Numerical Methods in Ergodic Optimal Stochastic Control and Application *

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Abstract

In Campillo *[4] we presented a numerical algorithm for the computation of the optimal feedback law in an ergodic stochastic optimal control problem. This method, based on the discretization of the associated Hamilton–Jacobi–Bellman equation, can be used only in low dimensions (2,4, or 6 in a parallel computer). For higher dimensional problems, we propose here to use a stochastic gradient algorithm in order to find the optimal feedback within a given subclass of parametrized controls. As is [4], we apply these techniques to the control of semi-active suspensions for road vehicles.*

1 Introduction

In this paper we consider numerical procedures for stochastic control problems. Given a real case study (here we consider semi-active control of vehicle suspensions) we can use different methods. For low dimensional problems, we can use optimal methods : we discretize the Hamilton–Jacobi–Bellman equation (this approach is proposed in [4]). In higher dimensions this approach is cumbersome or even impossible to implement, in this case we can look for the optimal feedback in a given subclass of parametrized controls using a stochastic gradient algorithm. The aim of this paper is to compute the stochastic gradient in the simplest two-dimensional model relevant in our application, and to give some numerical results.

In section 2 we introduce the stochastic control problem of ergodic type in $\mathbb{R}^2$ motivated by the application to the control of suspension systems (see Bellizzi et al [1]).

In section 3, we derive the stochastic gradient for the above problem.

2 A stochastic control problem

2.1 The problem

Let $\{X_t(\theta); t \geq 0\}$ be the solution of the following stochastic system in $\mathbb{R}^2$

$$dX_t^1(\theta) = X_t^2(\theta)\, dt \,, \tag{1}$$

$$dX_t^2(\theta) = -\Big[u(\theta, X_t(\theta))\, X_t^2(\theta) + \beta\, X_t^1(\theta) + \gamma\, \mathrm{sign}(X_t^2(\theta))\Big]\, dt + \sigma\, dW_t \tag{2}$$

where

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- β, γ are strictly positive constants,
- W is a standard Brownian motion,
- $u(\theta,x)$ is a feedback control parametrized by $\theta \in \Theta$ (Θ is a compact subset of $\mathbb{R}^d$, for some $d \geq 1$). $u(\theta,x)$ takes values in $U = [u_1, u_2]$, $0 < u_1 < u_2 < \infty$.

The components of $x \in \mathbb{R}^2$ (resp. of $X_t(\theta)$) are denoted by x^1 and x^2 (resp. by $X_t^1(\theta)$ and $X_t^2(\theta)$). For each $\theta \in \Theta$, we consider the following long time average cost functional

$$J(\theta) \triangleq \lim_{T\to\infty} \frac{1}{T} E \int_0^T f(\theta, X_t(\theta))\, dt \tag{3}$$

with

$$f(\theta,x) \triangleq \left[u(\theta,x)\, x^2 + \beta\, x^1 + \gamma\, \mathrm{sign}(x^2)\right]^2 .$$

A class of feedback controls : Let $\mathcal{U}$ denote the set of feedback functions $u(\theta,x)$ such that

- $u : \Theta \times \mathbb{R}^2 \to U = [u_1, u_2]$,
- $\forall x \in \mathbb{R}^2$, $\theta \to u(\theta,x)$ is C^1,
- $\forall \theta \in \Theta$, $x \to u(\theta,x)\, x^2$ is C^1 on $\mathbb{R} \times \mathbb{R} \setminus \{0\}$ with bounded derivatives.

This last condition allows u to be discontinuous at $x^2 = 0$; it is the case for the optimal control which we have computed in a previous work [4].

Note that in (2), the drift coefficient is C^1 on $\mathbb{R} \times \mathbb{R} \setminus \{0\}$, and it is discontinuous across $x^2 = 0$ with a jump of amplitude -2γ.

Lemma 2.1 *For any $\theta \in \Theta$, the system (1,2) admits a unique strong solution.*

Proof In view of Yamada–Watanabe's result (Karatzas–Shreve [8] proposition V–3.20), it is sufficient to prove that

(i) existence of a weak solution,

(ii) pathwise uniqueness

hold for system (1,2).

Part *(i)* has already been proved in Campillo et al [5] : using a Girsanov transformation, we remove the discontinuous terms in (1,2). We can rewrite (1,2) as

$$\begin{aligned} dX_t^1 &= X_t^2\, dt \\ dX_t^2 &= a(X_t^1, X_t^2)\, dt + b(X_t^2)\, dt + \sigma\, dW_t \end{aligned}$$

where a is Lipschitz continuous and b is decreasing. Now let X and X' be two solutions of this system and let $\bar{X} = X - X'$. We get

$$\begin{aligned} \frac{d}{dt}\bar{X}_t^1 &= \bar{X}_t^2 \\ \frac{d}{dt}\bar{X}_t^2 &= \left[a(X_t^1, X_t^2) - a(X_t'^1, X_t'^2) + b(X_t^2) - b(X_t'^2)\right] \end{aligned}$$

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so

$$\begin{aligned}\frac{d}{dt}|\bar{X}_t|^2 &= 2\,\bar{X}^1_t\,\frac{d\bar{X}^1_t}{dt}+2\,\bar{X}^2_t\,\frac{d\bar{X}^2_t}{dt}\\ &= 2\,[\bar{X}^1_t\,\bar{X}^2_t+\bar{X}^2_t\,(a(X^1_t,X^2_t)-a(X'^1_t,X'^2_t))+\underbrace{\bar{X}^2_t\,(b(X^2_t)-b(X'^2_t))}_{\leq 0}]\leq C\,|\bar{X}_t|^2\,.\end{aligned}$$

Hence, using Gronwall's lemma we prove the pathwise uniqueness. ∎

Note that the Lemma also follows from the monotonicity of the drift, and the results in Gyöngy–Krylov [6] and Jacod [7].

2.2 An example : a semi–active suspension system

In this section we present a damping control method for a nonlinear suspension of a road vehicle (comprising a spring, a shock absorber, a mass, and taking into account the dry friction, cf. figure 1). The aim is to improve the ride comfort.

Among alternatives to classical suspension systems (passive systems) we distinguish between active and semi–active techniques. An active suspension system consists of force elements in addition to a spring and a damper assembly. Force elements continuously vary the force according to some control law. In general, an active system is expensive, complicated, and requires an external power source. In contrast, a semi–active system requires no hydraulic power supply, and its hardware implementation is simpler and cheaper than a fully active system. A semi–active suspension system acts only on damping or spring laws, so it can only dissipate or store energy.

Here we consider a system with control on the damping law, the forces in the damper are generated by modulating its orifice for fluid flow. We use the simplest model which consists in a one degree–of–freedom model.

The equation of motion for a one degree–of–freedom model is (cf. figure 1 for the exact definition of the terms)

$$m\,\ddot{y}+c\,\dot{y}+k_s\,y+F_s\,\mathrm{sign}(\dot{y})=-m\,\ddot{e}\,. \tag{4}$$

a absolute displacement of mass m
y relative displacement ($y=a-e$)
e stochastic input (surface road acceleration)
m sprung mass
c shock–absorber damping constant (controlled)
k_s spring constant
F_s dry friction constant

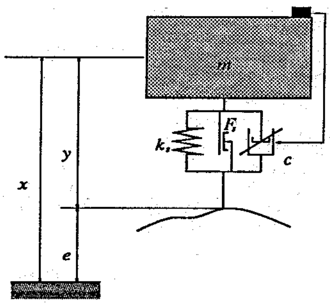

Figure 1: One degree–of–freedom model.

$\ddot{e}$ denotes the input acceleration, i.e. it models the roughness of the road surface. The restoring force $k_s\,y+F_s\,\mathrm{sign}(\dot{y})$, has a linear part $k_s\,y$, and a nonlinear part $F_s\,\mathrm{sign}(\dot{y})$ which describes the dry friction force. The damping force is $c\,\dot{y}$ where $c>0$ is the instantaneous damping coefficient (the control is acting on this term).

The problem is to compute a feedback law $c=c(y,\dot{y})$ such that the solution of the system (4) minimizes a criterion — related to the vibration comfort

$$J(u)\triangleq\lim_{T\to\infty}\frac{1}{T}\,E\int_0^T|\ddot{a}|^2\,dt=\lim_{T\to\infty}\frac{1}{T}\,E\int_0^T|\ddot{y}+\ddot{e}|^2\,dt\,.$$

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$\ddot{e}$ is supposed to be a white Gaussian noise process, $\ddot{e} = -\sigma\, dW/dt$ where W is a standard Wiener process.

Using

$$u = \frac{c}{m}\,,\ \beta = \frac{k_s}{m}\,,\ \gamma = \frac{F_s}{m}\,,\ \text{and } X \triangleq \begin{pmatrix} y \\ \dot{y} \end{pmatrix}\,,$$

equation (4) can be rewritten as (1), (2).

2.3 The "optimal" approach

This approach — presented in [4] — consists in discretizing the Hamilton–Jacobi–Bellman (HJB) equation associated with the ergodic stochastic control problem for the diffusion (1), (2) with the cost function (3) (but now the feedback function u is not parametrized)

The HJB equation is of the form

$$\min_{u \in [u_1, u_2]} (\mathcal{L}^u v(x) + f(u, x)) = \rho\,, \quad \forall x \in \mathbb{R}^2$$

where $v : \mathbb{R}^2 \to \mathbb{R}$ is defined up to an additive constant, ρ is a constant and $\mathcal{L}^u$ is the infinitesimal generator associated with the diffusion process (1), (2) (with $u(\theta, x)$ replaced by $u \in \mathcal{U}$).

3 Stochastic approximation algorithms

The problem is to find θ^* which minimizes the cost function (3). In [4] we have already proved that the cost function is of the form

$$J(\theta) = \int_{\mathbb{R}^2} f(\theta, x)\, \mu_\theta^X(dx) \tag{5}$$

where μ_θ^X is the unique invariant measure of the process $X_t(\theta)$. As pointed out in Ladelli [9], the gradient of $J(\theta)$ is *not* equal to

$$\int_{\mathbb{R}^2} \nabla_\theta f(\theta, x)\, \mu_\theta^X(dx)\,.$$

It is possible to use a Kiefer–Wolfowitz–type algorithm in order to minimize $J(\theta)$, thus avoiding the computation of the gradient of J. However, it seems that stochastic gradient algorithms often converge faster than Kiefer–Wolfowitz algorithms. Motivated by this remark, we shall compute here the gradient of $J(\theta)$, which involves differentiating $X_t(\theta)$.

3.1 The gradient process

In this section we investigate the regularity of the process $X_t(\theta)$ with respect to θ and we establish an equation for the gradient process

$$Y_t(i, \theta) \triangleq \frac{\partial}{\partial \theta_i} X_t(\theta)\,.$$

This derivation will give rise to the local time of the process $\{X_t^2(\theta)\}$ at 0.

Following Protter [11] we have the

Lemma 3.1 *The processes X_t^2 admits a local time at 0, which is the unique adapted, continuous, nondecreasing process L_t such that $L_0 = 0$, $\int_0^\infty 1_{\mathbb{R}\setminus\{0\}}(X_s^2)\, dL_s = 0$ a.s., and*

$$|X_t^2| = |X_0^2| + \int_0^t \operatorname{sign}(X_s^2)\, dX_s^2 + 2\, L_t\,. \tag{6}$$

Moreover,

$$L_t = \lim_{\varepsilon \to 0}\text{-a.s.}\ \frac{\sigma^2}{2\varepsilon} \int_0^t 1_{[-\varepsilon, \varepsilon]}(X_s^2)\, ds\,. \tag{7}$$

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The following result is proved in the Appendix below.

Proposition 3.2 *For each $t \geq 0$, $X_t(\theta)$ is mean–square differentiable w.r.t. the parameter θ_i ($i = 1, \dots, d$), and $Y_t(i,\theta) \triangleq \partial X_t(\theta)/\partial\theta_i$ is the solution of the following system*

$$dY_t(\theta) = A(\theta, X_t(\theta))\, Y_t(\theta)\, dt + B\, Y_t(\theta)\, dL_t + C(\theta, X_t(\theta))\, dt\ , \tag{8}$$

with

$$A(\theta,x) \triangleq \begin{pmatrix} 0 & 1 \\ -\beta - u_{x^1}(\theta,x)x^2 & -u(\theta,x) - u_{x^2}(\theta,x)x^2 \end{pmatrix}$$

$$B \triangleq \begin{pmatrix} 0 & 0 \\ 0 & -\frac{2\gamma}{\sigma^2} \end{pmatrix}\ , \qquad C(\theta,x) \triangleq \begin{pmatrix} 0 \\ -u_\theta(\theta,x)x^2 \end{pmatrix}\ .$$

$Y_0 \equiv 0$, and where L is the local time of X^2 at 0. We denote

$$Y_t(\theta) = \left[Y_t(1,\theta)|\cdots|Y_t(d,\theta)\right]\ .$$

For the sake of simplicity, from now on we suppose that θ is scalar, so we drop the subscript i. We have an explicit representation of $Y_t(\theta)$ in terms of $X_t(\theta)$. Let $\{\Phi_t(\theta)\,,\, t \geq 0\}$ be the solution of

$$d\Phi_t(\theta) \triangleq A(\theta, X_t(\theta))\, \Phi_t(\theta)\, dt + B\, \Phi_t(\theta)\, dL_t\ , \qquad \Phi_0(\theta) = I\ , \tag{9}$$

then

$$Y_t(\theta) = \int_0^t \Phi_t(\theta)\, \Phi_s(\theta)^{-1}\, C(\theta, X_s(\theta))\, ds\ . \tag{10}$$

3.2 Asymptotic properties of $(X_t(\theta), Y_t(\theta))$

We note

$$\xi_t(\theta) = \begin{pmatrix} X_t(\theta) \\ Y_t(\theta) \end{pmatrix}\ . \tag{11}$$

In [4] we showed that $X_t(\theta)$ admits a unique invariant measure μ_θ^X. We extend the process $X_t(\theta)$ for all $t \in \mathbb{R}$, such that it is stationary and μ_θ^X is the law of $X_t(\theta)$ for all $t \in \mathbb{R}$. We can then solve equation (9) and define $\Phi_t(\theta)$ for all $t \in \mathbb{R}$. It is easily seen that the Markov process $\xi_t(\theta)$ possesses a unique invariant measure μ_θ iff the following integral converges a.s.

$$\int_{-\infty}^0 \Phi_t(\theta)^{-1}\, C(\theta, X_t(\theta))\, dt\ .$$

A sufficient condition for that fact is given in the following lemma

Lemma 3.3 *Suppose that there exists $C > 0$ such that for all (θ, x)*

$$\beta + u_{x_1}(\theta,x)\, x^2 \geq C > 0\ ,$$
$$u(\theta,x) + u_{x^2}(\theta,x)\, x^2 \geq C > 0\ ,$$

then there exists $\lambda < 0$ such that

$$\limsup_{t\to\infty} \frac{1}{t} \log\left\|\Phi_{-t}(\theta)^{-1}\right\| \leq \lambda \text{ a.s.} \tag{12}$$

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Proof With the hypotheses of the Lemma and using the properties of the solution of (9), we can prove that

$$\Phi_t(\theta) \underset{t\to\infty}{\longrightarrow} 0 \text{ a.s.} \tag{13}$$

Moreover there exists C such that $\|\Phi_t(\theta)\| \leq C$, $\forall t \geq 0$, and by bounded convergence $E\|\Phi_t(\theta)\| \to 0$, $t \to \infty$. Hence, given $\alpha < 1$, there exists $t > 0$ such that

$$E\|\Phi_t(\theta)\| = \alpha\ . \tag{14}$$

Now, using the ergodicity property of $\Phi_t(\theta)$ as in Bougerol [2,3]

$$\begin{aligned}
\lim_{T\to\infty} \frac{1}{T} \log \|\Phi_T(\theta)\| &= \lim_{T\to\infty} \frac{1}{T} E\left(\log \|\Phi_T(\theta)\|\right) \\
&= \frac{1}{t} \lim_{n\to\infty} \frac{1}{n} E\left(\log \|\Phi_{nt}(\theta)\|\right) \\
&\leq \frac{1}{t} \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} E\left(\log \left\|\Phi_{kt,(k-1)t}(\theta)\right\|\right) \\
&= \frac{1}{t} E\left(\log \|\Phi_t(\theta)\|\right) \\
&\leq \frac{1}{t} \log E\|\Phi_t(\theta)\| \\
&= \frac{\log \alpha}{t} < 0
\end{aligned}$$

where $\{\Phi_{s,t}(\theta)\, ,\, t \geq s\}$ is the solution of (9) satisfying $\Phi_{s,s}(\theta) = I$.

Note that $\Phi_t(\theta)$ and $\Phi_{-t}(\theta)^{-1}$ are equal in law, so $E\|\Phi_t(\theta)\| = E\|\Phi_{-t}(\theta)^{-1}\|$, and similarly as above

$$\lim_{T\to\infty} \frac{1}{T} \log \left\|\Phi_{-T}(\theta)^{-1}\right\| = \lim_{T\to\infty} \frac{1}{T} E\left(\log \left\|\Phi_{-T}(\theta)^{-1}\right\|\right) = \lim_{T\to\infty} \frac{1}{T} E\left(\log \|\Phi_T(\theta)\|\right) < 0 \text{ a.s.}$$

■

3.3 The gradient of the cost functional

Let

$$g(\theta, x) \triangleq u(\theta, x)\, x^2 + \beta\, x^1 + \gamma\, \text{sign}(x^2)$$

so that $f(\theta, x) = [g(\theta, x)]^2$. We have

$$\begin{aligned}
f_\theta(\theta, x) &= 2\, g(\theta, x)\, u_\theta(\theta, x)\, x^2\ , \\
f_{x^1}(\theta, x) &= 2\, g(\theta, x)\, [u_{x^1}(\theta, x)\, x^2 + \beta]\ ,
\end{aligned}$$

and, in the sense of distributions, $\delta(x^2)$ denoting the Dirac measure "in the variable x^2,

$$\begin{aligned}
f_{x^2}(\theta, x) &= 2\, g(\theta, x)\, [u_{x^2}(\theta, x)\, x^2 + u(\theta, x) + 2\,\gamma\, \delta(x^2)] \\
&= 2\, g(\theta, x)\, [u_{x^2}(\theta, x)\, x^2 + u(\theta, x)] + 4\, \beta\, \gamma\, x^1\, \delta(x^2)\ .
\end{aligned}$$

Let

$$\begin{aligned}
\tilde{f}_{x^1}(\theta, x) &= f_{x^1}(\theta, x) \\
\tilde{f}_{x^2}(\theta, x) &= 2\, g(\theta, x)\, [u_{x^2}(\theta, x)\, x^2 + u(\theta, x)]
\end{aligned}$$

and

$$F(\theta, x, y) \triangleq f_\theta(\theta, x) + \tilde{f}_x(\theta, x)\, y\ .$$

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Formally, if we take the derivative of the cost functional (3) with respect to θ, and if we interchange the derivation and the limit as $T \to \infty$, we get formally

$$\lim_{T\to\infty} \frac{1}{T} E \left[\int_0^T F(\theta, X_t(\theta), Y_t(\theta))\, dt + \int_0^T 4\,\beta\,\gamma\, X_t^1(\theta)\, Y_t^2(\theta)\, \delta(X_t^2(\theta))\, dt \right]$$

and one can show rigorously that the gradient of the cost functional is given by

$$\nabla_\theta J(\theta) = \lim_{T\to\infty} \frac{1}{T} E \left[\int_0^T F(\theta, X_t(\theta), Y_t(\theta))\, dt + \int_0^T 4\,\beta\,\gamma\, X_t^1(\theta)\, Y_t^2(\theta)\, dL_t \right] . \tag{15}$$

It is possible to prove that the process $(X_t(\theta), Y_t(\theta))$ admits a unique invariant measure μ_θ which is regular with respect to the parameter θ, from which one can conclude that the limit (15) is well defined.

3.4 Stochastic gradient algorithm

In order to minimize (5), we want to find $\theta^* \in \Theta$ such that

$$\nabla_\theta J(\theta)\big|_{\theta=\theta^*} = 0 . \tag{16}$$

The associated stochastic gradient algorithm is the following : given $\Delta t > 0$ and $t_k \triangleq k\,\Delta t$, we solve equations (1),(2),(8) with

$$\theta = \theta_k \text{ for } t_k \le t < t_{k+1} ,$$

and θ_k is given by

$$\theta_{k+1} = \theta_k - \rho_k \left[F(\theta_k, X_{t_k}(\theta_k), Y_{t_k}(\theta_k))\, \Delta t + 4\,\gamma\,\beta\, X_{t_k}^1(\theta_k)\, Y_{t_k}^2(\theta_k)\, \Delta L_k \right] , \tag{17}$$

where

$$\Delta L_k = L_{t_{k+1}} - L_{t_k} ,$$

and where the sequence of positive gains $\{\rho_k\}$ satisfies appropriate conditions.

4 Computational aspects and Numerical Results

4.1 Time discretization

We approximate X_t by X_t^n given by the following Euler scheme

$$X_{k+1}^{n,1} = X_k^{n,1} + X_k^{n,2}\, \Delta t , \tag{18}$$

$$X_{k+1}^{n,2} = X_k^{n,2} - \left(u(\theta, X_t^n)\, X_k^{n,2} + \beta\, X_k^{n,1} + \gamma\, \mathrm{sign}(X_k^{n,2}) \right) \Delta t + \sigma\, \Delta W_k^n \tag{19}$$

where

$$\Delta t \triangleq t_{k+1} - t_k , \quad \Delta W_k^n \triangleq W_{t_{k+1}} - W_{t_k} \sim \mathcal{N}(0, t_{k+1} - t_k) .$$

For Y_t we also use an Euler scheme

$$Y_{k+1}^{n,1} = Y_k^{n,1} + Y_k^{n,2}\, \Delta t , \tag{20}$$

$$\begin{aligned} Y_{k+1}^{n,2} = {} & Y_k^{n,2} - \left(u_\theta(\theta, X_k^n) + u_x(\theta, X_k^n)\, Y_k^n \right) X_k^{n,2}\, \Delta t \\ & - u(\theta, X_k^n)\, Y_k^{n,2}\, \Delta t - \beta\, Y_k^{n,1}\, \Delta t - \frac{2\gamma}{\sigma^2} Y_k^{n,2}\, \Delta L_k^n \end{aligned} \tag{21}$$

where ΔL_k^n is an approximation of $L_{t_{k+1}} - L_{t_k}$ given by

$$\Delta L_k^n = \begin{cases} |X_{k+1}^{n,2}| & \text{if } X_k^{n,2}\, X_{k+1}^{n,2} < 0 , \\ 0 & \text{otherwise,} \end{cases}$$

(cf. proposition 4.2).

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Proposition 4.1 *For all $t \geq 0$ and all $\theta \in \Theta$*

$$X_t^n(\theta) \underset{n\to\infty}{\overset{L^2}{\to}} X_t(\theta) \ .$$

Proof We fixe $\theta \in \Theta$. We use the notation

$$dX_t = h(X_t)\,dt + G\,dW_t \tag{22}$$

where

$$h(x) = h_1(x) + h_2(x) \ , \quad h_2(x) = \begin{pmatrix} 0 \\ -\gamma\,\mathrm{sign}(x_2) \end{pmatrix}$$

h_1 is Lipschitz continuous and h_2 is discontinuous and *monotonic.* Let $\Delta t = 1/n$, $t_k = k\,\Delta t$ and $\phi_n(t) = t_k$ if $t \in [t_k, t_{k+1}[$. Let W_t^n be the polygonal interpolation of the Wiener process W_t, that is

$$W_t^n = W_{t_k} + (t - t_k)\,\frac{W_{t_{k+1}} - W_{t_k}}{\Delta t} \ , \quad t \in [t_k, t_{k+1}[$$

Then the Euler scheme (18) reads

$$dX_t^n = h(X_{\phi_n(t)}^n)\,dt + G\,dW_t^n \ . \tag{23}$$

Because X_t^n is not adapted to the filtration of the Wiener process and for technical simplification, we can replace W_t^n is this last equation by

$$W_t^n = W_{t_k} + (t - t_k)\,\frac{W_{t_{k+1}} - W_{t_k}}{\Delta t} \ ; \quad t \in [t_{k-1}, t_k[$$

(with the convention $W_t = 0$ for $t \leq 0$) in this case X_t^n is adapted.

The difference between (22) and (23) gives

$$d[X_t - X_t^n] = [h(X_t) - h(X_t^n)]\,dt + [h(X_t^n) - h(X_{\phi_n(t)}^n)]\,dt + G\,d[W_t - W_t^n]$$

and by Itô formula

$$\begin{aligned} |X_t - X_t^n|^2 \ = \ & 2\int_0^t (h(X_s) - h(X_s^n), X_s - X_s^n)\,ds + 2\int_0^t \left(h(X_s^n) - h(X_{\phi_n(s)}^n), X_s - X_s^n\right) ds \\ & +2\ (X_t - X_t^n, G\,(W_t - W_t^n)) + |G\,(W_t - W_t^n)|^2 \\ & +2\int_0^t (G\,(W_s - W_s^n), h(X_s) - h(X_s^n))\,ds \ . \end{aligned} \tag{24}$$

Now we have the following results

- By Lispchitz continuity of h_1 and monotonicity of h_2

$$\begin{aligned} & \int_0^t (h(X_s) - h(X_s^n), X_s - X_s^n)\,ds \\ & = \int_0^t (h_1(X_s) - h_1(X_s^n), X_s - X_s^n)\,ds + \int_0^t \underbrace{(h_2(X_s) - h_2(X_s^n), X_s - X_s^n)}_{\leq 0}\,ds \\ & \leq \int_0^t (h_1(X_s) - h_1(X_s^n), X_s - X_s^n)\,ds \\ & \leq C\int_0^t |X_s - X_s^n|^2 ds \ . \end{aligned}$$

- By Lispchitz continuity of h_1

$$\int_0^t \left(h(X_t^n) - h(X_{\phi_n(t)}^n), X_s - X_s^n\right) ds$$
$$\leq C \int_0^t |X_s - X_s^n|^2 ds + C \int_0^t \left(h(X_s^n) - h(X_{\phi_n(s)}^n), X_s - X_s^n\right) ds$$
$$\leq C \int_0^t |X_s - X_s^n|^2 ds + C \int_0^t |X_s^n - X_{\phi_n(s)}^n|^2 ds + C \int_0^t |\mathrm{sign}(X_s^{n,2}) - \mathrm{sign}(X_{\phi_n(s)}^{n,2})|^2 ds$$

and

$$E|\mathrm{sign}(X_t^{n,2}) - \mathrm{sign}(X_{\phi_n(t)}^{n,2})|^2 \leq C\, P(|X_s^{n,2}| \leq \alpha) + C\, E\left(\mathbf{1}_{(|X_s^{n,2}|>\alpha)} |\mathrm{sign}(X_t^{n,2}) - \mathrm{sign}(X_{\phi_n(t)}^{n,2})|^2\right)$$

and this last term tends to 0 as $n \to \infty$.

- For $0 < \rho < 1$

$$2\, |(X_t - X_t^n, G(W_t - W_t^n))| \leq \rho\, |X_t - X_t^n|^2 + \frac{\sigma^2}{\rho} |W_t - W_t^n|^2$$

- By Hölder inequality

$$E\, 2 \int_0^t (G(W_s - W_s^n), h(X_s) - h(X_s^n))\, ds \leq C \left(E \int_0^t |W_s - W_s^n|^2 ds\right)^{1/2}$$

So from (24) and Gronwall's Lemma

$$E|X_t - X_t^n|^2 \leq \left(\varepsilon_n + C \int_0^t P(|X_s^{n,2}| \leq \alpha)\, ds\right) e^{C\,t} \tag{25}$$

where $\varepsilon_n \to 0$, as $n \to \infty$.

It is simple to see that the sequence (X^n, W^n) is tight, so there exists a subsequence (also denoted (X^n, W^n)) and a process (Z, W) such that $(X^n, W^n) \to (Z, W)$ weakly. The process Y_t is of the from

$$Y_t = X_0 + \int_0^t \chi_s\, ds + G\, W_t$$

where W is a Wiener process, so we have $P(|Y_s^2| \leq \alpha) \to 0$ as $\alpha \to 0$. Then form (25) we deduce

$$\limsup_{n\to\infty} E|X_t - X_t^n|^2 \leq C \int_0^t P(|Y_s^2| \leq 2\,\alpha)\, ds\, e^{C\,t}$$

which tends to 0 as $\alpha \to 0$. Then by uniqueness of the limit we prove that $X^n \to X$ in L^2. ■

4.2 Approximation of the local time

From (6)

$$L_t = \frac{1}{2}|X_t^2| - \frac{1}{2}|X_0^2| - \frac{1}{2}\int_0^t \mathrm{sign}(X_s^2)\, dX_s^2\ . \tag{26}$$

We approximate L_t the following way : in (26) we replace X_t^2 by the polygonal interpolation of the discrete process $X_k^{n,2}$ given in (19)

$$X_t^{n,2} \triangleq \sum_{k\geq 0} \frac{(t_{k+1}^n - t)\, X_{n_k}^{n,2} + (t - t_k^n)\, X_{n_{k+1}}^{n,2}}{t_{k+1}^n - t_k^n} \mathbf{1}_{[t_k^n, t_{k+1}^n)}(t)$$

with $t_k^n = k/n$. So we get the approximation

$$L_t^n = \frac{1}{2}|X_t^{n,2}| - \frac{1}{2}|X_0^{n,2}| - \frac{1}{2}\int_0^t \mathrm{sign}(X_{\phi_n(s)}^{n,2})\, dX_s^{n,2}\ , \tag{27}$$

where $\phi_n(s) = t_k^n$ if $s \in [t_k^n, t_{k+1}^n)$.

Proposition 4.2

$$L_{t_{k+1}^n}^n - L_{t_k^n}^n = \begin{cases} |X_{t_{k+1}^n}^{n,2}| & \textit{if}\ X_{t_{k+1}^n}^{n,2} X_{t_k^n}^{n,2} < 0\ , \\ 0 & \textit{otherwise.} \end{cases}$$

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Proof Let $t \in [t_k^n, t_{k+1}^n)$, $\Delta t_k^n = t_{k+1}^n - t_k^n$, $\Delta X_k^{n,2} = X_{t_{k+1}^n}^{n,2} - X_{t_k^n}^{n,2}$,

$$\begin{aligned}\frac{dL_t^n}{dt} &= \frac{1}{2}\frac{d}{dt}|X_t^{n,2}| - \frac{1}{2}\operatorname{sign}(X_{t_k^n}^{n,2})\frac{\Delta X_k^{n,2}}{\Delta t_k^n}\,,\\ &= \frac{1}{2}\operatorname{sign}(X_t^{n,2})\frac{\Delta X_k^{n,2}}{\Delta t_k^n} - \frac{1}{2}\operatorname{sign}(X_{t_k^n}^{n,2})\frac{\Delta X_k^{n,2}}{\Delta t_k^n}\,,\\ &= \frac{1}{2}\frac{\Delta X_k^{n,2}}{\Delta t_k^n}\left[\operatorname{sign}(X_t^{n,2}) - \operatorname{sign}(X_{t_k^n}^{n,2})\right]\,,\end{aligned}$$

then

$$\operatorname{sign}(X_t^{n,2}) - \operatorname{sign}(X_{t_k^n}^{n,2}) = 2\operatorname{sign}(X_{t_{k+1}^n}^{n,2})\,\mathbf{1}_{(X_t^{n,2}X_{t_k^n}^{n,2}<0)}\,,$$

so

$$\frac{dL_t^n}{dt} = \operatorname{sign}(X_{t_{k+1}^n}^{n,2})\frac{\Delta X_k^{n,2}}{\Delta t_k^n}\mathbf{1}_{(X_t^{n,2}X_{t_k^n}^{n,2}<0)}\,,$$

and

$$L_t^n = \sum_{k\geq 0}\operatorname{sign}(X_{t_{k+1}^n}^{n,2})\frac{\Delta X_k^{n,2}}{\Delta t_k^n}\lambda(s\in[t_k^n,t_{k+1}^n)\cap[0,t];X_s^{n,2}X_{t_k^n}^{n,2}<0)$$

where λ is Lebesgue measure. Finally,

$$L_{t_{k+1}^n}^n - L_{t_k^n}^n = \operatorname{sign}(X_{t_{k+1}^n}^{n,2})\frac{\Delta X_k^{n,2}}{\Delta t_k^n}\lambda(s\in[t_k^n,t_{k+1}^n);X_s^{n,2}X_{t_k^n}^{n,2}<0)$$

and

$$\lambda(s\in[t_k^n,t_{k+1}^n);X_s^{n,2}X_{t_k^n}^{n,2}<0) = \begin{cases} X_{t_{k+1}^n}^{n,2}\dfrac{\Delta t_k^n}{\Delta X_k^{n,2}} & \text{if } X_{t_{k+1}^n}^{n,2}X_{t_k^n}^{n,2}<0\,,\\ 0 & \text{otherwise}\,.\end{cases}$$

■

Proposition 4.3 *For all* t

$$L_t^n \xrightarrow[n\to\infty]{L^2} L_t\,.$$

Proof Using the definitions (26), (27) of L_t and L_t^n, it is sufficient to prove that

$$\int_0^t \operatorname{sign}(X_s^2)\,dX_s^2 \xrightarrow[n\to\infty]{L^2} \int_0^t \operatorname{sign}(X_{\phi_n(s)}^{n,2})\,dX_s^{n,2}\,.$$

Using the notations of the proof of Proposition 4.1, we have

$$\begin{aligned}&E\left|\int_0^t \operatorname{sign}(X_s^2)\,dX_s^2 - \int_0^t \operatorname{sign}(X_{\phi_n(s)}^{n,2})\,dX_s^{n,2}\right|^2\\ &\leq C\int_0^t E|\phi(X_{\phi_n(s)}^n) - \phi(X_s)|^2\,ds\\ &\quad + C\int_0^t E|\operatorname{sign}(X_{\phi_n(s)}^{n,2}) - \operatorname{sign}(X_s^2)|^2\,ds + C\,E\left|\int_0^t \operatorname{sign}(X_{\phi_n(s)}^{n,2})\,d[W_s^n - W_s]\right|^2\end{aligned}$$

where $\phi(x) = \operatorname{sign}(x^2)\,h(x)$ satisfies $|\phi(x)| \leq C\,(1 + |x|)$. So the result follows from the same arguments used in the proof of Proposition 4.1. ■

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4.3 Numerical results

We consider the example of section 2.2. We use the following parameters

m	110
K	26000
F	85
σ	0.5

We use a feedback control of the form

$$u(\theta, x) = \theta^1 + \left[-\theta^2 x^1 \operatorname{sign}(x^2)\right]^+$$

where θ^1 and θ^2 are positive.

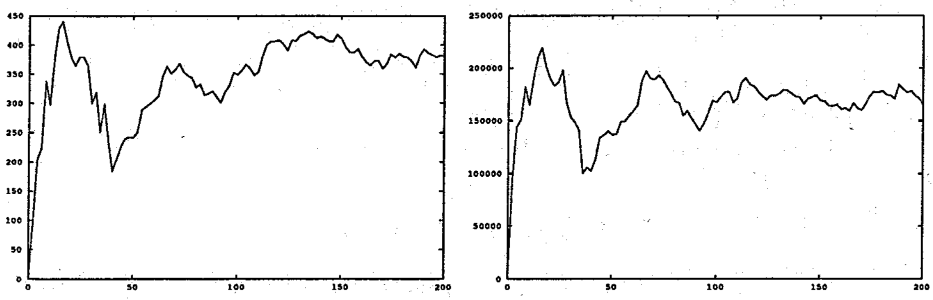
We do not exactly use the algorithm (17), but the following modification

$$\theta_{k+1} = \theta_k - \begin{pmatrix} \rho_k^1 & 0 \\ 0 & \rho_k^2 \end{pmatrix} \left[F\left(\theta_k, X_{t_k}(\theta_k), Y_{t_k}(\theta_k)\right) \Delta t + 4\,\beta\,\gamma\, X_{t_k}^1(\theta_k)\, Y_{t_k}^2(\theta_k)\, \Delta L_k\right] ,$$

that is we do not exactly use the direction of the gradient but another direction of minimization which is more convenient in practice. Moreover we do not take $\rho_k^i = 1/i$ (which is not very good in practice) but

$$\rho_k^i = a_i + \frac{b_i}{\max(1, k - k_i)} , \qquad i = 1, 2 ,$$

where a_i, b_i, k_i are given. In figure 2 we have an example of trajectories : $k \to \hat{\theta}_k^1$ and $k \to \hat{\theta}_k^2$.



$k \to \theta_k^1$ $\qquad$ $k \to \theta_k^2$

Figure 2: Case 3, the plot of θ^1

5 Conclusion

We have studied in this paper an ergodic stochastic control problem, where the control is to be chosen in a parametrized family of feedback laws. We have computed the gradient of its cost functional. The corresponding stochastic gradient algorithm has been implemented on an applied problem, and has proved to be efficient. It would be worthwhile to prove the convergence of that algorithm towards a global minimum of our cost functional. Unfortunately, the best results that we know in that direction are those of M. Métivier and P. Priouret [10], which required the cost functional to have a single local (and hence global) minimum. There is no reason for that assumption to hold in our case. However,

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using ideas from simulated annealing, one might perhaps show that our algorithm (or possibly a modified version of it) converges to a global minimum. Such a study is far beyond the scope of the present paper.

The authors wish to thank Prof. P. Bougerol for some helpful remarks used in the proof of Lemma 3.3.

Appendix : Regularity of $X_t(\theta)$

We use the notation of Lemma 2.1,

$$\begin{aligned} dX_t^1(\theta) &= X_t^2(\theta)\,dt\,, \\ dX_t^2(\theta) &= a(\theta, X_t(\theta))\,dt + b(X_t^2(\theta))\,dt + \sigma\,dW_t\,, \end{aligned}$$

where $b(x^2) = -\gamma\,\mathrm{sign}(x^2)$ is non increasing. In this section we suppose that θ is scalar and that the function $a(\theta, x)$ is regular enough in (θ, x) : differentiable with bounded derivatives,

$$\begin{aligned} a(\theta', x) - a(\theta, x) &= a_\theta(\theta, x)\,(\theta' - \theta) + \mathcal{O}^1_{\theta,x}(\theta' - \theta)\,, \\ a(\theta, x') - a(\theta, x) &= a_x(\theta, x)\,(x' - x) + \mathcal{O}^2_{\theta,x}(x' - x)\,, \end{aligned}$$

with $a_\theta(\theta, x)$, $a_x(\theta, x)$ bounded, continuous and

$$\sup_x \mathcal{O}^1_{\theta,x}(\theta)/|\theta| \to 0 \text{ as } |\theta| \to 0\,,$$

$$\sup_\theta \mathcal{O}^2_{\theta,x}(x)/|x| \to 0 \text{ as } |x| \to 0\,.$$

Define

$$\bar{X}_t^i = \frac{X_t^i(\theta + h) - X_t^i(\theta)}{h}\,, \quad i = 1, 2\,.$$

Lemma 5.1 *The process X_t^θ is continuous with respect to θ, more precisely : there exists a certain constant $C > 0$ such that*

$$\left|\bar{X}_t\right|^2 \le C\,e^{Ct}\,, \quad \forall t \ge 0\,. \tag{28}$$

Proof

$$\begin{aligned} \frac{d}{dt}|\bar{X}_t|^2 &= 2\,\bar{X}_t^1\,\frac{d}{dt}\bar{X}_t^1 + 2\,\bar{X}_t^2\,\frac{d}{dt}\bar{X}_t^2 \\ &= 2\,\bar{X}_t^1\,\bar{X}_t^2 + 2\,\bar{X}_t^2\,\frac{a(\theta + h, X_t(\theta + h)) - a(\theta, X_t(\theta))}{h} + 2\,\underbrace{\bar{X}_t^2\,\frac{b(X_t^2(\theta + h)) - b(X_t^2(\theta))}{h}}_{\le 0}\,, \\ &\le C_1\,|\bar{X}_t|^2 + C_2\,. \end{aligned}$$

The result follows from Gronwall's lemma. ∎

Proof of Proposition 3.2

$$\begin{aligned} d\bar{X}_t^1 &= \bar{X}_t^2\,dt\,, \\ d\bar{X}_t^2 &= \frac{a(\theta + h, X_t(\theta + h)) - a(\theta, X_t(\theta))}{h}\,dt + \frac{b(X_t^2(\theta + h)) - b(X_t^2(\theta))}{h}\,dt \end{aligned}$$

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and

$$
\begin{aligned}
dY_t^1 &= Y_t^2\,dt\,,\\
dY_t^2 &= (a_\theta(\theta,X_t(\theta))+a_x(\theta,X_t(\theta))\,Y_t)\;dt-\frac{2\gamma}{\sigma^2}Y_t^2\,dL_t\,,
\end{aligned}
$$

so

$$
\begin{aligned}
d[\bar X_t^1-Y_t^1] &= [\bar X_t^2-Y_t^2]\,dt\,,\\
d[\bar X_t^2-Y_t^2] &= \left[\frac{a(\theta+h,X_t(\theta))-a(\theta,X_t(\theta))}{h}-a_\theta(\theta,X_t(\theta))\right]dt\\
&\quad+\left[\frac{a(\theta+h,X_t(\theta+h))-a(\theta+h,X_t(\theta))}{h}-a_x(\theta+h,X_t(\theta))\,\bar X_t\right]dt\\
&\quad+[a_x(\theta+h,X_t(\theta))-a_x(\theta,X_t(\theta))]\,\bar X_t\,dt\\
&\quad+a_x(\theta,X_t(\theta))\,[\bar X_t-Y_t]\,dt\\
&\quad+\frac{b(X_t^2(\theta+h))-b(X_t^2(\theta))}{h}\,dt+\frac{2\gamma}{\sigma^2}Y_t^2\,dL_t\,.\\
&= \frac{\mathcal{O}^1_{\theta,X_t(\theta)}(h)}{h}\,dt+\frac{\mathcal{O}^2_{\theta+h,X_t(\theta)}(h\,\bar X_t)}{h}\,dt\\
&\quad+[a_x(\theta+h,X_t(\theta))-a_x(\theta,X_t(\theta))]\,\bar X_t\,dt\\
&\quad+a_x(\theta,X_t(\theta))\,[\bar X_t-Y_t]\,dt\\
&\quad+\frac{b(X_t^2(\theta+h))-b(X_t^2(\theta))}{h}\,dt+\frac{2\gamma}{\sigma^2}Y_t^2\,dL_t\,.
\end{aligned}
$$

Then

$$
\begin{aligned}
d|\bar X_t-Y_t|^2 &= 2\,(\bar X_t^1-Y_t^1)\,d[\bar X_t^1-Y_t^1]+2\,(\bar X_t^2-Y_t^2)\,d[\bar X_t^2-Y_t^2]\\
&= 2\,(\bar X_t^1-Y_t^1)\,(\bar X_t^2-Y_t^2)\,dt\\
&\quad+2\,(\bar X_t^2-Y_t^2)\,\frac{\mathcal{O}^1_{\theta,X_t(\theta)}(h)}{h}\,dt+2\,(\bar X_t^2-Y_t^2)\,\frac{\mathcal{O}^2_{\theta+h,X_t(\theta)}(h\,\bar X_t)}{h}\,dt\\
&\quad+2\,(\bar X_t^2-Y_t^2)\,[a_x(\theta+h,X_t(\theta))-a_x(\theta,X_t(\theta))]\,\bar X_t\,dt\\
&\quad+2\,(\bar X_t^2-Y_t^2)\,a_x(\theta,X_t(\theta))\,[\bar X_t-Y_t]\,dt\\
&\quad+2\,(\bar X_t^2-Y_t^2)\,\frac{b(X_t^2(\theta+h))-b(X_t^2(\theta))}{h}\,dt\\
&\quad+4\,(\bar X_t^2-Y_t^2)\,\frac{\gamma}{\sigma^2}Y_t^2\,dL_t\,,
\end{aligned}
$$

so we have the following inequality

$$
\begin{aligned}
|\bar X_t-Y_t|^2 &\le C\int_0^t|\bar X_s-Y_s|^2\,ds\\
&\quad+C\int_0^t\left|\frac{\mathcal{O}^1_{\theta,X_s(\theta)}(h)}{h}\right|^2ds+C\int_0^t\left|\frac{\mathcal{O}^2_{\theta+h,X_s(\theta)}(h\,\bar X_t)}{h}\right|^2ds\\
&\quad+\int_0^t|\bar X_s|^2\,|a_x(\theta+h,X_s(\theta))-a_x(\theta,X_s(\theta))|^2\,ds\\
&\quad+C\left|\int_0^t(\bar X_s^2-Y_s^2)\,\frac{b(X_s^2(\theta+h))-b(X_s^2(\theta))}{h}\,ds+\int_0^t\frac{2\gamma}{\sigma^2}(\bar X_s^2-Y_s^2)\,Y_s^2\,dL_s\right|\,.
\end{aligned}
$$

Let Δ_t^1 denotes the last term of the right hand side of this last inequality.

$$
\Delta_t^1 \le \gamma\left|\int_0^t(\bar X_s^2-Y_s^2)\,\frac{\mathrm{sign}(X_s^2(\theta+h))-\mathrm{sign}(X_s^2(\theta))}{h}\,ds\right.
$$

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$$
\begin{aligned}
& - \int_0^t (\bar{X}_s^2 - Y_s^2) \frac{\mathrm{sign}_\varepsilon(X_s^2(\theta+h)) - \mathrm{sign}_\varepsilon(X_s^2(\theta))}{h} ds \Big| \\
& + \gamma \left| \int_0^t (\bar{X}_s^2 - Y_s^2) \frac{\mathrm{sign}_\varepsilon(X_s^2(\theta+h)) - \mathrm{sign}_\varepsilon(X_s^2(\theta))}{h} ds - \int_0^t \frac{2\gamma}{\sigma^2} (\bar{X}_s^2 - Y_s^2) Y_s^2 \, dL_s \right| \\
\leq \; & 2 \int_0^t |\bar{X}_s - Y_s|^2 ds \\
& + 2 \int_0^t \left| \frac{\mathrm{sign}(X_s^2(\theta+h)) - \mathrm{sign}(X_s^2(\theta))}{h} - \frac{\mathrm{sign}_\varepsilon(X_s^2(\theta+h)) - \mathrm{sign}_\varepsilon(X_s^2(\theta))}{h} \right|^2 ds \\
& + \gamma \left| \int_0^t (\bar{X}_s^2 - Y_s^2) \frac{\mathrm{sign}_\varepsilon(X_s^2(\theta+h)) - \mathrm{sign}_\varepsilon(X_s^2(\theta))}{h} ds - \int_0^t \frac{2\gamma}{\sigma^2} (\bar{X}_s^2 - Y_s^2) Y_s^2 \, dL_s \right| .
\end{aligned}
$$

Let Δ_t^2 denotes the last term of the right hand side of this last inequality.

For h small enough

$$
\begin{aligned}
& \int_0^t (\bar{X}_s^2 - Y_s^2) \frac{\mathrm{sign}_\varepsilon(X_s^2(\theta+h)) - \mathrm{sign}_\varepsilon(X_s^2(\theta))}{h} ds \\
& \quad = \int_0^t (\bar{X}_s^2 - Y_s^2) \frac{\mathrm{sign}_\varepsilon(X_s^2(\theta+h)) - \mathrm{sign}_\varepsilon(X_s^2(\theta))}{h} 1_{[-\varepsilon,\varepsilon]}(X_s^2(\theta)) \, ds \\
& \quad = 2 \int_0^t (\bar{X}_s^2 - Y_s^2) \bar{X}_s^2 \frac{1}{2\varepsilon} 1_{[-\varepsilon,\varepsilon]}(X_s^2(\theta)) \, ds \\
& \quad \underset{\varepsilon \to 0}{\overset{\text{a.s.}}{\longrightarrow}} \frac{2}{\sigma^2} \int_0^t (\bar{X}_s^2 - Y_s^2) \bar{X}_s^2 \, dL_s \\
& \quad \underset{h \to 0}{\simeq} \frac{2}{\sigma^2} \int_0^t (\bar{X}_s^2 - Y_s^2) Y_s^2 \, dL_s \, .
\end{aligned}
$$

Then from (29),

$$
|\bar{X}_t - Y_t|^2 \leq C \int_0^t |\bar{X}_s - Y_s|^2 ds + \nu_t(\theta, h, \varepsilon) ,
$$

where

$$
\lim_{h \to 0} \lim_{\varepsilon \to 0} \nu_s(\theta, h, \varepsilon) = 0 \quad \text{a.s.}
$$

Then using Gronwall's inequality

$$
|\bar{X}_t - Y_t|^2 \leq \nu_t(\theta, h, \varepsilon) + C \int_0^t \nu_s(\theta, h, \varepsilon) e^{C(t-s)} ds
$$

so

$$
|\bar{X}_t - Y_t|^2 \to 0 \, .
$$

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