the solution of the HJB equation (8). Any function $\tilde{u}(x,t)$ which satisfies the relation

$$\mathcal{L}^{\tilde{u}(x,t)}v(x,t)+f(x,\tilde{u}(x,t))=\inf_{u\in U}\left[\mathcal{L}^u v(x,t)+f(x,u)\right]\ ,\quad\forall(x,t)\ , \tag{13}$$

is an optimal feedback control, i.e.

$$J(\tilde{u})=\inf_{u\in\mathcal{U}}J(u)\ . \tag{14}$$

Proof. We need the following lemma (cf. [23] for a precise statement)

244

Lemma 2 a Feynman–Kac formula. *Let $\Psi(x,s)$ be a solution of the following backward partial differential equation*

$$\frac{\partial}{\partial s}\Psi(x,s) + \mathcal{L}^u\,\Psi(x,s) + M(x,s) = 0 \ ,$$

then, for any $T > t$,

$$\Psi(x,t) = E_{x,t}^u\left[\int_t^T M(X_s,s)\,ds + \Psi(X_T,T)\right] \ .$$

Moreover, from (13) and (8) we get

$$\frac{\partial}{\partial t}\,v(x,t) + \mathcal{L}^{\tilde{u}(x,t)}\,v(x,t) + f(x,\tilde{u}(x,t)) = 0 \ ,$$

hence, using lemma 2 we find

$$v(x,t) = E_{x,t}^{\tilde{u}}\left[\int_t^r f(X_s,\tilde{u}(X_s,s))\,ds + v(X_r,r)\right] \ , \quad t < r \ ,$$

and for $r = T$ and $t = 0$, this last equality becomes

$$v(x,0) = E\left[\int_0^T f(X_s,\tilde{u}(X_s,s))\,ds + h(X_T)\right] \ .$$

The definitions of J and v lead to (14).

Rigorous statements [7] The main hypothesis is the nondegeneracy of the matrix $a(x) \stackrel{\triangle}{=} \sigma(x)\,\sigma(x)^*$. The case of degenerate diffusion is much more tricky. So we suppose that

$$a(x) \stackrel{\triangle}{=} \sigma(x)\,\sigma(x)^* \geq \sigma_0\,I > 0 \ , \quad \forall x \in \mathbb{R}^n \ . \tag{15}$$

Up to now, $\mathcal{U}$ was a class of Markovian controls (i.e. feedback controls); throughout this section, we use a wider class containing controls which are not necessarily Markovian (i.e. stochastic processes which are not only function of the current state value X_t). We define the class $\mathcal{U}$ of admissible controls as follow [5]

$$u = \{u(t); 0 \leq t \leq T\} \in \mathcal{U} \quad \Longleftrightarrow \quad \begin{cases} u \in L_{\mathcal{F}}^2(0,T;\mathbb{R}^k) \ , \\ u(t,\omega) \in U, \ t \text{ a.e.}, \ \omega \text{ a.s.} \ . \end{cases}$$

----

[5] the space $L_{\mathcal{F}}^2(0,T;\mathbb{R}^k)$ is defined by

$$z \in L_{\mathcal{F}}^2(0,T;\mathbb{R}^k) \quad \Longleftrightarrow \quad \begin{cases} (t,\omega) \to z(t,\omega) \text{ is measurable}, \\ E\int_0^T |z(t)|^2\,dt < \infty \ , \\ z(t) \in L^2(\Omega,\mathcal{F}_t,P) \ , \quad t \text{ a.e.} \end{cases}$$

where $\{\mathcal{F}_t\}$ is the filtration associated with the Wiener process w.

We also make the following hypotheses

- U is a closed convex subset of $\mathbb{R}^k$, (16)
- $f(x,u) \geq f_0\,|u|^2 - C_0$, $h(x) \geq -C_0$, $\forall\,(x,u) \in \mathbb{R}^n \times U$, (17)
- $b : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \to L(\mathbb{R}^n, \mathbb{R}^n)$, (18)

  b, σ are continuously differentiable, the derivatives are bounded,

  $|b(x,u)| \leq \bar{b}\,(1 + |x| + |u|)$, $|\sigma(x)| \leq \bar{\sigma}\,(1 + |x|)$,
- $f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$, $h : \mathbb{R}^n \to \mathbb{R}$, (19)

  f, h are continuously differentiable,

  $|f(x,u)| \leq \bar{f}\,(1 + |x|^2 + |u|^2)$, $|h(x)| \leq \bar{h}\,(1 + |x|^2)$,

  $|\partial f / \partial x_i|,\ |\partial f / \partial u_i| \leq \bar{f}\,(1 + |x| + |u|)$, $|\partial h / \partial x_i| \leq \bar{h}\,(1 + |x|)$.

Concerning the quasilinear partial differential equation (8), we have the following result

Theorem 3 Bensoussan[7]. *Under assumptions (15—19), there exists one and only one solution v of (8) such that ([6])*

$$v \in L^p(0, T; W^{2,p}_{loc}(\mathbb{R}^n)) \ , \quad \frac{\partial}{\partial t} v \in L^p(0, T; L^p_{loc}(\mathbb{R}^n)) \ , \tag{20}$$

for any $2 \leq p < \infty$, and

$$|v(x,t)| \leq C\,(1 + |x|^2) \ , \quad \left| \frac{\partial v}{\partial x_i} \right| \leq C\,(1 + |x|) \ . \tag{21}$$

This solution is explicitly given by (4).

Then there exists $\hat{u} : \mathbb{R}^n \times [0, \infty[\to \mathbb{R}^k$ such that

$$\mathcal{L}^{\hat{u}(x,t)}\,v(x,t) + f(x, \hat{u}(x,t)) = \inf_{u \in U}\,[\mathcal{L}^u\,v(x,t) + f(x,u)] \ .$$

Using the estimates (21), we can prove that

$$|\hat{u}(x,t)| \leq C\,(1 + |x|) \ . \tag{22}$$

This last theorem does not establish that the infimum in the right hand side of (4) is reached. The reason is that the feedback $\hat{u}(x,t)$ is not smooth enough to ensure that the s.d.e. (1) has a strong solution.

With additional assumptions we can prove that the feedback $\hat{u}(x,t)$ is optimal for the original problem (1,2). Indeed

---

[6] The space $L^p_{loc}(\mathbb{R}^n)$ is defined by

$$z \in L^p_{loc}(\mathbb{R}^n) \quad \Longleftrightarrow \quad z\phi \in L^p(\mathbb{R}^n),\ \forall\,\phi \in C_0^\infty(\mathbb{R}^n) \ ,$$

and the space $W^{2,p}_{loc}(\mathbb{R}^n)$ is defined by

$$z \in W^{2,p}_{loc}(\mathbb{R}^n) \quad \Longleftrightarrow \quad z,\ \frac{\partial z}{\partial x_i},\ \frac{\partial^2 z}{\partial x_i\,\partial x_j} \in L^p_{loc}(\mathbb{R}^n) \ .$$

Theorem 4 Bensoussan[7]. *Under the assumptions of Theorem 3, assume that*

- $\dfrac{\partial^2 f}{\partial x_i\,\partial u_q}$, $\dfrac{\partial^2 f}{\partial u_p\,\partial u_q}$, $\dfrac{\partial^2 b}{\partial x_i\,\partial u_q}$, $\dfrac{\partial^2 b}{\partial u_p\,\partial u_q}$ *exist and are continuous, bounded* (23)

- $\left[\dfrac{\partial^2 f}{\partial u_p\,\partial u_q}\right] \geq f_0\, I$, $\left|\dfrac{\partial^2 b}{\partial u_p\,\partial u_q}\right| \leq \dfrac{b_0}{1+|x|}$, $f_0 > b_0 \sup\limits_{x\in\mathbb{R}^n} \dfrac{|\nabla v|}{1+|x|}$. (24)

Then $\hat{u}(x,t)$ is an optimal feedback control for the problem (1,2).

2.2 Infinite–horizon problems

The discounted case We consider the stochastic control problem defined by the state equation

$$dX_t = b(X_t, u(X_t))\,dt + \sigma(X_t)\,dW_t , \quad t \geq 0 \tag{25}$$

and the cost function

$$J(u) \triangleq E \int_0^\infty e^{-\alpha t} f(X_t, u(X_t))\,dt . \tag{26}$$

Note that the feedback control does not depend on t anymore.

Formally, we can take $h \equiv 0$ and replace f by $e^{\alpha t} f$ in (2), we get

$$J_T(u) \triangleq E \int_0^T e^{-\alpha t} f(X_t, u(X_t))\,dt ,$$

and (8) becomes

$$\frac{\partial}{\partial t} v_\alpha^T(x,t) + \inf_{u\in U}\left[\mathcal{L}^u v_\alpha^T(x) + e^{-\alpha t} f(x,u)\right] = 0 , \quad v_\alpha^T(x,T) \equiv 0 . \tag{27}$$

Let

$$\tilde{v}_\alpha^T(x,t) \triangleq e^{\alpha t}\, v_\alpha^T(x,t) , \tag{28}$$

so (27) reads

$$\frac{\partial}{\partial t} \tilde{v}_\alpha^T(x,t) + \inf_{u\in U}\left[\mathcal{L}^u \tilde{v}_\alpha^T(x,t) + f(x,u)\right] = \alpha\, \tilde{v}_\alpha^T(x,t) , \quad e^{-\alpha T}\, \tilde{v}_\alpha^T(x,T) \equiv 0 ,$$

taking $T \to \infty$ in this last equation yields

$$\frac{\partial}{\partial t} \tilde{v}_\alpha(x,t) + \inf_{u\in U}\left[\mathcal{L}^u \tilde{v}_\alpha(x,t) + f(x,u)\right] = \alpha\, \tilde{v}_\alpha(x,t) , \quad t \geq 0 , \tag{29}$$

where $\tilde{v}_\alpha(x,t) = \tilde{v}_\alpha^\infty(x,t)$.

In fact $\tilde{v}_\alpha(x,t)$ does not depend on t, this can be checked using (28) and the definition of $v_\alpha^T(x,t)$, indeed

$$\tilde{v}_\alpha(x,t) = e^{\alpha t} \, v_\alpha^\infty(x,t)$$

$$= e^{\alpha t} \min_{u \in \mathcal{U}} E^u_{x,t} \int_t^\infty e^{-\alpha s} f(X_s, u(X_s)) \, ds$$

$$= \min_{u \in \mathcal{U}} E^u_{x,t} \int_t^\infty e^{-\alpha (s-t)} f(X_s, u(X_s)) \, ds$$

$$= \min_{u \in \mathcal{U}} E \int_0^\infty e^{-\alpha s} f(X_s, u(X_s)) \, ds$$

this last equality comes from the fact that equation (25) is homogeneous in t. Hence $\tilde{v}_\alpha(x,t)$ is now denoted $\tilde{v}_\alpha(x)$ and (29) can be rewritten as follows

$$\inf_{u \in U} [\mathcal{L}^u \, \tilde{v}_\alpha(x) + f(x,u)] = \alpha \, \tilde{v}_\alpha(x) \,, \quad t \geq 0 \,. \tag{30}$$

Equation (30) is the HJB for the infinite–horizon control problem associated with state equation (25) and the discounted cost function (26). The optimal feedback control is given by

$$\hat{u}_\alpha(x) \in \operatorname{Arg} \min_{u \in U} [\mathcal{L}^u \, \tilde{v}_\alpha(x) + f(x,u)]$$

The undiscounted case We consider the diffusion process (25). We want to minimize an *average cost* of the form

$$J(u) \triangleq \liminf_{T \to \infty} \frac{1}{T} E \int_0^T f(X_t, u(t)) \, dt \,. \tag{31}$$

Most frequently the *discounted cost function* (26) is used. In many applications, the cost functional (31) is more realistic than (26) because it represents a time average while (26) involves a discount factor which is often difficult to evaluate and not always relevant. In general, (31) implies that the control stabilizes the system. In fact we need to suppose some recurrence and stability conditions (cf. [27]). From the mathematical viewpoint, the discounted problem is easier than the undiscounted one, since the former avoid considering the behavior of the controlled process as t goes to infinity.

For the average case (31), we want to find a pair (v, ρ) where ρ is a constant and v is a smooth function, such that the following HJB equation is satisfied

$$\inf_{u \in U} [\mathcal{L}^u \, v(x) + f(x,u)] = \rho \,. \tag{32}$$

Let (v, ρ) be a solution of (32), then the optimal feedback is given by

$$\hat{u}(x) \in \operatorname{Arg} \min_{u \in U} [\mathcal{L}^u \, v(x) + f(x,u)]$$

and ρ is the optimal cost, i.e.

$$\rho = J(\hat{u}) = \inf_{u \in U} J(u) \,.$$

The average cost problem can be viewed as a limit model for the discounted problem as $\alpha \to 0$, indeed

$$\alpha \, \tilde{v}_\alpha(\cdot) \underset{\alpha \to 0}{\to} \rho \ ,$$

$$\tilde{v}_\alpha(\cdot) - \tilde{v}_\alpha(x_0) \underset{\alpha \to 0}{\to} v(\cdot) \ ,$$

(in a suitable sense, see [7] for a rigorous proof).

3 A Class of Ergodic Stochastic Control Problems

We present a class of models which derive from a problem of control for semi-active suspension systems. In these models — like in most realistic models — difficulties of the following type are met: the coefficients of the diffusion which we want to control are discontinuous and strongly nonlinear.

In section 3.1 we introduce the class $\mathcal{C}$ of problems. Then, in section 3.2, we consider μ_u the invariant measure associated to a system of the class $\mathcal{C}$ for a given admissible control u, and we prove that μ_u exists and is unique. Finally, in section 3.3, we present the original semi-active suspensions problem.

3.1 The problem

Let us consider the following stochastic system

$$dX_t = b\big(u(X_t), X_t\big) \, dt + \begin{pmatrix} 0 \\ \sigma \end{pmatrix} dW_t \ , \tag{33}$$

where X is a process which takes values in $\mathbb{R}^2$, W is a real standard Wiener process and $\sigma > 0$. b maps $\mathbb{R} \times \mathbb{R}^2$ in $\mathbb{R}^2$ and is defined by

$$b(u,x) \triangleq \begin{pmatrix} b_1(u,x) \\ b_2(u,x) \end{pmatrix} \triangleq \begin{pmatrix} x_2 \\ -u \, x_2 - \gamma_1 \, x_1 - \gamma_2 \, \mathrm{sign}(x_2) \end{pmatrix} \ , \quad x \triangleq \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \ ,$$

where γ_1, γ_2 are strictly positive constants. In (33), u is a feedback control which belongs to the class $\mathcal{U}$ of admissible controls defined by (fix $\underline{u}$, $\overline{u}$ such that $0 < \underline{u} < \overline{u} < \infty$)

$$u \in \mathcal{U} \iff \begin{array}{l} u : \mathbb{R}^2 \to [\underline{u}, \overline{u}] \text{ and there exists a finite number of subman-} \\ \text{ifolds of } \mathbb{R}^2 \text{ with dimension less than or equal to 1 outside} \\ \text{of which } u \text{ is continuous.} \end{array}$$

We are concerned with an ergodic type control problem, whose cost functional is

$$J(u) \triangleq \lim_{T \to \infty} \frac{1}{T} E \int_0^T f\big(u(X_t), X_t\big) \, dt \ , \quad \forall u \in \mathcal{U} \ , \tag{34}$$

where the instantaneous cost function f is defined by

$$f(u,x) \triangleq \big(u \, x_2 + \gamma_1 \, x_1 + \gamma_2 \, \mathrm{sign}(x_2)\big)^2 \ . \tag{35}$$

$$dX_t = b(u(X_t), X_t)\, dt + \begin{pmatrix} 0 \\ \sigma \end{pmatrix} dW_t$$

$$b(u, x) \triangleq \begin{pmatrix} b_1(u, x) \\ b_2(u, x) \end{pmatrix} \triangleq \begin{pmatrix} x_2 \\ -u\, x_2 - \gamma_1\, x_1 - \gamma_2\, \mathrm{sign}(x_2) \end{pmatrix} , \quad \gamma_1, \gamma_2 > 0$$

$$J(u) \triangleq \lim_{T \to \infty} \frac{1}{T}\, E \int_0^T f(u(X_t), X_t)\, dt$$

$$f(u, x) \triangleq (u\, x_2 + \gamma_1\, x_1 + \gamma_2\, \mathrm{sign}(x_2))^2$$

Table 1. The class $\mathcal{C}$ of ergodic control problems

¿From now on, we denote

$$b^u(x) \triangleq b(u(x), x) , \quad f^u(x) \triangleq f(u(x), x) , \quad \forall u \in \mathcal{U} .$$

The Hamilton–Jacobi–Bellman equation for the ergodic control problem (33) (34) can be formally written as (see section 2)

$$\min_{u \in [\underline{u}, \overline{u}]} \left(\mathcal{L}^u v(\cdot) + f(u, \cdot) \right) = \rho \quad \text{on } \mathbb{R}^2, \tag{36}$$

where $v : \mathbb{R}^2 \to \mathbb{R}$ is defined up to an additive constant, ρ is a constant and $\mathcal{L}^u$ is the infinitesimal generator associated with (33).

$$\mathcal{L}^u \phi(x) \triangleq b_1^u(x)\, \frac{\partial \phi(x)}{\partial x_1} + b_2^u(x)\, \frac{\partial \phi(x)}{\partial x_2} + \frac{\sigma^2}{2}\, \frac{\partial^2 \phi(x)}{\partial^2 x_2} . \tag{37}$$

Remark. The arguments presented bellow may be applied to a wider class of problems. Indeed, we can consider a system of the form

$$d \begin{pmatrix} X_t^1 \\ X_t^2 \end{pmatrix} = \begin{pmatrix} b_1(X_t) \\ b_2(u, X_t) \end{pmatrix} dt + \begin{pmatrix} 0 \\ \sigma \end{pmatrix} dW_t ,$$

where X_t^1 (resp. X_t^2) takes values in $\mathbb{R}^{n_1}$ (resp. $\mathbb{R}^{n_2}$) and W is a standard Wiener process. The main hypotheses are

(i) the discontinuous terms appear only in the "noisy part" of the system, that is $b_1(x)$ is smooth and $\sigma\sigma^* > 0$,

(ii) the system satisfies a stability property (e.g. $E|X_t|^2 \leq C$, $\forall t \geq 0$).

Point (i) permits us to use a Girsanov transformation to remove the discontinuous terms.

Remark. In this case the choice of the value of the function "sign" at point 0 is not important. Indeed, in (33) the noise is added to the second component, so we can prove that

$$P(X_t^2 = 0) = 0 , \quad \forall t . \tag{38}$$

Property (38) implies that, if we change the value of sign(0), the (weak) solution of (33) will not be changed. If (38) was false, we should use differential inclusion techniques to give a meaning to the stochastic differential equation (33).

3.2 The invariant probability measure

The cost function (34) can be rewritten as

$$J(u) = \langle f^u, \mu_u \rangle, \quad \forall u \in \mathcal{U}, \tag{39}$$

where μ_u is the invariant probability measure associated with system (33). In this section we present an existence and uniqueness property for μ_u which gives a meaning to expressions (34,39). For the results presented in this section all the details can be found in [13].

Proposition 5. *For any $u \in \mathcal{U}$, the diffusion process (33) admits an invariant probability measure μ_u.*

Proof. We fix $u \in \mathcal{U}$. By means of usual techniques (e.g. [16] th. 9.3 ch. 4), it is sufficient to prove the following properties

(i) There exists a constant C such that

$$E |X_t|^2 \leq C, \quad \forall t \geq 0. \tag{40}$$

(ii) The process X_t solution of (33) has the Feller property, i.e. for any $t \geq 0$ and $\phi \in C_b(\mathbb{R}^2)$, the function

$$\mathbb{R}^2 \ni x \longrightarrow E\phi(X_t^x) \tag{41}$$

is continuous, where $\{X_t^x\}$ denotes the diffusion process (33) starting at point x at time 0.

proof of (i) We define

$$V(t) \triangleq E\,\mathcal{V}(X_t), \qquad \mathcal{V}(x) \triangleq \gamma_1\,(x_1)^2 + \varepsilon\,x_1\,x_2 + (x_2)^2.$$

There exists $\varepsilon_0 > 0$ such that for any $\varepsilon_0 > \varepsilon > 0$

$$\mathcal{V}(x) \geq \frac{1}{2}\left(\gamma_1\,(x_1)^2 + (x_2)^2\right).$$

Hence, it is sufficient to show that $V(t) \leq$ Cte for any $t \geq 0$.

We can check [13] that there exist strictly positive constants ε and δ such that

$$\frac{d}{dt}V(t) \leq -C(\varepsilon,\delta)\,V(t) + \frac{\varepsilon}{2\,\delta} + \sigma^2,$$

where $C(\varepsilon,\delta) > 0$, which yields the conclusion.

proof of (ii) In (33), the drift coefficient can be written as

$$b(u, x) = \overline{B} x + \begin{pmatrix} 0 \\ -u\, x_2 - \gamma_2 \operatorname{sign}(x_2) \end{pmatrix}$$

$$\triangleq \begin{pmatrix} 0 & 1 \\ \gamma_1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -u\, x_2 - \gamma_2 \operatorname{sign}(x_2) \end{pmatrix} .$$

Let

$$\overline{W}_t \triangleq W_t + \int_0^t \psi(X_s^x)\, ds ,$$

$$\psi(x) \triangleq -\frac{1}{\sigma} \left(u(x)\, x_2 + \gamma_2 \operatorname{sign}(x_2) \right) ,$$

$$Z_t^x \triangleq \exp\left(\int_0^t \psi(X_s^x)\, d\overline{W}_s - \frac{1}{2} \int_0^t \psi(X_s^x)^2\, ds \right) . \tag{42}$$

We define a new probability law

$$\left. \frac{d\overline{P}}{dP} \right|_{\mathcal{F}_t} \triangleq (Z_t^x)^{-1} , \quad t \geq 0 .$$

X^x satisfies

$$dX_t^x = \overline{B}\, X_t^x\, dt + \begin{pmatrix} 0 \\ \sigma \end{pmatrix} d\overline{W}_t , \tag{43}$$

where — from Girsanov's theorem — $\overline{W}_t$ is a real standard Wiener process under the probability law $\overline{P}$.

For any sequence $x_n \to x$, we want to prove that

$$E\phi(X_t^{x_n}) = \overline{E}\left[\phi(X_t^{x_n})\, Z_t^{x_n} \right] \underset{n \to \infty}{\to} E\phi(X_t^x) = \overline{E}\left[\phi(X_t^x)\, Z_t^x \right] , \tag{44}$$

where $\overline{E}$ denotes the expectation with respect to $\overline{P}$. We can check that it is sufficient to prove that

$$X_t^{x_n} \underset{n \to \infty}{\to} X_t^x \quad \overline{P}\text{-a.s.} , \tag{45}$$

$$Z_t^{x_n} \underset{n \to \infty}{\to} Z_t^x \quad \text{in } \overline{P}\text{-probability.} \tag{46}$$

Under the probability law $\overline{P}$, X_t is the solution of a *linear* stochastic differential system, so (45) is obvious. For (46), we show that

$$\overline{E} \int_0^t \left[u(X_s^{x_n})\, X_s^{x_n,2} - u(X_s^x)\, X_s^{x,2} \right]^2 ds \underset{n \to \infty}{\to} 0 , \tag{47}$$

$$\overline{E} \int_0^t \left[\operatorname{sign}(X_s^{x_n,2}) - \operatorname{sign}(X_s^{x,2}) \right]^2 ds \underset{n \to \infty}{\to} 0 . \tag{48}$$

The difficulty comes from the discontinuity of the functions $\operatorname{sign}(\cdot)$ and $u(\cdot)$, but using the definition of $\mathcal{U}$ we know that theses function are continuous a.e., which, using standard arguments, is enough to conclude.

Proposition 6. *For any* $u \in \mathcal{U}$, *the diffusion process (33) admits a unique invariant measure* μ_u.

Proof. ¿From now on, we fix $u \in \mathcal{U}$ and we suppose that μ denotes an invariant probability measure of the system (33), and X_t is the solution of this system with μ as initial law (i.e. X_0 has law μ). We also define Z_t by (42) where X^x is replaced by X. It is sufficient to prove that

> μ has a density $p(x)$ with respect to Lebesgue measure, and $p(x) > 0$ for almost all x. $\qquad(49)$

Indeed (49) implies that if there exist two invariant measures, they are equivalent. So there exits at most one extremal invariant measure, which establishes the proposition.

We first prove the following result

> Under $\overline{P}$, for any $t > 0$, the law of X_t has a density $\overline{p}(t,x)$ such that $\overline{p}(t,x) > 0$, $\forall x$. $\qquad(50)$

Under $\overline{P}$, consider the system (43) where $d\overline{W}$ is replaced by $v\,dt$ ($v \in L^2(\mathbb{R}^+)$), we get

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -\beta\,x_1 \end{pmatrix} + \begin{pmatrix} 0 \\ \sigma \end{pmatrix} v , \quad x(0) = x . \qquad(51)$$

Let $x^{x,v}(t)$ denote the solution of this last equation. We define the reachability set

$$\mathcal{A}(t,x) \triangleq \left\{ x^{x,v}(t) \; ; \; \forall v \in L^2(\mathbb{R}^+) \right\} .$$

(51) can be rewritten as $\dot{x} = A\,x + B\,v$ and the matrix $[B|A\,B]$ has full rank. Hence this system is controllable [29]. So

$$\forall t > 0 , \quad \forall x \in \mathbb{R}^2 , \quad \mathcal{A}(t,x) = \mathbb{R}^2 . \qquad(52)$$

Using [33] §3.6.1, we prove that — under $\overline{P}$ — the law of X_t is absolutely continuous with respect to Lebesgue measure and that its density $\overline{p}(t,x)$ is strictly positive for any $t > 0$ and x.

Now we prove (49). For any $\phi \in C_b(\mathbb{R}^2)$

$$\langle \mu , \phi \rangle = \overline{E}[\phi(X_t)\,Z_t] ,$$

$$= \overline{E}\left[\phi(X_t)\,\overline{E}[Z_t|X_t] \right] ,$$

$$= \int_{\mathbb{R}^2} \phi(x)\,\overline{E}[Z_t|X_t = x]\,\overline{p}(t,x)\,dx .$$

Since $\overline{E}[Z_t|X_t] > 0$ $\overline{P}$–a.s. and under $\overline{P}$ the law of X_t is equivalent to Lebesgue measure, we get $\overline{E}[Z_t|X_t = x] > 0$ $\forall x$–a.e. . Using (50) and the last inequality, we prove that μ has a density

$$q(x) \triangleq \overline{E}[Z_t|X_t = x]\,\overline{p}(t,x) ,$$

and that this density is strictly positive for almost all $x \in \mathbb{R}^2$.

3.3 An example: a semi–active suspension system

In this section we present a damping control method for a nonlinear suspension of road vehicle (comprising a spring, a shock absorber, a mass, and taking into account the dry friction, cf. figure 1). The aim is to improve the ride comfort.

Among alternatives to classical suspension systems (passive systems) we distinguish between active and semi–active techniques. An active suspension system consists in force elements in addition to a spring and a damper assembly. Force elements continuously vary the force according to some control law. In general, an active system is costly, complex, and requires an external power source [19]. In contrast, a semi–active system requires no hydraulic power supply, and the implementation of its hardware is simpler and cheaper than a fully active system. A semi–active suspension system acts only on damping or spring laws, so it can only dissipate or store energy.

Here we consider a system with control on the damping law, the forces in the damper are generated by modulating its orifice for fluid flow [2, 37]. We use the simplest model which consists in a one degree–of–freedom model (this model can be represented as a problem of the class $\mathcal{C}$).

The equation of motion for a one degree–of–freedom model is

$$m\,\ddot{y} + c\,\dot{y} + k_s\,y + z = -m\,\ddot{e}\;, \tag{53}$$

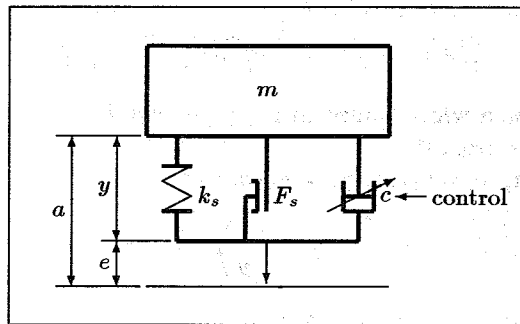(cf. figure 1 and table 2 for the exact definition of the terms).

Fig. 1. One degree–of–freedom model.

$\ddot{e}$ denotes the input acceleration. The restoring force $k_s\,y + z$, has a linear part $k_s\,y$, and a nonlinear part z which describes the dry friction force [11, 12] defined by

$$\dot{z} + \beta\left(|\dot{y}|z + \dot{y}|z|\right) = (k - k_s)\dot{y}\;, \tag{54}$$

where $\beta > 0$ and $k > k_s > 0$. For "large displacements", z degenerate to a Coulomb friction force

$$z = F_s\,\mathrm{sign}(\dot{y})\;, \tag{55}$$

a absolute displacement of mass m
y absolute displacement ($y = a - e$)
e stochastic input (surface road acceleration)
m sprung mass
c shock–absorber damping constant (controlled)
k_s spring constant
F_s dry friction constant

Table 2. Notations.

with $F_s = (k - k_s)/2\beta$. The damping force is $c\dot{y}$ where $c > 0$ is the instantaneous damping coefficient (the control is acting on this term).

The general model is described by equations (53,54). The problem is to compute a feedback law $c = c(y, \dot{y})$ such that the solution of the system (53,54) minimizes a criterion — related to the vibration comfort —

$$J(u) \triangleq \lim_{T \to \infty} \frac{1}{T} E \int_0^T |\ddot{a}|^2 \, dt = \lim_{T \to \infty} \frac{1}{T} E \int_0^T |\ddot{y} + \ddot{e}|^2 \, dt \ .$$

This model leads to a control problem for a 3–dimensional diffusion process. If we want to obtain a 2–dimensional problem, we must use the system given by equations (53,55) (i.e. we use a Coulomb force term). We get the following simplified model

$$m\ddot{y} + c\dot{y} + k_s x + F_s \operatorname{sign}(\dot{y}) = -m\ddot{e} \ , \qquad (56)$$

$\ddot{e}$ is supposed to be a white Gaussian noise process, $\ddot{e} = -\sigma \, dW/dt$ where W is a standard Wiener process.

Using $u = c/m$, $\gamma_1 = k_s/m$, $\gamma_2 = F_s/m$ and

$$X \triangleq \begin{pmatrix} y \\ \dot{y} \end{pmatrix} \ ,$$

equation (56) can be rewritten as (33).

We get the following system

$$dX_t = b(u(X_t), X_t) \, dt + \begin{pmatrix} 0 \\ \sigma \end{pmatrix} dW_t \ , \qquad (57)$$

where

$$b(u, x) = \begin{pmatrix} b_1(u, x) \\ b_2(u, x) \end{pmatrix} \triangleq \begin{pmatrix} x_2 \\ -(u \, x_2 + \gamma_1 \, x_1 + \gamma_2 \operatorname{sign}(x_2)) \end{pmatrix} \ . \qquad (58)$$

Hence the instantaneous cost function is

$$f(u, x) \triangleq |\ddot{y} + \ddot{e}|^2 = |u \, x_2 + \gamma_1 \, x_1 + \gamma_2 \operatorname{sign}(x_2)|^2 \ . \qquad (59)$$

4 Numerical Approximation

We use the following procedure: we do not discretize directly the HJB equation but we transform the original ergodic control problem to a control problem for a Markov process in continuous time and finite state space (section 4.1). Then, for the discrete case, we can write a dynamic programming equation (section 4.2); this equation is solved numerically via an iterative algorithm (section 4.3).

We describe the approximation procedure in the case of a diffusion process defined by

$$dX_t = b(u(X_t), X_t) dt + \sigma(X_t) dW_t ,\tag{60}$$

and with the following cost function

$$J(u) = \liminf_{T\to\infty} \frac{1}{T} E \int_0^T f(u(X_t), X_t) dt ,\tag{61}$$

where

$$
\begin{aligned}
b &: \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n , \\
\sigma &: \qquad \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^d , \\
f &: \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^+ .
\end{aligned}
$$

X takes values in $\mathbb{R}^n$ and W in $\mathbb{R}^d$. u belongs to a given class $\mathcal{U}$ of applications from $\mathbb{R}^n$ to $U \subset \mathbb{R}^k$. We suppose that, for any $u \in \mathcal{U}$, the solution X_t of (60) admits a unique invariant probability measure, so the cost function (61) is well defined.

The infinitesimal generator associated with (60) is

$$\mathcal{L}^u \phi(x) \triangleq \sum_{i=1}^n b_i(x, u) \frac{\partial \phi(x)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 \phi(x)}{\partial x_i \partial x_j} ,\tag{62}$$

where $a(x) \triangleq \sigma(x) \sigma(x)^*$. We note $b_i^u(x) = b_i(u, x)$ and $f^u(x) = f(u, x)$.

In section 4.4, we apply this approximation technique to the class $\mathcal{C}$. For this class we also present some convergence results which were proved in [4, 13].

4.1 The finite state space problem

In a first step we approximate the solution X_t of (60) by a controlled Markov process X_t^h in continuous time and discrete (but infinite) state space. In a second step, X_t is approximated by a controlled Markov process $X_t^{h,D}$ in continuous time and finite state space.

$$\frac{\partial \phi(x)}{\partial x_i} \simeq \begin{cases} \dfrac{\phi(x + e_i\,h_i) - \phi(x)}{h_i} & \text{if } b_i^u(x) > 0 \\[2ex] \dfrac{\phi(x) - \phi(x - e_i\,h_i)}{h_i} & \text{if } b_i^u(x) < 0 \end{cases}$$

$$\frac{\partial^2 \phi(x)}{\partial x_i^2} \simeq \frac{\phi(x + e_i\,h_i) - 2\,\phi(x) + \phi(x - e_i\,h_i)}{h_i^2}$$

$$\frac{\partial^2 \phi(x)}{\partial x_i\,\partial x_j} \simeq \begin{cases} \dfrac{2\,\phi(x) + \phi(x + e_i\,h_i + e_j\,h_j) + \phi(x - e_i\,h_i - e_j\,h_j)}{2\,h_i\,h_j} \\[2ex] \quad - \dfrac{\phi(x + e_i\,h_i) + \phi(x - e_i\,h_i) + \phi(x + e_j\,h_j) + \phi(x - e_j\,h_j)}{2\,h_i\,h_j} \\[2ex] \hspace{10cm} \text{if } a_{ij}(x) > 0 \\[3ex] -\,\dfrac{2\,\phi(x) + \phi(x + e_i\,h_i - e_j\,h_j) + \phi(x - e_i\,h_i + e_j\,h_j)}{2\,h_i\,h_j} \\[2ex] \quad + \dfrac{\phi(x + e_i\,h_i) + \phi(x - e_i\,h_i) + \phi(x + e_j\,h_j) + \phi(x - e_j\,h_j)}{2\,h_i\,h_j} \\[2ex] \hspace{10cm} \text{if } a_{ij}(x) < 0 \end{cases}$$

$$i, j = 1, \ldots, n\,, \ i \neq j\,, \quad e_i \text{ unit vector in the } i\text{th coordinate direction}$$

Table 3. Finite difference schemes.

first step: discrete state space Let h_i (resp. e_i) denote the finite difference interval (resp. the unit vector) in the ith coordinate direction and $h = (h_1, \ldots, h_n)$. We define $\mathbb{R}_h^n$, the h-grid on $\mathbb{R}^n$, by

$$\mathbb{R}_h^n \triangleq \{x \in \mathbb{R}^n\,;\, x_i = n_i\,h_i + h_i/2\,,\ i = 1, \ldots, n\,,\ n_i \in \mathbf{Z}\}\ .$$

The infinitesimal generator (62) is approximated using finite difference schemes given in table 3. The reason for the choices in the schemes will be explained below.

$\mathcal{L}^u$ is approximated by an infinite dimensional matrix $\mathcal{L}_h^u$ of $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ given as follows

$$\mathcal{L}^u \phi(x) \simeq \mathcal{L}_h^u \phi(x) \triangleq \sum_{y \in \mathbb{R}_h^n} \mathcal{L}_h^u(x, y)\,\phi(y)\,, \quad \forall x \in \mathbb{R}_h^n$$

the terms $\mathcal{L}_h^u(x, y)$ of this matrix are detailled in table 4.

The matrix $\mathcal{L}_h^u$ has the following property

$$\sum_{y \in \mathbb{R}_h^n} \mathcal{L}_h^u(x, y) = 0\,, \quad \forall x \in \mathbb{R}_h^n\ .$$

$$\mathcal{L}_h^u(x,x) \triangleq -\sum_i \left(\frac{a_{ii}(x)}{h_i^2} - \frac{1}{2} \sum_{k;k\neq i} \frac{|a_{ik}(x)|}{h_i\,h_k} \right) - \sum_i \frac{|b_i^u(x)|}{h_i}$$

$$\mathcal{L}_h^u(x,x+e_i\,h_i) \triangleq \frac{1}{2}\left(\frac{a_{ii}(x)}{h_i^2} - \sum_{k;k\neq i} \frac{|a_{ik}(x)|}{h_i\,h_k} \right) + \frac{(b_i^u(x))^+}{h_i}$$

$$\mathcal{L}_h^u(x,x-e_i\,h_i) \triangleq \frac{1}{2}\left(\frac{a_{ii}(x)}{h_i^2} - \sum_{k;k\neq i} \frac{|a_{ik}(x)|}{h_i\,h_k} \right) + \frac{(b_i^u(x))^-}{h_i}$$

$$\mathcal{L}_h^u(x,x+e_i\,h_i+e_j\,h_j) \triangleq \mathcal{L}_h^u(x,x-e_i\,h_i-e_j\,h_j) \triangleq \frac{a_{ij}^+(x)}{2\,h_i\,h_j}$$

$$\mathcal{L}_h^u(x,x+e_i\,h_i-e_j\,h_j) \triangleq \mathcal{L}_h^u(x,x-e_i\,h_i+e_j\,h_j) \triangleq \frac{a_{ij}^-(x)}{2\,h_i\,h_j}$$

$$i,j = 1,\ldots,n,\ i\neq j$$

Table 4. The discrete infinitesimal generator.

Suppose that

$$a_{ii}(x) - \sum_{j;j\neq i} |a_{ij}(x)| \geq 0, \qquad \forall x \in \mathrm{I\!R}_h^n,\ i=1,\ldots,n, \tag{63}$$

then

$$\mathcal{L}_h^u(x,y) \geq 0, \qquad \forall x,y \in \mathrm{I\!R}_h^n,\ x\neq y.$$

Remark. The choice of the finite difference schemes we use (cf. table 3) depends on the sign of the drift coefficients of the diffusion process. The reason for the choice is the following: if (63) is true then $\{\mathcal{L}_h^u(x,y)\,;\,x,\,y\in\mathrm{I\!R}_h^n\}$ can be viewed as the infinitesimal generator of a continuous–time Markov process X_t^h with discrete state space $\mathrm{I\!R}_h^n$ [18]. We will see later why this is important.

So we get a stochastic control problem for a Markov process X_t^h with infinitesimal generator $\mathcal{L}_h^u$, and the following cost function

$$J_h(u) \triangleq \lim_{T\to\infty} E\,\frac{1}{T}\int_0^T f^u(X_t^h)\,dt. \tag{64}$$

u belongs to the class $\mathcal{U}_h$ defined by

$$u \in \mathcal{U}_h \iff u \text{ is an application from } \mathrm{I\!R}_h^n \text{ to } U.$$

second step: finite state space X_t^h has a discrete but infinite state space; if we want to perform computations it is necessary to work on a finite state space. We consider a bounded domain D of $\mathbb{R}^n$. We define a new state space

$$\mathbb{R}_{h,D}^n \stackrel{\triangle}{=} \mathbb{R}_h^n \cap D = \{x^1, x^2, \ldots, x^N\} \ , \quad N \stackrel{\triangle}{=} \mathrm{Card}\left(\mathbb{R}_{h,D}^n\right) \ . \tag{65}$$

Because we are working on a bounded domain, we must specify boundary conditions. In practice, D will be chosen large enough so that the process will rarely reach the border. Hence, the choice of the boundary conditions is of little importance, provided that all the states communicate. Example of such conditions (usually reflecting conditions) will be given later for the suspension problem.

So we get an approximation $\mathcal{L}_{h,D}^u$ to $\mathcal{L}_h^u$

$$\mathcal{L}_{h,D}^u \, \phi(x) = \sum_{y \in \mathbb{R}_{h,D}^n} \mathcal{L}_{h,D}^u(x,y) \, \phi(y) \ ,$$

$\mathcal{L}_{h,D}^u$ is a $N \times N$–matrix.

Remark. The choices in the finite difference schemes (cf. table 3) imply that

$$\sum_{y \in \mathbb{R}_{h,D}^n} \mathcal{L}_{h,D}^u(x,y) = 0 \ , \quad \forall x \in \mathbb{R}_{h,D}^n \ ,$$

moreover, hypothesis (63) implies that

$$\mathcal{L}_{h,D}^u(x,y) \geq 0 \ , \quad \forall x, y \in \mathbb{R}_{h,D}^n \ , x \neq y \ .$$

Hence $\mathcal{L}_{h,D}^u$ can be interpreted as the infinitesimal generator of a controlled Markov process $X_t^{h,D}$ in continuous time and finite state space. $X_t^{h,D}$ is described by the following terms

- a sequence $\{\Delta_l^{h,D} ; l \geq 0\}$ where the random variable $\Delta_l^{h,D}$ denotes the elapsed time between the lth and the $(l+1)$th jump,
- a Markov chain $\{\xi_l^{h,D} ; l \geq 0\}$ with state space $\mathbb{R}_{h,D}^n$, $\xi_l^{h,D}$ denotes the state of the process between the lth and the $(l+1)$th jump.

The law of the random variable $\Delta_l^{h,D}$ and the transition probabilities of the Markov chain $\{\xi_l^{h,D} ; l \geq 0\}$ are defined as follows

- the pair $(\Delta_l^{h,D}, \xi_{l+1}^{h,D})$ depends only on $\xi_l^{h,D}$,
- under the conditional probability law $P(\cdot | \xi_l^{h,D})$, the random variables $\Delta_l^{h,D}$ and $\xi_{l+1}^{h,D}$ are independant.

And for any $x \in \mathbb{R}_{h,D}^n$, under the conditional probability law $P(\cdot | \xi_l^{h,D} = x)$

- the random variable $\Delta_l^{h,D}$ obeys an exponential law of parameter $\delta_l^{h,D}(x)$ where

$$\delta_l^{h,D}(x) \stackrel{\triangle}{=} -\frac{1}{\mathcal{L}_{h,D}^u(x,x)} \ ,$$

— the transition probabilities $\{\pi^{h,D}(x,y)\,;\,y \in \mathbb{R}^n_{h,D}\}$ are defined by

$$\pi^{h,D}(x,y) \triangleq -\frac{\mathcal{L}^u_{h,D}(x,y)}{\mathcal{L}^u_{h,D}(x,x)}\,, \quad x \neq y\,.$$

With a suitable choice of boundary conditions (usually reflecting conditions) and with the finite differences schemes we used, we have: $\mathcal{L}^u_{h,D}(x,x) < 0$ for all $x \in \mathbb{R}^n_{h,D}$.

With remark 4.1, the discretized problem can be viewed as a control problem for a Markov process $X^{h,D}_t$ in continuous time, finite state space, and infinitesimal generator $\mathcal{L}^u_{h,D}$. The cost function is

$$J_{h,D}(u) \triangleq \lim_{T \to \infty} E\,\frac{1}{T} \int_0^T f^u(X^{h,D}_t)\,dt\,, \tag{66}$$

and u belongs to a class $\mathcal{U}_{h,D}$ of control defined by

$$u \in \mathcal{U}_{h,D} \quad \Longleftrightarrow \quad u \text{ is an application from } \mathbb{R}^n_{h,D} \text{ to } U.$$

The solution to this problem is given by the dynamic programming equation.

Remark. Let $\mu^{h,D}_u$ be the invariant measure of the process $X^{h,D}_t$. Using $\mu^{h,D}_u$, the cost function (66) can be rewritten as

$$J_{h,D}(u) = \sum_{x \in \mathbb{R}^n_{h,D}} f^u(x)\,\mu^{h,D}_u(x)\,.$$

The measure $\mu^{h,D}_u$ is solution of the following linear system

$$\begin{cases} \displaystyle\sum_{y \in \mathbb{R}^n_{h,D}} \mathcal{L}^u_{h,D}(y,x)\,\mu^{h,D}_u(y) = 0\,, \quad \forall x \in \mathbb{R}^n_{h,D}\,, \\[2em] \displaystyle\sum_{y \in \mathbb{R}^n_{h,D}} \mu^{h,D}_u(y) = 1\,. \end{cases}$$

4.2 The "discrete" Hamilton–Jacobi–Bellman equation

Associated with the control problem defined in the last section we have the following dynamic programming equation

$$\min_{u \in U} \left[\sum_{y \in \mathbb{R}^n_{h,D}} \mathcal{L}^u_{h,D}(x,y)\,v(y) + f^u(x) \right] = \rho\,, \quad \forall x \in \mathbb{R}^n_{h,D}\,, \tag{67}$$

where ρ is a strictly positive constant and $v : \mathbb{R}^n_{h,D} \to \mathbb{R}$ (i.e. $v \in \mathbb{R}^N$) is defined up to an additive constant.

If (v, ρ) is a solution to (67) then

$$\hat{u}(x) \in \underset{u \in U}{\text{Arg}\min} \left[\sum_{y \in \mathbb{R}^n_{h,D}} \mathcal{L}^u_{h,D}(x,y)\, v(y) + f^u(x) \right] , \quad x \in \mathbb{R}^n_{h,D} \qquad (68)$$

is an optimal feedback control law, and ρ is the minimal cost

$$\rho = J_{h,D}(\hat{u}) = \min_{u \in \mathcal{U}_{h,D}} J_{h,D}(u) .$$

Equation (67) can be viewed as an approximation to the HJB equation (36). Equation (67) gives the solution to the ergodic control problem for the Markov process $X_t^{h,D}$.

4.3 The policy iteration algorithm

In order to solve (67), we use the policy iteration algorithm [15, 22]: suppose that $u^0 \in \mathcal{U}_{h,D}$ — the initial policy — is given. Starting with u^0 we generate a sequence $\{u^j; j \geq 1\}$. The iteration $u^j \to u^{j+1}$ proceeds in two steps (cf. table 5).

1	compute (v^j, ρ^j)	we compute $(v^j, \rho^j) \in \mathbb{R}^N \times \mathbb{R}^+$ the solution of the linear system

$$\sum_{y \in \mathbb{R}^n_{h,D}} \mathcal{L}^{u^j}_{h,D}(x,y)\, v^j(y) + f^{u^j}(x) = \rho^j , \quad \forall x \in \mathbb{R}^n_{h,D}$$

stopping test $\big| |\rho^{j+1} - \rho^j| \leq \varepsilon.$

2	compute u^{j+1}	we solve the N following optimization problems: $\forall x \in \mathbb{R}^2_{h,D}$

$$u^{j+1}(x) \in \text{Arg} \min_{u \in [\underline{u}, \overline{u}]} \left(\sum_{y \in \mathbb{R}^2_{h,D}} \mathcal{L}^{h,D}_u(x,y)\, v^j(y) + f^u(x) \right) .$$

Table 5. The policy iteration algorithm, iteration $u^j \to u^{j+1}$.

Remark. The first step of this algorithm leads to a linear system of dimension N. Let $\mathbb{R}^2_{h,D} = \{x^i ; i = 1, \ldots, N\}$, then the unknown parameters are

$$v(x^2), \ v(x^3), \ldots, v(x^N), \ \rho ,$$

and we take $v(x^1) = 0$.

Remark. For the second step, the optimization problems are nonlinear and they are solved by means of iterative routines. The nonlinearity comes from the discretization technique we use. Indeed, the choice of finite difference approximation (cf. table 3) depends on u. Instead of the schemes of the table 3, we can use central difference approximation (so that it does not depend on u), in which case the second step becomes explicit because the functions to be optimized are now quadratic in u. On the other hand, with this kind of difference approximation, a certain condition on the parameter h has to be fulfilled (h must be small enough) for the matrix $\mathcal{L}_u^{h,D}$ to be the generator of a Markov process. See [26] p.175–179 for further considerations.

4.4 Application to the class of problem $\mathcal{C}$

The approximation In this example, the discretized state space are $\mathbb{R}_h^2$ and $\mathbb{R}_{h,D}^2$ where $h = (h_1, h_2)$ and D is of the form

$$D = [-\overline{x}_1, \overline{x}_1] \times [-\overline{x}_2, \overline{x}_2] ,$$

so

$$\mathbb{R}_{h,D}^2 = \left\{ x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(N_1)} \right\} \times \left\{ x_2^{(1)}, x_2^{(2)}, \ldots, x_2^{(N_2)} \right\} ,$$

with

$$x_1^{(i)} = -\overline{x}_1 + 2\,\overline{x}_1\,\frac{i-1}{N_1-1} , \quad h_1 = \frac{1}{N_1-1} ,$$

$$x_2^{(j)} = -\overline{x}_2 + 2\,\overline{x}_2\,\frac{j-1}{N_2-1} , \quad h_2 = \frac{1}{N_2-1}$$

(cf. figure 2).

The matrix $a(x)$ is degenerate

$$a(x) = \begin{pmatrix} 0 & 0 \\ 0 & \sigma^2 \end{pmatrix} .$$

Condition (63) is fulfilled. The finite difference schemes of table 3 are simplified, they are presented on table 6; the terms of the matrix $\mathcal{L}_h^u$ are presented on table 7.

For this example we give explicit boundary conditions. We define

$$\Gamma_{h,D} \triangleq \left\{ x_1^{(1)}, x_1^{(N_1)} \right\} \times \left\{ x_2^{(1)}, x_2^{(2)}, \ldots, x_2^{(N_2)} \right\}$$

$$\cup \left\{ x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(N_1)} \right\} \times \left\{ x_2^{(1)}, x_2^{(N_2)} \right\} .$$

$\Gamma_{h,D}$ the set of points on the border. We chose very simple reflecting conditions, we obtain the matrix $\mathcal{L}_{h,D}^u$ described table 8.
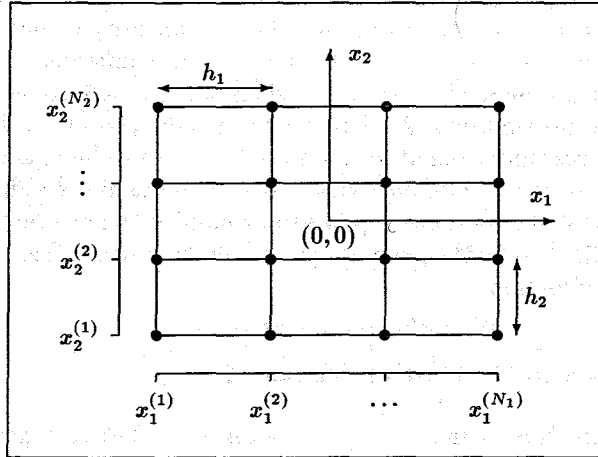
Fig. 2. Discretized state space.

$$\frac{\partial \phi(x)}{\partial x_i} \simeq \begin{cases} \dfrac{\phi(x + e_i\, h_i) - \phi(x)}{h_i} & \text{if } b_i^u(x) > 0 \\[2ex] \dfrac{\phi(x) - \phi(x - e_i\, h_i)}{h_i} & \text{if } b_i^u(x) < 0 \end{cases}$$

$$\frac{\partial^2 \phi(x)}{\partial x_2^2} \simeq \frac{\phi(x + e_2\, h_2) - 2\,\phi(x) + \phi(x - e_2\, h_2)}{h_2^2}$$

$$i = 1,2\,, \quad e_i \text{ unit vector in the } i\text{th coordinate direction}$$

Table 6. Finite difference schemes (class $\mathcal{C}$).

The convergence results We present two kinds of results. Firstly, consider the discrete HJB equation (67), we can prove that it admits a unique solution and that the policy iteration algorithm converges to this unique solution. Secondly, we can also prove a convergence result for the approximation as the discretization step h tends to 0. These results are presented for the class $\mathcal{C}$.

existence and uniqueness of a solution to the discrete HJB equation
We have the following results

Theorem 7. *The HJB equation (67) (with $v(x^1) = 0$) admits a unique solution $(v, \rho) \in \mathbb{R}^N \times \mathbb{R}^+$.*

For the existence part of theorem (7), we use the following

$$\mathcal{L}_h^u(x,x) \triangleq -\frac{\sigma^2}{h_2^2} - \sum_{i=1,2} \frac{|b_i^u(x)|}{h_i}$$

$$\mathcal{L}_h^u(x, x + e_1\, h_1) \triangleq \frac{(b_1^u(x))^+}{h_1}$$

$$\mathcal{L}_h^u(x, x - e_1\, h_1) \triangleq \frac{(b_1^u(x))^-}{h_1}$$

$$\mathcal{L}_h^u(x, x + e_2\, h_2) \triangleq \frac{\sigma^2}{2\, h_2^2} + \frac{(b_2^u(x))^+}{h_2}$$

$$\mathcal{L}_h^u(x, x - e_2\, h_2) \triangleq \frac{\sigma^2}{2\, h_2^2} + \frac{(b_2^u(x))^-}{h_2}$$

Table 7. Discrete infinitesimal generator (class $\mathcal{C}$).

for $x \in \mathbb{R}_{h,D}^2 \setminus \Gamma_{h,D}$ et $y \in \mathbb{R}_{h,D}^2$ $\quad \mathcal{L}_{h,D}^u(x,y) = \mathcal{L}_h^u(x,y)$

for $x \in \Gamma_{h,D}$ such that $x_1 = x_1^{(1)}$ $\quad \mathcal{L}_{h,D}^u(x,x) = \mathcal{L}_h^u(x,x)$
$$\mathcal{L}_{h,D}^u(x, x + h_1\, e_1) = -\mathcal{L}_h^u(x,x)$$

for $x \in \Gamma_{h,D}$ such that $x_1 = x_1^{(N_1)}$ $\quad \mathcal{L}_{h,D}^u(x,x) = \mathcal{L}_h^u(x,x)$
$$\mathcal{L}_{h,D}^u(x, x - h_1\, e_1) = -\mathcal{L}_h^u(x,x)$$

for $x \in \Gamma_{h,D}$ such that $x_2 = x_2^{(1)}$ $\quad \mathcal{L}_{h,D}^u(x,x) = \mathcal{L}_h^u(x,x)$
$$\mathcal{L}_{h,D}^u(x, x + h_2\, e_2) = -\mathcal{L}_h^u(x,x)$$

for $x \in \Gamma_{h,D}$ such that $x_2 = x_2^{(N_2)}$ $\quad \mathcal{L}_{h,D}^u(x,x) = \mathcal{L}_h^u(x,x)$
$$\mathcal{L}_{h,D}^u(x, x - h_2\, e_2) = -\mathcal{L}_h^u(x,x)$$

all other terms are null

Table 8. Discrete infinitesimal generator $\mathcal{L}_{h,D}^u$ (class $\mathcal{C}$).

Corollary 8. *The policy iteration algorithm converge to an optimal feedback control.*

These results are proved in [4], but one can find the same kind of results in a more general setup in [10].

Approximation: a convergence result We present a convergence result concerning the approximation, when the discretization parameter h tends to 0 and when the domain D tends to $\mathbb{R}^2$ (for the complete proof of this result cf. [13]).

Theorem 9. *We consider two strictly increasing sequences*

$$\{\overline{x}_1^h; h > 0\} \quad and \quad \{\overline{x}_2^h; h > 0\}$$

such that $\overline{x}_i^h > 0$ *and* $\overline{x}_i^h \to \infty$ *as* $h \to 0$. *We define*

$$D_h = [-\overline{x}_1^h, \overline{x}_1^h] \times [-\overline{x}_2^h, \overline{x}_2^h] .$$

We suppose that

$$\lim_{h \to 0} h \, \delta_h = 0 , \quad where \quad \delta_h \stackrel{\triangle}{=} radius(D_h) . \tag{69}$$

Then, for any $u \in \mathcal{U}$,

$$J_{h,D_h}(u) \underset{h \to 0}{\to} J(u) .$$

Remark. Theorem 7 proves the existence of an optimal feedback control law for the discretized problem. With such a control, we can associate a feedback control law $\hat{u}_h$ for the continuous state space problem, where $\hat{u}_h$ is piecewise constant. Using theorem 9 we can easily conclude that

$$\limsup_{h \to 0} J_{h,D_h}(\hat{u}_h) \leq \inf_{u \in \mathcal{U}} J(u) .$$

We would like to prove the stronger result that the sequence $\{\hat{u}_h; h > 0\}$ is a minimizing sequence for the functional J, i.e.

$$J(\hat{u}_h) \to \inf_{u \in \mathcal{U}} J(u) , \quad when \ h \to 0 .$$

A numerical example

parameters As an example, we use values which roughly correspond to a suspension system for the seat of a truck: $m = 60(\text{kg})$, $k_s = 3500(\text{N/m})$, $F_s = 40(\text{N})$. These values have already been used in [4]. We also set $\sigma = 0.5$.

We use the following discretization parameters

$$\overline{x}_1 = y_{max} = -y_{min} = 0.1 \ (\text{m}) ,$$
$$\overline{x}_2 = \dot{y}_{max} = -\dot{y}_{min} = 1 \ (\text{m/s}) ,$$
$$n_1 = n_2 = 30 .$$

So we get a $30 \times 30 = 900$ points grid.

optimal feedback control[4] The approximated optimal feedback control (68) (plotted on figure 3) is computed using the policy iteration algorithm. The value of the minimal cost is given below.

Now we present suboptimal control laws, for a more general discussion concerning these techniques one can consult [4, 3].
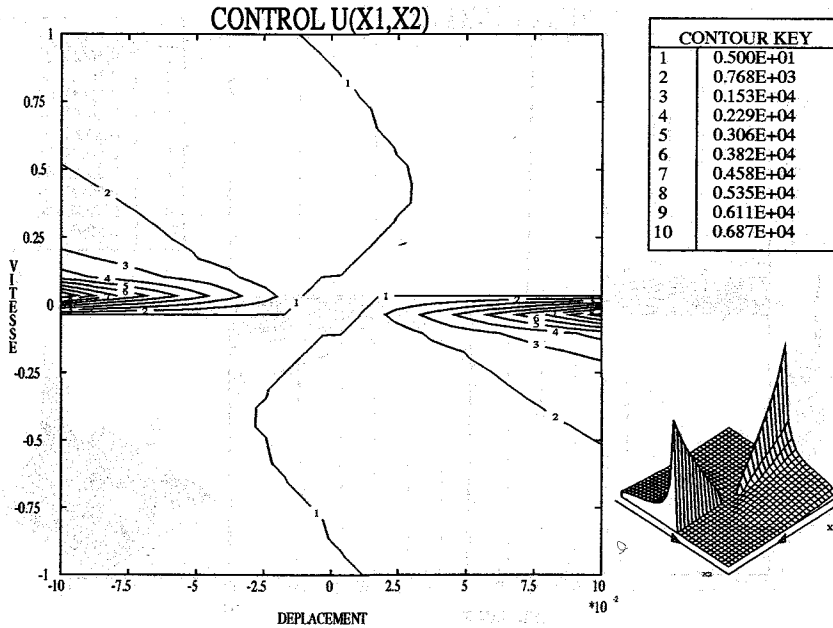
Fig. 3. The optimal feedback control.

suboptimal feedback control #1 ne possibility is to find a feedback control which minimizes the instantaneous cost function (59). We obtain

$$\tilde{u}(x) = \frac{-k_s\, x_1 \mathrm{sign}(x_2) - F_s}{|x_2|} .$$

To take into account the constraint $\underline{u} \leq u \leq \overline{u}$, we use the following control law

$$u(x) = (\tilde{u}(x) \vee \underline{u}) \wedge \overline{u} ,$$

(cf. figure 4) (we take $\underline{u} = 0$ and $\overline{u}$ large).

suboptimal feedback control #2 The previous results lead us to the class of suboptimal feedback controls — parametrized by $\theta \in \mathbb{R}^2$ — of the following form

$$u_\theta(x) \stackrel{\triangle}{=} [(\theta_1 + \theta_2\, x_1 \mathrm{sign}(x_2)) \vee \underline{u}] \wedge \overline{u} , \quad \theta = (\theta_1, \theta_2) \in \mathbb{R}^2 . \tag{70}$$

The techniques presented above can also be applied to compute the suboptimal feedback control $u_{\hat{\theta}}$ such that

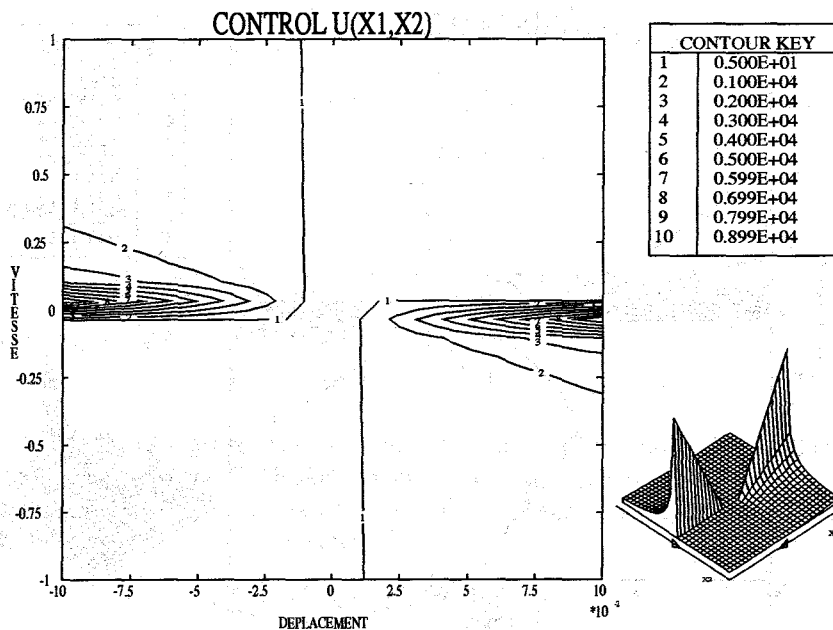$$J_{h,D_h}(u_{\hat{\theta}}) = \min_{\theta \in \Theta} J_{h,D_h}(u_\theta) ,$$

Fig. 4. The suboptimal feedback control #1.

where $\Theta = \{\theta \in \mathbb{R}^2 \; ; \; u_\theta \in \mathcal{U}\}$. We get

$$\hat{\theta}_1 = 137.2, \quad \hat{\theta}_2 = -12130. \tag{71}$$

The control law $u_{\hat{\theta}}(x)$ is plotted on figure 5. A feedback control where the sign of the product $x_1 x_2$ (i.e. $y \dot{y}$) appears has already been proposed in [37].

comparison of the feedback controls Now we compare the three feedback controls presented above to the constant control $u(X) \equiv u_0$. The optimal constant u_0 (i.e. the constant which minimizes the cost) is 188. The different values of the cost are given in the following table

control type	cost
constant control	2.93
suboptimal feedback control #1	2.68
suboptimal feedback control #2	2.37
optimal control	2.22

References

1. M. AKIAN, J.P. CHANCELIER, and J.P. QUADRAT. Dynamic programming complexity and application. In *Proceedings of 27th Conference on Decision and Control*, pages 1551–1558, Austin, Texas, December 1988. IEEE.
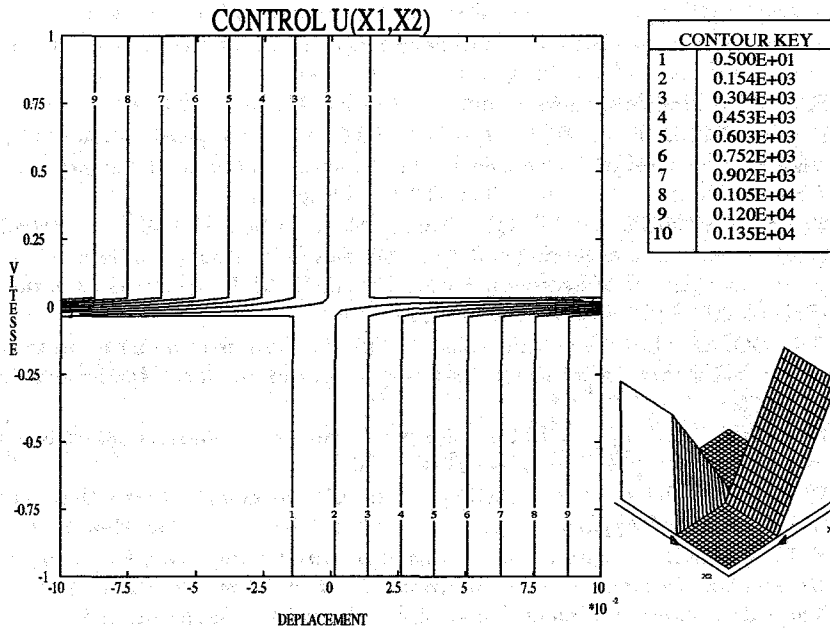
Fig. 5. The suboptimal feedback control #2.

2. J. ALANOLY and S. SANKAR. Semi–active force generators for shock isolation. *Journal of Sound and Vibration*, **126**(1):145–156, 1988.

3. S. BELLIZZI and R. BOUC. Adaptive suboptimal parametric control for non-linear stochastic systems — Application to semi–active vehicle suspensions. In *Effective Stochastic, P. Krée & W. Wedig (eds.)*, 1989. To appear.

4. S. BELLIZZI, R. BOUC, F. CAMPILLO, and E. PARDOUX. Contrôle optimal semi–actif de suspension de véhicule. In *Analysis and Optimization of Systems, A. Bensoussan and J.L. Lions (eds.)*. INRIA, Antibes, 1988. Lecture Notes in Control and Information Sciences 111, 1988.

5. A. BENSOUSSAN. *Stochastic Control by Functional Analysis Methods*, volume 11 of *Studies in Mathematics and its Applications*. North–Holland, Amsterdam, 1982.

6. A. BENSOUSSAN. Discretization of the Bellman equation and the corresponding stochastic control problem. In *26th Conference on Decision and Control*, pages 2251–2254, Los Angeles, CA, December 1987.

7. A. BENSOUSSAN. *Perturbation Methods in Optimal Control*. John Wiley & Sons, New–York, 1988.

8. A. BENSOUSSAN and J.L. LIONS. *Applications des Inéquations Variationnelles en Contrôle Stochastique*, volume 6 of *Méthodes Mathématiques pour l'Informatique*. Dunod, Paris, 1978.

9. A. BENSOUSSAN and J.L. LIONS. *Contrôle Impulsionnel et Inéquations Quasi Variationnelles*, volume 11 of *Méthodes Mathématiques pour l'Informatique*. Dunod, Paris, 1982.

10. D.P. BERTSEKAS. *Dynamic Programming and Stochastic Control*, volume 123 of *Mathematics in Science and Engineering*. Academic Press, New–York, 1976.

11. R. BOUC. Forced vibration of mechanical system with hysteresis. In *Proceedings of 4th conference ICNO*, Prague, 1967. Résumé.

12. R. BOUC. Modèle mathématique d'hystérésis. *Acustica*, 24(3):16–25, 1971.

13. F. CAMPILLO, F. LE GLAND, and E. PARDOUX. Approximation d'un problème de contrôle ergodique dégénéré. In *Colloque International Automatique Non Linéaire*, Nantes, 13–17 Juin 1988. CNRS. To appear.

14. F. DELEBECQUE and J.P. QUADRAT. Contribution of stochastic control singular perturbation averaging and team theories to an example of large–scale systems: management of hydropower production. *IEEE Transactions on Automatic Control*, AC–23(2):209–221, April 1978.

15. B.T. DOSHI. Continuous time control of Markov processes on an arbitrary state space: average return criterion. *Stochastic Processes and their Applications*, 4:55–77, 1976.

16. N. ETHIER and T.G. KURTZ. *Markov Processes – Characterization and Convergence*. J. Wiley & Sons, New–York, 1986.

17. W.H. FLEMING and R.W. RISHEL. *Deterministic and Stochastic Optimal Control*, volume 1 of *Applications of Mathematics*. Springer–Verlag, New York, 1975.

18. F. LE GLAND. *Estimation de Paramètres dans les Processus Stochastiques, en Observation Incomplète — Applications à un Problème de Radio–Astronomie*. Thèse de Docteur–Ingénieur, Université de Paris IX – Dauphine, 1981.

19. R.M. GOODDALL and W. KORTUM. Active controls in ground transportation — A review of the state–of–the–art and future potential. *Vehicle systems dynamics*, 12:225–257, 1983.

20. U.G. HAUSSMANN. *A Stochastic Maximum Principle for Optimal Control of Diffusions*, volume 151 of *Pitman Research Notes in Mathematics Series*. Longman, Harlow, UK, 1986.

21. R.H.W. HOPPE. Multi–grid methods for Hamilton–Jacobi–Bellman equations. *Numerische Mathematik*, 49:239–254, 1986.

22. R.A. HOWARD. *Dynamic Programming and Markov Processes*. J. Wiley, New–York, 1960.

23. I. KARATZAS and S.E. SHREVE. *Brownian Motion and Stochastic Calculus*, volume 113 of *Graduate Texts in Mathematics*. Srpinger–Verlag, New–York, 1988.

24. N. EL KAROUI. *Les aspects probabilistes du contrôle stochastique. Ecole d'Eté de Probabilité de Saint–Flour IX–1979, P.L. Hennequin (ed.)*, volume 876 of *Lecture Notes in Mathematics*. Springer–Verlag, Berlin, 1981.

25. H.J. KUSHNER. *Introduction to Stochastic Control*. Holt, Rinehart and Winston Inc., New York, 1971.

26. H.J. KUSHNER. *Probability Methods for Approximations in Stochastic Control and for Elliptic Equations*, volume 129 of *Mathematics in Science and Engineering*. Academic Press, New–York, 1977.

27. H.J. KUSHNER. Optimality conditions for the average cost per unit time problem with a diffusion model. *SIAM Journal of Control and Optimization*, 16(2):330–346, March 1978.

28. H.J. KUSHNER. Numerical methods for stochastic control problems in continuous time. Technical report, Lefschetz Center for Dynamical Systems, 1988.

29. H. KWAKERNAAK and R. SIVAN. *Linear Optimal Control Systems*. John Wiley & Sons, New York, 1972.

30. P.L. LIONS. *Generalized solutions of Hamilton–Jacobi equations,* volume 69 of *Research Notes in Mathematics.* Pitman, Boston, 1982.

31. P.L. LIONS. On the Hamilton–Jacobi–Bellman equations. *Acta Applicandae Mathematicae,* 1:17–41, 1983.

32. P.L. LIONS and B. MERCIER. Approximation numérique des équations de Hamilton–Jacobi–Bellman. *R.A.I.R.O. Numerical Analysis,* 14(4):369–393, 1980.

33. D. MICHEL and E. PARDOUX. An introduction to Malliavin's calculus and some of its applications. To appear.

34. J.P. QUADRAT. Existence de solution et algorithme de résolution numérique, de problèmes de contrôle optimal de diffusion stochastiques dégénérée ou non. *SIAM Journal of Control and Optimization,* 18(2):199–226, March 1980.

35. J.P. QUADRAT. *Sur l'Identification et le Contrôle de Systèmes Dynamiques Stochastiques.* Thèse, Université de Paris IX – Dauphine, 1981.

36. M. ROBIN. Long–term average cost control problems for continuous time Markov processes: a survey. *Acta Applicandae Mathematicae,* 1:281–299, 1983.

37. S. TAKAHASHI, T. KANEKO, and K. TAKAHASHI. A damping force control method which reduce energy to the vehicle body. *JSAE Review,* 8(3):95–98, 1987.